The internal structure of natural numbers and one method for the definition of large prime numbers

Emmanuil Manousos

APM Institute for the Advancement of Physics and Mathematics, 13 Pouliou str., 11 523

Athens, Greece

Abstract

It holds that every product of natural numbers can also be written as a sum. The inverse does not hold when 1 is excluded from the product. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article. We prove that primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa. The following theorem is proven: ''Every natural number, except for 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2 with the coefficients of the linear combination being -1 or +1.'' This theorem reveals a symmetry between the distribution of the prime numbers and the composite prime numbers. From this symmetry a method for identifying large prime numbers is derived. This method is not associated with the sieve of Eratosthenes and other relevant methods. Starting from a pair of prime numbers we can obtain a set of different prime numbers which are extremely larger that the initially chosen pair.

1. Introduction

It holds that every product of natural numbers can also be written as a sum. The inverse (i.e. each sum of natural numbers can be written as a product) does not hold when 1 is excluded from the product. This is due to prime numbers p which can be written as a product only in the form of $p=1\cdot p$. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article.

We prove that each natural number can be written as a sum of three or more consecutive natural numbers except of the powers of 2 and the prime numbers. Each power of 2 and each prime number cannot be written as a sum of three or more consecutive natural numbers. Primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa.

We now prove a theorem which plays the role of the fundamental theorem of arithmetic when we study natural numbers based on their sum: ''Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.'' This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived. Starting from a pair (q,Q) of prime numbers, $q\neq 3,5$ and $Q\neq 3,5$, we obtain a set of different prime numbers which are extremely larger than q and Q.

2. THE SEQUENCE $\mu(k,n)$

We consider the sequence of natural numbers

$$\mu(k,n)=k+(k+1)+(k+2)+\ldots+(k+n)=\frac{(n+1)(2k+n)}{2}$$
$$k\in\mathbb{N}^*=\{1,2,3,\ldots\} \quad . \qquad (2.1)$$
$$n\in A=\{2,3,4,\ldots\}$$

For the sequence $\mu(k,n)$ the following theorem holds:

Theorem 2.1.

'' For the sequence $\mu(k,n)$ the following hold:

1. $\mu(k,n)\in\mathbb{N}^*$.
2. No element of the sequence is a prime number.
3. No element of the sequence is a power of 2.
4. The range of the sequence is all natural numbers that are not primes and are not powers of 2.

Proof.

1. $\mu(k,n)\in\mathbb{N}^*$ as a sum of natural numbers.

2. $n\in A=\{2,3,4,\ldots\}$ and therefore it holds that

$$n\geq 2$$
$$n+1\geq 3 .$$

Also we have that

$$2k+n\geq 4$$
$$\frac{2k+n}{2}\geq\frac{3}{2}>1$$

since $k\in\mathbb{N}^*$ and $n\in A=\{2,3,4,\ldots\}$. Thus, the product

$$\frac{(n+1)(2k+n)}{2}=\mu(k,n)$$

is always a product of two natural numbers different than 1, thus the natural number $\mu(k,n)$ cannot be prime.

3. Let that the natural number $\mu(k,n)=\dfrac{(n+1)(2k+n)}{2}$ is a power of 2. Then, it exists $\lambda\in\mathbb{N}$ such as

$$\frac{(n+1)(2k+n)}{2}=2^{\lambda}$$

$$(n+1)(2k+n)=2^{\lambda+1}\quad. \tag{2.2}$$

Equation (2.2) can hold if and only if there exist $\lambda_1,\lambda_2\in\mathbb{N}$ such as

$$n+1=2^{\lambda_1}\wedge 2k+n=2^{\lambda_2}$$

and equivalently

$$\left.\begin{aligned} n&=2^{\lambda_1}-1\\ n&=2^{\lambda_2}-2k\end{aligned}\right\}. \tag{2.3}$$

We eliminate n from equations (2.3) and we obtain

$$2^{\lambda_1}-1=2^{\lambda_2}-2k$$

and equivalently

$$2k-1=2^{\lambda_2}-2^{\lambda_1}$$

which is impossible since the first part of the equation is an odd number and the second part is an even number. Thus, the range of the sequence $\mu(k,n)$ does not include the powers of 2.

