

Schwarzschild Cosmology

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Abstract

The internal Schwarzschild solution is examined in the context of a cosmological model. An examination of the Hubble parameter and redshift is made and it is shown that the model agrees with cosmological data in predicting the transition redshift when the expansion of the Universe changes from deceleration to acceleration. Distance modulus is plotted against redshift and compared to cosmological data. The consequences for an observer in freefall in a local gravitational field are then examined in the context of this model. It is shown that the model predicts an accelerated expansion that agrees with current observations of the expansion history of our Universe, namely that the initial expansion is infinitely fast, and then the expansion slows for some time followed by an accelerated expansion.

Expansion Along a Timelike Dimension

The current Big Bang model of the Universe says that the Universe expanded from an infinitely dense gravitational singularity at some time in the past. Current cosmological data suggests that this expansion was slowing down for some time, but is now continuing to expand at an accelerated rate. The Cosmological Principle suggests that from any reference frame in the Universe, the mass distribution is spherically symmetric and isotropic. It is proposed that the observed expansion of the Universe is the result of a freefall in the time dimension. To analyze the spherically symmetric Universe freefalling through the time dimension, we need the Schwarzschild solution where the radial coordinate is the timelike coordinate. The interior ($r < 1$) solution of the Schwarzschild field (throughout the paper, we will work in units with Schwarzschild radius equal to 1) gives us precisely that. For $r < 1$, the signature of the Schwarzschild metric flips and the radial coordinate becomes a dimension measuring time while the t coordinate becomes a dimension measuring space.

But the Schwarzschild metric is a vacuum solution to Einstein's equations. Given that the Universe is full of energy, further justification is required to use the vacuum solution. Consider an isotropically collapsing spherical star (such that the surface of the star is in freefall). The external Schwarzschild solution is valid outside the surface of the star and is static as the star collapses. The spacetime inside the star is not governed by the Schwarzschild metric due to the energy inside the star. However, the metric inside the star and the metric outside the star must agree at the surface of the star and therefore the particles on the surface of the star see the Schwarzschild metric on the surface. For the $r < 1$ Schwarzschild metric, it will be shown that at a fixed r , we have a spacelike 3-dimensional slice of the Universe (the space of the Universe at a given cosmological time). Therefore, if we assume that in our frame of reference the future spacetime of the

Universe is empty (i.e. the Universe for cosmological times greater than our current cosmological time is a vacuum), we see that the $r < 1$ Schwarzschild metric can be applied to our cosmology for the same reasons that we can apply the external Schwarzschild metric to the surface of a collapsing star. Thus, whereas in the case of the collapsing star, the boundary was a 2-sphere collapsing in space, in the cosmological case the boundary (present time) is a 3-sphere expanding in time (in both cases, the boundary is defined by a given value of r). In fact, if we begin with the assumption that the future spacetime is a vacuum, then according to Birkhoff's theorem, the $r < 1$ Schwarzschild solution is the only possible cosmological solution. Thus, the Schwarzschild solution as a cosmological model is based on 2 assumptions:

1. The Universe is spacetime and its geometry is governed by the equations of General Relativity
2. The 'present' of the Universe is defined as the state of the Universe in which space is uniformly filled with energy at the present time and void of energy at all later times

That energy does not fill all times can also be justified by looking at the momentum 4-vector:

$$P = m_0 \frac{dX}{d\tau} = \begin{bmatrix} m_0 \frac{dt}{d\tau} \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} = \begin{bmatrix} E \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} \quad (1)$$

As can be seen from Equation 1, the rest energy comes from the velocity of the mass through the time dimension. So energy can't just exist statically at some particular place and time, as would be the case if the future were already filled with energy. For the energy to exist, the matter must be flowing through time, as though the present state of energy in the Universe is freefalling through the time dimension into the future vacuum. Equation 1 will also help us understand the extreme points of this freefall.

So let us take the center of our galaxy as the origin of an inertial reference frame. We can draw a line through the center of the reference frame that extends infinitely in both directions radially outward. This line will correspond to fixed angular coordinates (θ, ϕ) . There are infinitely many such lines, but since we have an isotropic, spherically symmetric Universe, we only need to analyze this model along one of these lines, and the result will be the same for any line.

The radial distance in this frame is kind of a compound dimension. It is a distance in space as well as a distance in time. The farther away a galaxy is from us, the farther back in time the light we currently receive from it was emitted. Fortunately the $r < 1$ spacetime of the Schwarzschild solution plotted in Kruskal-Szekeres coordinates provides us with a method to understand this radial direction. Figure 1 shows the $r < 1$ solution on a Kruskal-Szekeres coordinate chart where, in this model, the hyperbolas of

constant r represent spacelike slices of constant cosmological time and the rays of t represent radial distances (each point on this plot is a 2-sphere and each hyperbola is a 3-sphere).