4. We now prove that the range of the sequence $\mu(k,n)$ includes all natural numbers that are not primes and are not powers of 2. Let a random natural number N which is not a prime nor a power of 2. Then, N can be written in the form

$$N=\chi\psi$$

where at least one of the χ,ψ is an odd number ≥ 3. Let χ be an odd number ≥ 3. We will prove that there are always exist $k\in\mathbb{N}$ and $n\in A=\{2,3,4,\ldots\}$ such as

$$N=\chi\cdot\psi=\mu(k,n).$$

We consider the following two pairs of k and n:

$$\begin{aligned} &\chi\leq 2\psi-1,\ \chi,\psi\in\mathbb{N}\\ &k=k_1=\frac{2\psi+1-\chi}{2}\\ &n=n_1=\chi-1\end{aligned} \tag{2.4}$$

$$\begin{aligned} &\chi\geq 2\psi+1,\ \chi,\psi\in\mathbb{N}\\ &k=k_2=\frac{\chi+1-2\psi}{2}\quad.\\ &n=n_2=2\psi-1\end{aligned} \tag{2.5}$$

For every $\chi, \psi \in \mathbb{N}$ it holds either the inequality $\chi \le 2\psi - 1$ or the inequality $\chi \ge 2\psi + 1$. Thus, for each pair of naturals (χ, ψ), where χ is odd, at least one of the pairs (k_1, n_1), (k_2, n_2) of equations (2.4), (2.5) is defined. We now prove that "when the natural number k_1 of equation (2.4) is $k_1 = 0$ then the natural number k_2 of equation (2.5) is $k_2 = 1$ and additionally it holds that $n_2 > 2$.". For $k_1 = 0$ from equations (2.4) we take

$$\chi = 2\psi + 1$$

and from equations (2.5) we have that

$$k_2 = \frac{(2\psi + 1) + 1 - 2\psi}{2} = 1$$

$$n_2 = 2\psi - 1$$

and because $\psi \ge 2$ we obtain

$$k_2 = 1$$

$$n_2 = 2\psi - 1 \ge 3 > 2.$$

We now prove that when $k_2 = 0$ in equations (2.5), then in equations (2.4) it is $k_1 = 1$ and $n_1 > 2$. For $k_2 = 0$, from equations (2.5) we obtain

$$\chi = 2\psi - 1$$

and from equations (2.4) we get

$$k_1 = \frac{2\psi + 1 - (2\psi - 1)}{2} = 1.$$

$$n_1 = \chi - 1 = 2\psi - 2 \ge 2$$

We now prove that at least one of the k_1 and k_2 is positive. Let

$$k_1 < 0 \wedge k_2 < 0.$$

Then from equations (2.4) and (2.5) we have that

$$2\psi + 1 - \chi < 0 \wedge \chi + 1 - 2\psi < 0. \tag{2.6}$$

Taking into account that $\chi > 1$ is odd, that is $\chi = 2\rho + 1, \rho \in \mathbb{N}$, we obtain from inequalities (2.6)

$$2\psi + 1 - (2\rho - 1) < 0 \wedge (2\rho + 1) + 1 - 2\psi < 0$$

$$2\psi - 2\rho < 0 \wedge 2\rho - 2\psi + 2 > 0$$

$$\psi < \rho \wedge \psi > \rho + 1$$

which is absurd. Thus, at least one of k_1 and k_2 is positive.

For equations (2.4) we take

$$\mu(k_1,n_1)=\frac{(n_1+1)(2k_1+n_1)}{2}$$
$$=\frac{(\chi-1+1)\left(2\frac{2\psi+1-\chi}{2}+\chi-1\right)}{2}=\frac{\chi(2)\psi}{2}=\chi\psi=N.$$

For equations (2.5) we obtain

$$\mu(k_2,n_2)=\frac{(n_2+1)(2k_2+n_2)}{2}$$
$$=\frac{(2\psi-1+1)\left(2\frac{\chi+1-2\psi}{2}+2\psi-1\right)}{2}=\frac{2\psi\chi}{2}=\chi\psi=N.$$

Thus, there are always exist $k\in\mathbb{N}^*$ and $n\in A=\{2,3,4,\ldots\}$ such as

$N=\chi\psi=\mu(k,n)$ for every N which is not a prime number and is not a power of 2. □

Example 2.1. For the natural number $N=40$ we have

$N=40=5\cdot 8$

$\chi=5$

$\psi=8$

and from equations (2.4) we get

$k=k_1=\frac{16+1-5}{2}=6$

$n=n_1=5-1=4$

thus, we obtain

$40=\mu(6,4)$.

Example 2.2. For the natural number $N=51$,

$N=51=3\cdot 17=17\cdot 3$

there are two cases. First case:

$N=51=3\cdot 17$

$\chi=3$

$\psi=17$

and from equations (2.4) we obtain

$k=k_1=\frac{34+1-3}{2}=16$

$n=n_1=3-1=2$

thus,

$51=\mu(16,2)$.

Second case:

$N = 51 = 17 \cdot 3$
$\chi = 17$
$\psi = 3$
and from equations (2.5) we obtain

$$k = k_2 = \frac{17+1-6}{2} = 6$$

$n = n_2 = 6-1 = 5$
thus,
$51 = \mu(6,5)$.

The second example expresses a general property of the sequence $\mu(k,n)$. The more composite an odd number that is not prime (or an even number that is not a power of 2) is, the more are the $\mu(k,n)$ combinations that generate it.

Example 2.3.

$$135 = 15 \cdot 9 = 27 \cdot 5 = 9 \cdot 15 = 45 \cdot 3 = 5 \cdot 27 = 3 \cdot 45$$
$$135 = \mu(2,14) = \mu(9,9) = \mu(11,8) = \mu(20,5) = \mu(25,4) = \mu(44,2)$$

a. $135 = 9 \cdot 15 = \mu(2,14) = \mu(11,8)$

$135 = 2+3+4+......+15+16 = 11+12+13.....+18+19$.

b. $135 = 5 \cdot 27 = \mu(9,9) = \mu(25,4)$

$135 = 9+10+11+.....+17+18 = 25+26+27+28+29$.

c. $135 = 3 \cdot 45 = \mu(20,5) = \mu(44,2)$

$135 = 20+21+22+23+24+25 = 44+45+46$.

In the transitive property of multiplication, when writing a composite odd number or an even number that is not a power of 2 as a product of two natural numbers, we use the same natural numbers $\chi, \psi \in \mathbb{N}$:

$$\Phi = \chi \cdot \psi = \psi \cdot \chi .$$

On the contrary, the natural number Φ can be written in the form $\Phi = \mu(k,n)$ using different natural numbers $k \in \mathbb{N}^*$ and $n \in A = \{2,3,4,...\}$, through equations (2.4), (2.5). This difference between the product and the sum can also become evident in example 2.3:

$$135 = 3 \cdot 45 = 45 \cdot 3$$
$$135 = 44+45+46 = 20+21+22+23+24+25 .$$

From Theorem 2.1 the following corollary is derived:

Corollary 2.1. "1. Every natural number which is not a power of 2 and is not a prime can be written as the sum of three or more *consecutive* natural numbers.

2. Every power of 2 and every prime number cannot be written as the sum of three or more *consecutive* natural numbers."

Proof. Corollary 2.1 is a direct consequence of Theorem 2.1.□

3. The concept of rearrangement

In this paragraph, we present the concept of rearrangement of the composite odd numbers and even numbers that are not power of 2. Moreover, we prove some of the consequences of the rearrangement in the Diophantine analysis. The concept of rearrangement is given from the following definition:

Definition. ''We say that the sequence $\mu(k,n), k\in\mathbb{N}^*, n\in A=\{2,3,4,...\}$ is rearranged if there exist natural numbers $k_1\in\mathbb{N}^*, n_1\in A$, $(k_1,n_1)\neq(k,n)$ such as

$$\mu(k,n)=\mu(k_1,n_1).'' \tag{3.1}$$

From equation (2.1) written in the form of

$$\mu(k,n)=k+(k+1)+(k+2)+.....+(k+n)$$

two different types of rearrangement are derived: The "compression", during which n decreases with a simultaneous increase of k. The «decompression», during which n increases with a simultaneous decrease of k. The following theorem provides the criterion for the rearrangement of the sequence $\mu(k,n)$.