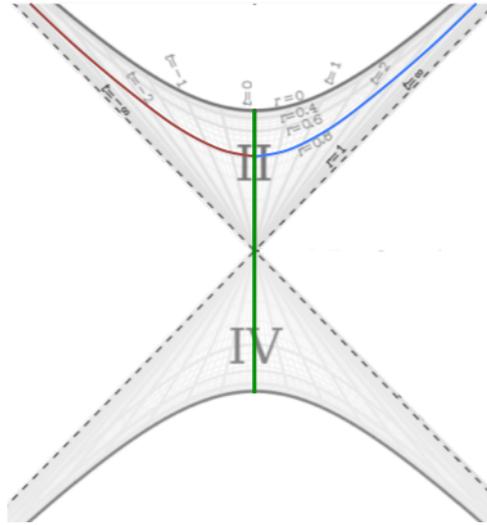


Figure 1 – Freefall Through Cosmological Time¹

We must first determine the paths of inertial observers in the spacetime. For this we need the internal Schwarzschild metric and the geodesic equations for the internal Schwarzschild metric [1]:

$$d\tau^2 = \frac{r}{1-r} dr^2 - \frac{1-r}{r} dt^2 - r^2 d\Omega^2 \quad (2)$$

$$\frac{d^2 t}{d\tau^2} = \frac{1}{r(1-r)} \frac{dr}{d\tau} \frac{dt}{d\tau} \quad (3)$$

$$\frac{d^2 r}{d\tau^2} = \frac{1}{2r^2} \left[\frac{1-r}{r} \left(\frac{dt}{d\tau} \right)^2 - \frac{r}{1-r} \left(\frac{dr}{d\tau} \right)^2 \right] \quad (4)$$

In Equations 2, 3, and 4, we use units where $c = 1$ and the constant in the Schwarzschild metric is 1 and equations 3 and 4 assume no angular motion. Looking at points $0 < r < 1$, then by inspection of Equation 3 it is clear that an inertial observer at rest at t ($\frac{d^2 t}{d\tau^2} = 0$ if $\frac{dt}{d\tau} = 0$). Also, we see that if an observer is moving inertially with some initial $\frac{dt}{d\tau}$, then if $\frac{dr}{d\tau} < 0$, the coordinate speed of the observer will be reduced over time (the coordinates are expanding beneath her) and if $\frac{dr}{d\tau} > 0$, the coordinate speed will be increased over time (the coordinates are collapsing beneath her).

¹ Diagram modified from: "Kruskal diagram of Schwarzschild chart" by Dr Greg. Licensed under CC BY-SA 3.0 via Wikimedia Commons - http://commons.wikimedia.org/wiki/File:Kruskal_diagram_of_Schwarzschild_chart.svg#/media/File:Kruskal_diagram_of_Schwarzschild_chart.svg

Let us therefore examine Equation 4 for an inertial observer at rest at t . From Equation 1, we see that this corresponds to $\frac{dt}{d\tau} = 0$ and $\frac{dr}{d\tau} = \pm \sqrt{\frac{1-r}{r}}$. Plugging these expressions into Equations 4 gives:

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2r^2} \quad (5)$$

Notice that the observer's acceleration through cosmological time is similar to the form of Newton's law of gravity, where r (a time coordinate) varies from 1 to 0 (If the Schwarzschild constant was $2GM$, as it would be in the external solution, Equation 5 would be Newton's gravity). Comparing the bracketed term in Equation 4 to Equation 1, we see that Equation 5 will hold for all inertial observers (with $d\Omega = 0$) since the bracketed term will always be -1.

So we will use Figure 1 to describe the freefall of the galaxies through the cosmological time dimension where galaxies (or galaxy clusters) follow lines of constant t (and any such observer can choose $t = 0$ as their coordinate). The 'Big Bang' will have occurred at the center of Figure 1 at $r = 1$. We know this because the above analysis showed that space expands if $\frac{dr}{d\tau}$ is negative, so for our current cosmological time, our worldlines must be moving toward $r = 0$.

How we see the Universe

It is important at this point to re-emphasize that we are not modeling the Universe as a fluid that fills the entire spacetime described by the metric. The energy of the Universe only exists on a 3-sphere defined by a particular hyperbola of constant r at a given instant, and the rest of the spacetime is a vacuum.

Looking at Figure 1, we should note that light signals travel on 45-degree angles. So when we look out at the Universe, we can imagine that we are seeing light emitted from concentric 2-spheres from when the energy of the Universe was at the particular coordinate time corresponding to a particular 2-sphere. They are 2-spheres because each sphere represents a specific coordinate time in the past and distance from us, they are not independent. We can choose to observe the Universe at any arbitrary past time, but we cannot choose to observe the Universe at an arbitrary distance at a time, the distance from us we observe depends on the present age of the Universe and the age of the 2-sphere we observe. Nonetheless, each 2-sphere will appear to us to be spatially homogeneous and isotropic and this is reflected in Equation 2 (if we fix the r of a 2-sphere, the space will be homogenous and isotropic).