Theorem 3.1. '' '1. The sequence $\mu(k_1,n_1), (k_1,n_1)\in\mathbb{N}^*\times A$ can be compressed

$$\mu(k_1,n_1)=\mu(k_1+\varphi,n_1-\omega) \tag{3.2}$$

if and only if there exist $\varphi,\omega\in\mathbb{N}^*, \omega\leq n_1-2$ which satisfies the equation

$$\begin{aligned}&\omega^2-(2k_1+2n_1+1+2\varphi)\omega+2(n_1+1)\varphi=0\\&\varphi,\omega\in\mathbb{N}^*\\&\omega\leq n_1-2\end{aligned} \tag{3.3}$$

2. The sequence $\mu(k_2,n_2), (k_2,n_2)\in\mathbb{N}^*\times A$ can be decompressed

$$\mu(k_2,n_2)=\mu(k_2-\varphi,n_2+\omega) \tag{3.4}$$

if and only if there exist $\varphi,\omega\in\mathbb{N}^*, \varphi\leq k_2-1$ which satisfies the equation

$$\begin{aligned}&\omega^2+(2k_2+2n_2+1-2\varphi)\omega-2(n_2+1)\varphi=0\\&\varphi,\omega\in\mathbb{N}^*\\&\varphi\le k_2-1\end{aligned} \quad . \tag{3.5}$$

3. The odd number $\Pi\neq 1$ is prime if and only if the sequence

$$\begin{aligned}&\mu(k,n)=\Pi\cdot 2^l\\&l,k\in\mathbb{N}^*,n\in A\end{aligned} \tag{3.6}$$

cannot be rearranged.

4. The odd Π is prime if and only if the sequence

$$\mu\left(\frac{\Pi+1}{2},\Pi-1\right)=\Pi^2 \tag{3.7}$$

cannot be rearranged.''

Proof. 1,2. We prove part 1 of the corollary and similarly number 2 can also be proven. From equation (4.1) we conclude that the sequence $\mu(k_1,n_1)$ can be compressed if and only if there exist $\varphi,\omega\in\mathbb{N}^*$ such as

$$\mu(k_1,n_1)=\mu(k_1+\varphi,n_1-\omega).$$

In this equation the natural number $n_1-\omega$ belongs to the set $A=\{2,3,4,...\}$ and thus $n_1-\omega\ge 2\Leftrightarrow\omega\le n_1-2$. Next, from equations (2.1) we obtain

$$\mu(k_1,n_1)=\mu(k_1+\varphi,n_1-\omega)$$

$$\frac{(n_1+1)(2k_1+n_1)}{2}=\frac{(n_1-\omega+1)\left[2(k_1+\varphi)+n_1-\omega\right]}{2}$$

and after the calculations we get equation (3.3).

3. The sequence (3.6) is derived from equations (2.4) or (2.5) for $\chi=\Pi$ and $\psi=2^l$. Thus, in the product $\chi\psi$ the only odd number is Π. If the sequence $\mu(k,n)$ in equation (3.6) cannot be rearranged then the odd number Π has no divisors. Thus, Π is prime. Obviously, the inverse also holds.

4. First, we prove equations (3.7). From equation (2.1) we obtain:

$$\mu\left(\frac{\Pi+1}{2},\Pi-1\right)=\frac{(\Pi-1+1)\left(2\frac{\Pi+1}{2}+\Pi-1\right)}{2}=\Pi^2.$$

In case that the odd number Π is prime in equations (2.4), (2.5) the natural numbers χ,ψ are unique $\chi=\Pi\wedge\psi=\Pi$, and from equation (2.5) we get $k=\frac{\Pi+1}{2}\wedge n=\Pi-1$. Thus, the sequence $\mu(k,n)=\mu\left(\frac{\Pi+1}{2},\Pi-1\right)$ cannot be rearranged. Conversely, if the sequence $\mu\left(\frac{\Pi+1}{2},\Pi-1\right)=\Pi^2=\Pi\cdot\Pi$ cannot be rearranged the odd number Π cannot be composite and thus Π is prime.□

We now prove the following corollary:

Corollary 3.1. ''1. The odd number Φ,

$$\begin{aligned}&\Phi=\Pi^2=\mu\left(\frac{\Pi+1}{2},\Pi-1\right)\\&\Pi=odd\\&\Pi\neq 1\end{aligned}\tag{3.8}$$

is decompressed and compressed if and only if the odd number Π is composite.

2. The even number α_1,

$$\begin{aligned}&\alpha_1=2^l\Pi=\mu\left(2^l-\frac{\Pi-1}{2},\Pi-1\right)\\&\Pi=odd\\&3\leq\Pi\leq 2^l-1\\&l\in\mathbb{N},l\geq 2\end{aligned}\tag{3.9}$$

cannot be decompressed, while it compresses if and only if the odd number Π is composite.

3. The even number α_2,

$$\begin{aligned}&\alpha_2=2^l\Pi=\mu\left(\frac{\Pi+1}{2}-2^l,2^{l+1}-1\right)\\&\Pi=odd\\&\Pi\geq 2^{l+1}+1\\&l\in\mathbb{N}^*\end{aligned}\tag{3.10}$$

cannot be compressed, while it decompresses if and only if the odd number Π is composite.

4. Every even number that is not a power of can be written either in the form of equation (3.9) or in the form of equation (3.10).''

Proof.

1. It is derived directly through number (4) of Theorem 3.1. A second proof can be derived through equations (2.4), (2,5) since every composite odd Π can be written in the form of $\Pi = \chi\psi$, $\chi,\psi \in \mathbb{N}$, χ,ψ odds.

2,3.

Let the even number α ,

$$\begin{aligned} &\alpha = 2^l \Pi \\ &\Pi = odd\,. \\ &l \in \mathbb{N}^* \end{aligned} \tag{3.11}$$

From equation (2.4) we obtain

$$\begin{aligned} &k = \frac{2 \cdot 2^l + 1 - \Pi}{2} = 2^l - \frac{\Pi - 1}{2} \\ &n = \Pi - 1 \end{aligned} \tag{3.12}$$

and since $k,n \in \mathbb{N}, k \geq 1 \wedge n \geq 2$ we get

$$\begin{aligned} &\frac{2 \cdot 2^l + 1 - \Pi}{2} \geq 1 \\ &\Pi - 1 \geq 2 \end{aligned}$$

and equivalently

$$3 \leq \Pi \leq 2^{l+1} - 1.$$

In the second of equations (3.12) the natural number n obtains the maximum possible value of $n = \Pi - 1$, and thus the natural number k takes the minimum possible value in the first of equations (3.12). Thus, the even number

$$\alpha_1 = \mu\left(2^l - \frac{\Pi - 1}{2}, \Pi - 1\right)$$

cannot decompress. If the odd number Π is composite then it can be written in the form of $\Pi = \chi\psi$, $\chi,\psi \in \mathbb{N}^*$, χ,ψ odds, $\chi,\psi < \Pi$, $\alpha_1 = 2^l \chi\psi$. Therefore, the natural number $\alpha_1 = 2^l \chi\psi$ decompresses since from equations (3.11) it can be written in the form of $\alpha_1 = \mu(k,n)$ with $n = \chi - 1 < \Pi - 1$. Similarly, the proof of 3 is derived from equations (2.5).

4. From the above proof process it follows that every even number that is not a power of 2 can be written either in the form of equation (3.9) or in the form of equation (3.10). □

By substituting $\Pi = P = prime$ in equations of Theorem 3.1 and of corollary 3.1 four sets of equations are derived, each including infinite *impossible* diophantine equations.

Example 3.1. The odd number $P = 999961$ is prime. Thus, combining (1) of Theorem 3.1 with (1) of corollary 3.1 we conclude that there is no pair $(\omega, \varphi) \in \mathbb{N}^2$ with $\omega \leq 999958$ which satisfies the diophantine equation

$\omega^2 - (2999883 + 2\varphi)\omega + 1999922\varphi = 0$.

We now prove the following corollary:

Corollary 3.2 ''The square of every prime number can be uniquely written as the sum of consecutive natural numbers.''

Proof. For $\Pi = P = prime$ in equation (3.5) we obtain

$$P^2 = \mu\left(\frac{P+1}{2}, P-1\right). \qquad (3.13)$$

According with 4 of Theorem 3.1 the odd P^2 cannot be rearranged. Thus, the odd can be uniquely written as the sum of consecutive natural numbers, as given from equation (3.13).□

Example 3.2. The odd $P = 17$ is prime. From equation (3.13) for $P = 17$ we obtain

$289 = \mu(9,16)$

and from equation (2.1) we get

$289 = 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 + 21 + 22 + 23 + 24 + 25$

which is the only way in which the odd number 289 can be written as a sum of consecutive natural numbers.