Normally, the r in the angular term of the metric would be a spacelike Euclidean radius that would range from 0 to infinity. However, the fact that this metric is describing a cosmological geometry makes the temporal characteristic of the radius more understandable. What makes having a time coordinate as the radius makes sense in this context is that we can't see farther back in time than the Big Bang, so even though the

Universe is ever expanding, we will always only have a finitely observable Universe. This is because since we are looking back in time as we look out in space, the past of the Universe is finite and thus the maximum radius must be finite, in this case, $r = 1$ (as time goes on, a particular r will correspond to greater and greater distances from us, but the range of r will always be finite).

Analysis of the Expansion

Expressions for the proper time interval along lines of constant t and Ω and the proper distance interval along hyperbolas of constant r and Ω from Equation 1 are:

$$\frac{dr}{d\tau} = \pm \sqrt{\frac{1-r}{r}} \quad (6)$$

$$\frac{ds}{dt} = \pm \sqrt{\frac{1-r}{r}} \quad (7)$$

First we should notice that neither Equation 6 nor 7 depend on the t coordinate. This is good because the t coordinate marks the position of other galaxies relative to ours. Since all galaxies are freefalling in time inertially, the particular position of any one galaxy should not matter. The proper velocity and proper distance only depends on the cosmological time r . When $r = 1$, Equations 6 and 7 are both 0. At this point (the Big Bang), it is our proper velocity in time that is zero. So at that instant, we are no longer moving through time and therefore all points in space are coincident (the observer can reach every point in space without moving through time). This is analogous to being at rest in space where all points of time essentially converge to a single point in space (the observer can reach every point in time without moving through space). So this why the proper distance goes to zero there and why the lines of t in Figure 1 converge at that point; it is an instant where our velocity through cosmological time goes to zero as our speed through cosmological time changes from positive to negative (we can see that if we draw a worldline through the center point, $\frac{dr}{d\tau}$ will change signs as it passes the $r = 1$ point).

At $r = 0$, both equations 6 and 7 are infinite. So when the worldlines enter or exit one of the $r = 0$ hyperbolas, they do so at infinite proper speed *through the time dimension*. If something is travelling through space at the speed to light, the proper distance between points in space is zero. In this case, since we have infinite proper velocity in the time dimension, the proper distance between points in space will be infinite, because you would traverse an infinite amount of time in order to move through an infinitesimal amount of space. What we see then is that at $r = 0$ space will be infinitely expanded.

We can also understand these extreme points using Equation 1. At $r = 1$, $\frac{dr}{d\tau} = 0$ and according to Equation 1, this would mean that massive objects no longer have rest energy. Thus there are no inertial reference frames in the Universe at that point; all worldlines are lightlike at that instant. Since lightlike objects see space infinitely

contracted, we see that at $r = 1$ space will be infinitely contracted in all frames. At $r = 0$, $\frac{dr}{d\tau} = \infty$, meaning that the rest energies of massive objects is effectively infinite. We can think of this as objects having infinite inertia. With infinite inertia, objects at rest cannot be accelerated and thus space is effectively infinitely expanded since motion is no longer possible. We saw previously that inertial observers with non-zero initial $\frac{dt}{d\tau}$ will be slowed down during expansion and this was explained by saying that the coordinates were expanding beneath her. But we can also think of it as her inertia increasing as a result of her increasing $\frac{dr}{d\tau}$, stopping her motion completely at $r = 0$. Note that according to Equation 3, in the first moments following the Big Bang the acceleration opposing peculiar motion is quite high and there would therefore be very little peculiar motion in those first moments as well, meaning the initial energy distributions would be very smooth.

A plot of $\frac{dr}{d\tau}$ vs. r during the expansion ($\frac{dr}{d\tau} < 0$) is given in Figure 2 below:

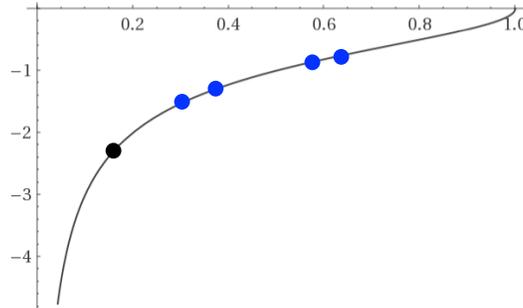


Figure 2 - $\frac{dr}{d\tau}$ vs. r

In Figure 2, time moves forward as we move right to left along the diagram. So the leftmost dot represents us at our current cosmological time in the Universe. The other dots represent galaxies at various distances from us (the farther they are to the right of the diagram, the greater the distance from us) whose signals we are currently receiving. Since the signals we receive now were emitted in the past, they were emitted from galaxies with a lower proper velocity than we have currently. Thus, the signals should all be redshifted since our proper velocity is currently greater than the velocities of the galaxies when they emitted the signals and the magnitude of the redshift should be proportional to the difference in velocity. Now note that in the region where the other dots are, the difference in velocity of the two more distant emitters is less than the difference in velocity of the closer emitters. This difference means that when we get the signals from the galaxies, the difference in redshift for the two closer galaxies will be greater than the difference in redshift from the two more distant galaxies, which looks like accelerated expansion. But on this graph, there is actually an inflection point at $r = 0.75$. That means that from $r = 1$ to $r = 0.75$, it would appear as if the Universe is expanding, but the expansion is slowing down. Then from $r = 0.75$ to $r = 0$, the Universe will look like it is expanding at an accelerated rate. This change from a negatively accelerating expansion to a positively accelerating expansion is consistent

with current cosmological data. Note that Equation 7, which describes the proper distance between coordinate points at a given r , has the same form as the function plotted in Figure 2 such that the rate of increase in proper distance decreases from $r = 1$ to $r = 0.75$ and then increases afterwards. Thus, from this equation, we can see that the rate at which wavelengths of light are stretched will first decrease and then increase, just as was discussed above.

Infinite Curvature

The timelike part of the metric is finite and ends at $r = 0$ where the curvature is infinite. Since r is the time coordinate, it seems curious that there should only be a finite amount of coordinate time from $r = 1$ to $r = 0$. But we can make sense of this from Equation 6. It tells us the velocity through the time dimension of an observer at rest relative to the CMB. First, we should note that the rest observer is the ‘slowest’ possible observer in the spacetime. Anyone moving with a velocity relative to the CMB will move faster through the time dimension than the rest observer. But we see from Equation 6 that the velocity of the rest observer through the time dimension goes to infinity in her own frame as she approaches $r = 0$. Thus, we can imagine that there is indeed an infinite amount of coordinate time, but since the velocity of all observers through time increases in their own frames over time, the future time coordinates get increasingly contracted in all frames. As their velocities approach infinity, the infinite future coordinates contract to a single coordinate position at $r = 0$ (a more correct interpretation would be that the time coordinates are increasingly contracted due to the intrinsic geometry, resulting in the increasing speed through time for timelike observers). So when observers reach $r = 0$, they are essentially traversing an infinite amount of coordinate time in zero proper time.

Redshift and the Hubble Parameter

We can use the fact that $\frac{1-r}{r}$ is the square of the scale factor and get the expression for cosmological redshift caused by the expansion [1]:

$$z = \sqrt{\frac{r_{emit}}{(1-r_{emit})}} \sqrt{\frac{(1-r)}{r}} - 1 \quad (8)$$

We can use Equation 8 to predict the redshift of the Universe at the time the expansion changed from decelerating to accelerating. First, we must find the value of the constant of the Schwarzschild metric, which we will call u (we have heretofore set this constant equal to 1). For the external metric, this constant has the value of the Schwarzschild radius of a mass given by $2GM$. For the interior metric, this constant will need to be a time; specifically, it will be the coordinate time in years from the ‘Big Bang’ to $r = 0$. We can use the known Hubble parameter and current age of the Universe to find this constant. The Hubble parameter is given by:

$$H = \frac{\dot{a}}{a} = \frac{d}{dr} \left(\sqrt{\frac{u-r}{r}} \right) \sqrt{\frac{r}{u-r}} = \frac{u}{2r(u-r)} \quad (9)$$

We know that the Universe is around 13.8 billion years old, so in Equation 9 we can make the substitution $r = u - 13.8$ (because the Big Bang occurs at $r = u$). The Hubble parameter at this time has been measured to be around 67.8 (km/s)/Mpc. Converting that value to units of 1/(billion years), setting Equation 9 equal to that value and solving for u we get an approximate value of:

$$u \approx 28.8 \text{ billion years} \quad (10)$$

We can now express r in units of billions of years from $r = 0$ (the Big Bang occurs at $r = 28.8$). A plot of Equation 9 with the value $u = 28.8$ and the Λ CDM model [2] with $\Lambda = 0.013$ is given in Figure 3 below (our current time is shown as the dashed vertical line):