4. NATURAL NUMBERS AS LINEAR COMBINATION OF CONSECUTIVE POWERS OF 2

According to the fundamental theorem of arithmetic, every natural number can be uniquely written as a product of powers of prime numbers. The previously presented study reveals a correspondence between odd prime numbers and the powers of 2. Thus, the question arises whether there exists a theorem for the powers of 2 corresponding to the fundamental theorem of arithmetic. The answer is given by the following theorem:

Theorem 4.1. 'Every natural number, with the exception of 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.''

Proof. Let the odd number Π as given from equation

$$\Pi = \Pi(\nu,\beta_i) = 2^{\nu+1} + 2^{\nu} \pm 2^{\nu-1} \pm 2^{\nu-2} \pm .... \pm 2^1 \pm 2^0 = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i$$

$$\beta_i = \pm 1, i = 0,1,2,........,\nu-1 \quad . \qquad (4.1)$$

$$\nu \in \mathbb{N}$$

From equation (4.1) for $\nu = 0$ we obtain

$$\Pi = 2^1 + 2^0 = 2+1 = 3.$$

We now examine the case where $\nu \in \mathbb{N}^*$. The lowest value that the odd number Π of equation (4.1) can obtain is

$$\Pi_{\min} = \Pi(\nu) = 2^{\nu+1} + 2^{\nu} - 2^{\nu-1} - 2^{\nu-1} - .... 2^1 - 1$$

$$\Pi_{\min} = \Pi(\nu) = 2^{\nu+1} + 1. \qquad (4.2)$$

The largest value that the odd number Π of equation (4.1) can obtain is

$$\Pi_{\max} = \Pi(\nu) = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... 2^1 + 1$$

$$\Pi_{\max} = \Pi(\nu) = 2^{\nu+2} - 1. \qquad (4.3)$$

Thus, for the odd numbers $\Pi = \Pi(\nu,\beta_i)$ of equation (4.1) the following inequality holds

$$\Pi_{\min} = 2^{\nu+1} + 1 \le \Pi(\nu,\beta_i) \le 2^{\nu+2} - 1 = \Pi_{\max} . \qquad (4.4)$$

The number $N(\Pi(\nu,\beta_i))$ of odd numbers in the closed interval $\left[2^{\nu+1}+1, 2^{\nu+2}-1\right]$ is

$$N(\Pi(\nu,\beta_i)) = \frac{\Pi_{\max} - \Pi_{\min}}{2} + 1 = \frac{(2^{\nu+2}-1) - (2^{\nu+1}+1)}{2} + 1$$

$$N(\Pi(\nu,\beta_i)) = 2^{\nu} . \qquad (4.5)$$

The integers $\beta_i, i = 0,1,2,........,\nu-1$ in equation (4.1) can take only two values, $\beta_i = -1 \vee \beta_i = +1$, thus equation (4.1) gives exactly $2^{\nu} = N(\Pi(\nu,\beta_i))$ odd numbers. Therefore, for every $\nu \in \mathbb{N}^*$ equation (4.1) gives all odd numbers in the interval $\left[2^{\nu+1}+1, 2^{\nu+2}-1\right]$.

We now prove the theorem for the even numbers. Every even number α which is a power of 2 can be uniquely written in the form of $\alpha = 2^{\nu}, \nu \in \mathbb{N}^*$. We now consider the case where the even number α is not a power of 2. In that case, according to corollary 3.1 the even number α is written in the form of

$$\alpha = 2^l \Pi, \Pi = \text{odd}, \Pi \ne 1, l \in \mathbb{N}^* . \qquad (4.6)$$

We now prove that the even number α can be uniquely written in the form of equation (4.6). If we assume that the even number α can be written in the form of

$$\begin{aligned}
&\alpha = 2^{l}\Pi = 2^{l^{'}}\Pi^{'} \\
&l \neq l^{'} (l > l^{'}) \\
&\Pi \neq \Pi^{'} \\
&l, l^{'} \in \mathbb{N}^{*} \\
&\Pi, \Pi^{'} = odd
\end{aligned} \tag{4.7}$$

the we obtain

$$\begin{aligned}
&2^{l}\Pi = 2^{l^{'}}\Pi' \\
&2^{l-l^{'}}\Pi = \Pi^{'}
\end{aligned}$$