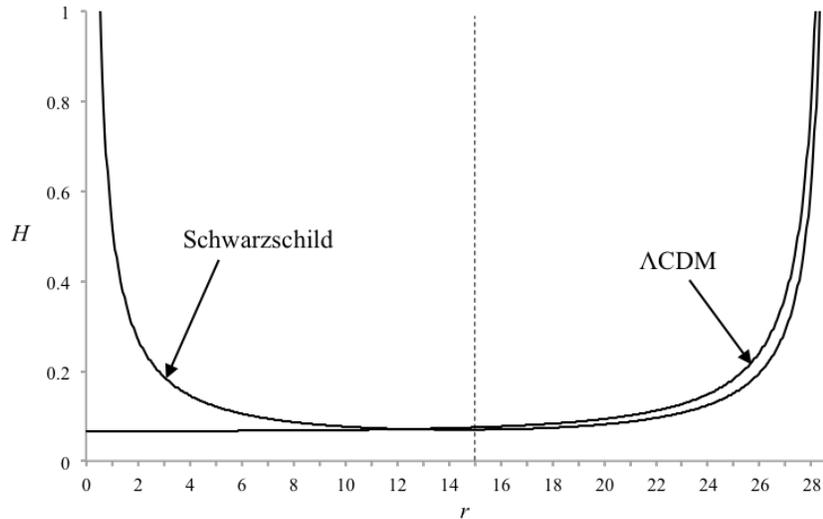


Figure 3 – Hubble Parameter vs. r ($u = 28.8$, $\Lambda = 0.013$)

Equation 8 can be used to find the transition redshift, which is the redshift we observe at the point when the Universe transitioned from a decelerating expansion to an accelerating expansion. In Equation 8, this transition occurs at $r_{emit} = 0.75$ and now that we know u , we know that our current dimensionless r in Equation 8 is $r = 0.52$. Plugging those values into Equation 8 we get an estimated transition redshift of:

$$z_t = 0.66 \quad (11)$$

This value is within the 2σ bound for the parameter [3,4], and therefore it does appear to be in agreement with cosmological measurements. A plot of redshifts measured at our current time vs. time is given in Figure 4 below:

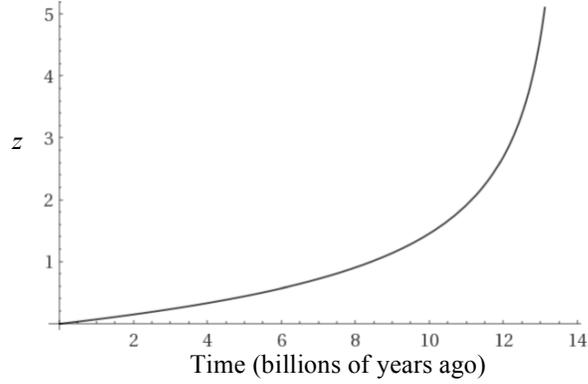


Figure 4 – Redshift vs. Time

Finally, the deceleration parameter is given by:

$$q = \frac{\ddot{a}a}{\dot{a}^2} = \frac{4r}{u} - 3 = \frac{r}{7.2} - 3 \quad (12)$$

A plot of the deceleration parameter is given in Figure 5 below:

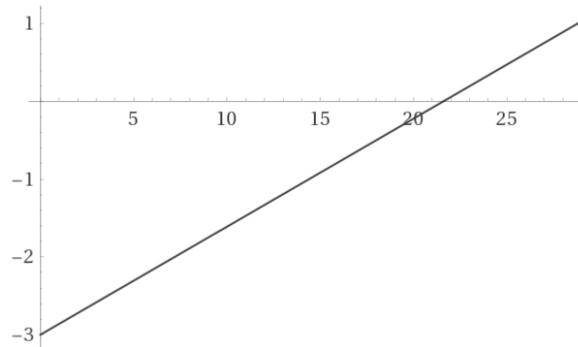


Figure 5 – Deceleration Parameter vs. r

Coordinate Distance & Distance Modulus

Figure 1 is a plot of the metric on a Kruskal-Szekeres coordinate chart where the T -axis is the vertical axis and the X -axis is the horizontal axis. The definition of T and X are given below for $u = 28.8$:

$$X = \sinh\left(\frac{t}{57.6}\right) \sqrt{(28.8 - r)e^{\frac{r}{28.8}}} \quad (13)$$

$$T = \cosh\left(\frac{t}{57.6}\right) \sqrt{(28.8 - r)e^{\frac{r}{28.8}}} \quad (14)$$

Light travels on 45-degree lines in these coordinates so if we consider our current reference frame at $t = 0$ and $r = 15$, we can find the coordinate distance t of some

galaxy we observe along the 45-degree line at some r by setting $\Delta X = -\Delta T$ and solving for t . When we do this, we get:

$$t = 28.8 \ln\left(\frac{23.23}{28.8-r}\right) - r \quad (15)$$

Where t is in billions of light years and $15 \leq r \leq 28.8$. Note that Equation 15 is only valid for the current cosmological time. The 23.23 constant is specific to this time so for some other time, a different constant would be required and is given by the value $C = (28.8 - r_0)e^{\frac{r_0}{28.8}}$. We can also use Equation 8 to find r_{emit} as a function of z and substitute that into Equation 15 to get the coordinate distance as a function of redshift. If we set $r = 15$ for $u = 28.8$ in Equation 8 and solve for r_{emit} we get:

$$r_{emit} = 28.8 \frac{z^2 + 2z + 1}{z^2 + 2z + 1.92} \quad (16)$$

Substituting Equation 16 into 15 will give the coordinate distance as a function of measured redshift. A commonly used parameter in cosmology is the distance modulus, μ , which is defined as:

$$\mu = 5 \log_{10}\left(\frac{d}{10}\right) \quad (17)$$

Where d is the distance measured in parsecs. A plot of distance modulus vs. redshift obtained by combining Equations 15, 16, and 17 (where we use t measured in parsecs for d in Equation 17) is shown in Figure 6 below plotted over data obtained from the Supernova Cosmology Project [6]:

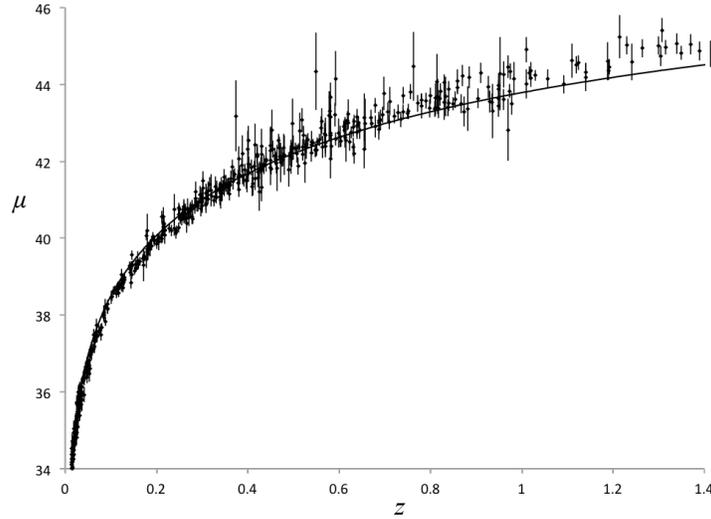


Figure 6 – Distance Modulus vs. Redshift

Note that all these predictions only required the spherical symmetry assumptions of the Schwarzschild metric and calculation of a single parameter, u , from cosmological data; it requires no information regarding the detailed energy distribution within the Universe. In fact, the value of u only determines the units we are working in; it does not affect the

form of the model. This reflects the fact that the details of the expansion are the result of the vacuum solution alone. Thus, we should expect that the model is accurate for the vacuum-dominated era of the Universe and less so for the matter and radiation dominated eras. We see this in Figure 6 where the model starts to under predict the distance modulus at high redshifts.

Proper Time of the Rest Observer

Figure 7 shows the past light cone of an inertial observer at a given time during the expansion:

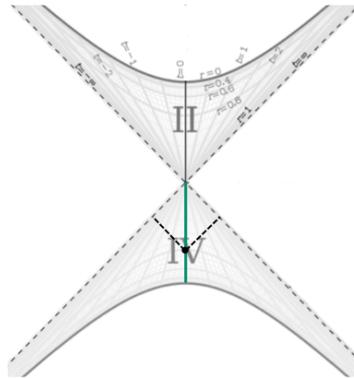


Figure 7 – Past Light Cone of Inertial Observer During the Expansion²

Notice that at all times during the expansion, the past light cone includes the entire width of the Universe. This means that (if there were no CMB to block the view) the observable Universe is the entire Universe. We see however from Equation 7 that the proper distance to the edge of the observable Universe when $r < u$ is infinite (because t ranges from negative infinity to infinity).

We can calculate the duration of the expansion of the Universe in the frame of an inertial observer at rest by integrating Equation 6 from 0 to 1. This integral yields a value of $\frac{\pi}{2}$, but this is in units where the Schwarzschild constant is 1. The total time of expansion is therefore:

$$\tau = \frac{\pi}{2} u \quad (18)$$

Where τ is measured in billions of years. Equation 18 tells us that in the frame of an observer at rest at t , the time elapsed from the Big Bang to $r = 0$ measured by her clock would be around 45.2 billion years and there is only about 8.8 billion years of proper time between now and $r = 0$ for her. If we imagine the Universe as a pendulum, τ in Equation 18 may also be thought of as $\frac{1}{4}$ of its period.

² Diagram modified from: "Kruskal diagram of Schwarzschild chart" by Dr Greg. Licensed under CC BY-SA 3.0 via Wikimedia Commons - http://commons.wikimedia.org/wiki/File:Kruskal_diagram_of_Schwarzschild_chart.svg#/media/File:Kruskal_diagram_of_Schwarzschild_chart.svg

Thinking of τ as a ‘Universal Period’ allows us to define a Universal constant $U = \frac{\pi}{2}u$ for time and space. Equation 18 is the maximum amount of time that can be measured between the Big Bang and $r = 0$. So if we set $U = \frac{\pi}{2}u = c = 1$ then we are working in units where space and time have the same units and all measurable times will be between 0 and 1. When working in these units, the constant in the interior Schwarzschild metric will be $u = \frac{2}{\pi}$.

Schwarzschild Metric and Geodesics in Terms of the Hubble Parameter

We can re-express equations 2-5 in terms of the scale factor $a = \sqrt{\frac{u-r}{r}}$ and the Hubble parameter:

$$d\tau^2 = a^{-2}dr^2 - a^2dt^2 - r^2d\Omega^2 \quad (19)$$

$$\frac{d^2t}{d\tau^2} = 2H \frac{dr}{d\tau} \frac{dt}{d\tau} \quad (20)$$

$$\frac{d^2r}{d\tau^2} = a^2H \left[a^2 \left(\frac{dt}{d\tau} \right)^2 - a^{-2} \left(\frac{dr}{d\tau} \right)^2 \right] \quad (21)$$

$$\frac{d^2r}{d\tau^2} = -a^2H \quad (22)$$

Equation 22 gives us a quantity analagous to the surface gravity used in the external solution. The non-zero Christoffel symbols of the model (for $d\Omega = 0$) in terms of H are:

$$\Gamma_{rr}^r = -H \quad (23)$$

$$\Gamma_{tr}^t = \Gamma_{rt}^t = H \quad (24)$$

$$\Gamma_{tt}^r = a^4H \quad (25)$$

The ‘Big Bounce’

A plot of τ vs. r from the uppermost to lowermost hyperbola in Figure 1 is given in Figure 8 below. It illustrates well the relationship to typical spatial projectile motion (for $u = 1$).

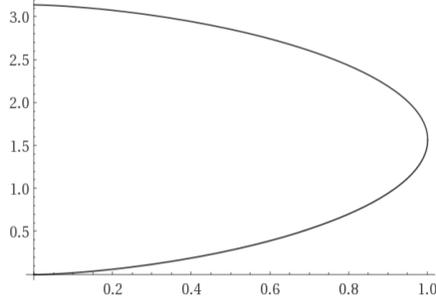


Figure 8 - τ vs. r

Consider a perfectly rigid and elastic ball in simple Newtonian mechanics. If we throw it straight up in the air with initial velocity $\frac{dx}{d\tau}$, the velocity will continuously decrease until at some height $\frac{dx}{d\tau} = 0$, at which point the ball will reverse direction and fall with increasingly negative $\frac{dx}{d\tau}$ until it returns to the ground. When it hits the ground, since it is perfectly rigid and elastic, it will experience an infinite acceleration that will bounce it back toward its maximum height and this cycle will continue ad infinitum. So there are two turnaround points for the ball. One point is maximum height, where the ball does not experience any special acceleration; it just stops moving through space as it turns around. The second point is a hard acceleration that the ball can really feel a (infinite) force changing its direction.

Likewise, we can see that the Schwarzschild cosmology is a similar situation except that the Universe is the ball and the acceleration is through time rather than space. The Big Bang corresponds to maximum height, where the Universe's velocity through time changes sign. The Hyperbolas are the 'bounce'. When the ball bounced, it experienced an infinite acceleration. In the cosmological case, when $r = 0$ the curvature of the spacetime is infinite [1]. This infinite curvature may be a point in time where the worldlines of the Universe turn back on themselves as if the spacetime is folded there and the worldlines go up one side and down the other (the infinite curvature is at the fold). Equation 22 is the inertial coordinate acceleration through time (proper acceleration is zero). If we express that equation in terms of proper acceleration we get:

$$A^r = \frac{d^2r}{d\tau^2} + \frac{1}{\pi r^2} \quad (26)$$

Immediately before $r = 0$, the coordinate velocity $\frac{dr}{d\tau}$ for the rest observers will be negatively infinite and immediately after the bounce the coordinate velocity will be infinitely positive. Therefore the inertial coordinate acceleration will need to be instantaneously zero at $r = 0$ and there will be a real proper acceleration given by:

$$A^r = \frac{1}{\pi r^2} \quad (27)$$

As was the case when the ball bounced, the proper acceleration will be infinite in the direction of increasing r . It is this instantaneous acceleration that would reverse the flow of the geodesics when the Universe is infinitely cold and dilute.

What we see from this paper is that the Schwarzschild solution describes two different scenarios, neither of which is the so-called black hole. The solution for $r > 1$ (in units where u is 1) describes the gravitational field when the gravitational source is a location in space for all time whereas the solution for $r < 1$ describes the gravitational field when the gravitational source is a location in time for all space, where the gravitational center is at $r = 1$ in both cases. This is why the metric signature flips at $r = 1$.