which is impossible, since the first part of this equation is even and the second odd. Thus, it is $l = l^{'}$ and we take that $\Pi = \Pi^{'}$ from equation (4.7). Therefore, every even number α that is not a power of 2 can be uniquely written in the form of equation (4.6). The odd number Π of equation (4.6) can be uniquely written in the form of equation (4.1), thus from equation (4.6) it is derived that every even number α that is not a power of 2 can be uniquely written in the form of equation

$$\begin{aligned}
&\alpha = \alpha\left(l, \nu, \beta_{i}\right) = 2^{l}\left(2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1}\beta_{i}2^{i}\right) \\
&l \in \mathbb{N}^{*}, \nu \in \mathbb{N} \\
&\beta_{i} = \pm 1, i = 0, 1, 2, ..., \nu - 1
\end{aligned} \tag{4.8}$$

and equivalently

$$\begin{aligned}
&\alpha = \alpha\left(l, \nu, \beta_{i}\right) = 2^{l+\nu+1} + 2^{l+\nu} + \sum_{i=0}^{\nu-1}\beta_{i}2^{l+i} \\
&l \in \mathbb{N}^{*}, \nu \in \mathbb{N} \quad . \\
&\beta_{i} = \pm 1, i = 0, 1, 2, ..., \nu - 1
\end{aligned} \tag{4.9}$$

For 1 we take

$$\begin{aligned}
&1 = 2^{0} \\
&1 = 2^{1} - 2^{0}
\end{aligned}$$

thus, it can be written in two ways in the form of equation (4.1). Both the odds of equation (4.1) and the evens of the equation (4.8) are positive. Thus, 0 cannot be written either in the form of equation (4.1) or in the form of equation (4.8). □

In order to write an odd number $\Pi \neq 1, 3$ in the form of equation (4.1) we initially define the $\nu \in \mathbb{N}^{*}$ from inequality (4.4). Then, we calculate the sum

$2^{\nu+1}+2^{\nu}$.

If it holds that $2^{\nu+1}+2^{\nu}<\Pi$ we add the $2^{\nu-1}$, whereas if it holds that $2^{\nu+1}+2^{\nu}>\Pi$ then we subtract it. By repeating the process exactly ν times we write the odd number Π in the form of equation (4.1). The number of ν steps needed in order to write the odd number Π in the form of equation (4.1) is extremely low compared to the magnitude of the odd number Π, as derived from equation (4.1).

Example 4.1. For the odd number $\Pi=23$ we obtain from inequality (4.4)

$$2^{\nu+1}+1<23<2^{\nu+2}-1$$
$$2^{\nu+1}+2<24<2^{\nu+2}$$
$$2^{\nu}<12<2^{\nu+1}$$

thus $\nu=3$. Then, we have

$2^{\nu+1}+2^{\nu}=2^4+2^3=24>23$ (thus 2^2 is subtracted)

$2^4+2^3-2^2=20<23$ (thus 2^1 is added)

$2^4+2^3-2^2+2^1=22<23$ (thus $2^0=1$ is added)

$2^4+2^3-2^2+2^1+1=23$.

Fermat numbers F_s can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\min}$,

$$F_s=2^{2^s}+1=\Pi_{\min}\left(2^s-1\right)=2^{2^s}+2^{2^s-1}-2^{2^s-2}-2^{2^s-3}-........-2^1-1 \quad s\in\mathbb{N}^* . \qquad (4.10)$$

Mersenne numbers M_p can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\max}$,

$$M_p=2^p-1=\Pi_{\max}\left(p-2\right)=2^{p-1}+2^{p-2}+2^{p-3}+........+2^1+1 \quad p=prime . \qquad (4.11)$$

In order to write an even number α that is not a power of 2 in the form of equation (4.1), initially it is consecutively divided by 2 and it takes the of the form of equation (4.6). Then, we write the odd number Π in the form of equation (4.1).

Example 4.2. By consecutively dividing the even number $\alpha=368$ by 2 we obtain $\alpha=368=2^4\cdot 23$. Then, we write the odd number $\Pi=23$ in the form of equation (4.1), $23=2^4+2^3-2^2+2^1+1$, and we get

$$368=2^4\left(2^4+2^3-2^2+2^1+1\right)$$
$$368=2^8+2^7-2^6+2^5+2^4 .$$

This equation gives the unique way in which the even number $\alpha = 368$ can be written in the form of equation (4.9).