Freefall In Space

Let us consider a meter stick at rest at the center of a collapsing spherically symmetric collapsing shell. According to Birkhoff's theorem, the spacetime inside the shell, where the meter stick resides, will be flat Minkowski spacetime. The meter stick inside the shell stretches from the center of the shell out to a distance $2GM$ (the shell is at a radius greater than $2GM$ so the entire stick is in flat spacetime). An observer in freefall on the collapsing shell does so with speed (in natural units measured by her clock) [5]:

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r}} \quad (28)$$

Therefore, the freefall observer will see observers at rest at r moving past her at the speed given in Equation 28. Since the meter stick is also at rest relative to observers at rest at any r , Equation 28 will also give the relative velocity between the freefall observer and the meter stick when the shell is at r . Since the spacetime between the freefall observer and central observer is flat, they will each see the other's clock dilated by the Special Relativity Relationship:

$$d\tau = dt\sqrt{1 - V^2} = dt\sqrt{1 - \frac{2GM}{r}} \quad (29)$$

Because the meter stick will appear to be moving in the frame of the freefalling observer, its length in her frame would be:

$$L = 2GM\sqrt{1 - \frac{2GM}{r}} \quad (30)$$

We see from Equation 30 that as the freefalling observer approaches $r = 2GM$ the length of the meter stick in her frame will contract to zero length. So observers in freefall will see the space beyond $r = 2GM$ fully contracted as they approach $r = 2GM$.

Let us make a radial coordinate change for the freefalling observer. We choose R such that $\frac{dR}{dr} = \frac{r}{r-2GM}$. This coordinate varies identically to the r coordinate for large r and then diverges from it at the horizon. Note that $R \rightarrow \infty$ as $r \rightarrow \infty$ and $R \rightarrow -\infty$ as

$r \rightarrow 2GM$. The coordinate velocity of the freefalling observer with this coordinate is given by:

$$\frac{dR}{dt} = -\sqrt{\frac{2GM}{r}} = -\sqrt{\frac{2GM}{W(e^{R-1})+1}} \quad (31)$$

Where W is the product-log function. This coordinate choice is also useful because the speed of light in these coordinates is 1 independent of R and t . The external Schwarzschild metric with the new coordinate becomes:

$$d\tau^2 = \frac{r-2GM}{r} [dt^2 - dR^2] \quad (32)$$

A plot of the integral of Equation 31 is given in Figure 9 below:

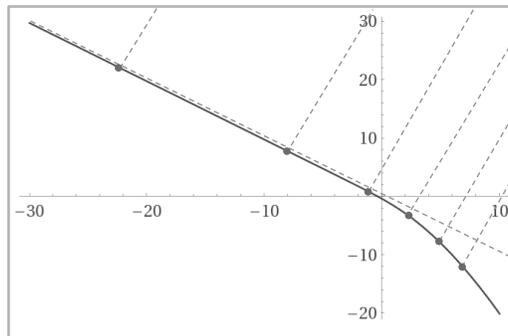


Figure 9 – Light Signals on t - R Chart

Figure 9 is a t - R chart that shows a single infalling signal representing the signal to which the freefall worldline is asymptotic. The freefalling observer will never receive this signal or any subsequent signal before the Universe reaches $r = 0$. The dots in Figure 9 represent intervals of equal proper time along the worldline and we can see that rest observers will receive signals from the freefalling observer at longer and longer intervals.

In the frame of the freefalling observer, rest observers will be moving away from her at the speed given in Equation 28. Therefore, she will see the external Universe accelerating away from her at an even faster rate than observers at infinity see other observers at infinity accelerating away from them, their signals increasingly redshifted as time passes. Nonetheless, the freefalling observer will never fall into a ‘black hole’, she will simply reach the cosmological $r = 0$ more quickly than observers far from any local gravitational fields according to her clock. In fact, as observers on the shell approach $r = 2GM$, their freefall velocity will be opposed by the spatial acceleration defined by Equation 3 such that they will never reach full collapse (Equation 3 acts as a time-dependent damping force opposing the radial motion during expansion and increasing the radial motion during collapse). We know this because the proper time of an observer at rest in the internal solution is the coordinate time of the external solution. Since it takes a freefalling observer an infinite amount of coordinate time to reach the horizon in the external solution, but there is only a finite amount of proper time to $r = 0$ in the internal solution, the freefaller can never reach the horizon during the expansion.

However, when the Universe is in the collapsing phase, an observer will reach $r = 2GM$ in a finite time and, as has been discussed, the $r < 2GM$ space will be fully contracted in her frame. However, when she reaches $r = 2GM$, the entire Universe will be fully contracted (it will have reached the $r = 1$ state described in the previous sections) as though everything in the Universe has collapsed to the same $r = 2GM$, and the observer as well as the entire Universe will have reached the next ‘Big Bang’ state at which point it will begin its expansion once more.

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