We now prove the following corollary of theorem 4.1:

Corollary 4.1. ''1. Every odd number $\Pi \neq 1$ can be written either in the form of equation

$$\begin{aligned} &\Pi = \Pi(\nu, \beta_i) = \Phi_1 - 2^{\xi}\Phi_2 = 2^x D_1 - D_2 \\ &\Phi_1, \Phi_2, D_1, D_2 = odd \\ &\Phi_1, D_1, D_2 > 1 \\ &\Phi_2 \geq 1 \\ &\xi, x \in \mathbb{N}^* \end{aligned} \tag{4.12}$$

or in the form of equation

$$\Pi = 2^{\nu+2} - 3, \nu \in \mathbb{N} \tag{4.13}$$

or in the form of equation

$$\Pi = 2^{\nu+2} - 1, \nu \in \mathbb{N}. \tag{4.14}$$

2. Every even number α is either a power of 2,

$$\alpha = 2^{\nu}, \nu \in \mathbb{N}^* \tag{4.15}$$

or can be written in the form of equation

$$a = 2^l \left(\Phi_1 - 2^{\xi}\Phi_2\right) = 2^l \left(2^x D_1 - D_2\right), l \in \mathbb{N}^* \tag{4.16}$$

or in the form of equation

$$a = 2^l \left(2^{\nu+2} - 3\right), l \in \mathbb{N}^*, \nu \in \mathbb{N} \tag{4.17}$$

or in the form of equation

$$a = 2^l \left(2^{\nu+2} - 1\right), l \in \mathbb{N}^*, \nu \in \mathbb{N}. \tag{4.18}$$

3. For the odd numbers Φ_1, Φ_2, D_1, D_2 the following inequalities hold

$$\left.\begin{aligned} &3 \cdot 2^{\nu} - 1 \leq \Phi_1 \leq 2^{\nu+2} - 3 \\ &1 \leq \Phi_2 \leq 2^{\nu-1} - 1 \\ &3 \leq D_1 \leq 2^{\nu+1} - 1 \\ &3 \leq D_2 \leq 2^{\nu+1} - 3 \end{aligned}\right. .'' \tag{4.19}$$

Proof. In equation (4.1) is either $\beta_0 = -1$ or $\beta_0 = +1$. We first examine the case in which $\beta_0 = -1$. From equation (4.1) for $\beta_0 = -1$ we obtain

$$\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{i=1}^{\nu-1} \beta_i 2^i - 1 \ . \qquad (4.20)$$

We first study the case that there exists at least one $\beta_s = -1, s \in \{1, 2, 3, ..., \nu - 1\}$. In that case, we separate the positive and the negative $\beta_i = \pm 1, i = 1, 2, 3, ..., \nu - 1$ and we get

$$\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{\beta_j = +1} \beta_j 2^j + \sum_{\beta_k = -1} \beta_k 2^k - 1. \qquad (4.21)$$

In equation (4.21) we also include $\beta_0 = -1$ in the sum $2^{\nu+1} + 2^{\nu} + \sum_{\beta_j = +1} \beta_j 2^j$ and expressing by Φ_1 the odd number

$$\Phi_1 = 2^{\nu+1} + 2^{\nu} + \sum_{\beta_j = +1} \beta_j 2^j - 1 \qquad (4.22)$$

we take

$$\Pi = \Phi_1 + \sum_{\beta_k = -1} \beta_k 2^k$$

and equivalently

$$\Pi = \Phi_1 - \left(- \sum_{\beta_k = -1} \beta_k 2^k \right). \qquad (4.23)$$

Expressing by 2^{ξ} the lowest power of the sum $-\sum_{\beta_k = -1} \beta_k 2^k$ in equation (4.23) we take

$$\Pi = \Phi_1 - 2^{\xi} \left(- \sum_{\beta_k = -1} \beta_k 2^{k-\xi} \right) . \qquad (4.24)$$

In equation (4.24) exactly one of the differences $k - \xi$ is equal to zero. Therefore, the expression $-\sum_{\beta_k = -1} \beta_k 2^{k-\xi}$ is a sum of powers of 2 with the lowest being the $2^0 = 1$; thus it is an odd number and expressing by Φ_2,

$$\Phi_2 = -\sum_{\beta_k = -1} \beta_k 2^{k-\xi} \qquad (4.25)$$

we take

$$\Pi = \Phi_1 - 2^{\xi} \Phi_2$$

which is the first of equations (4.12).

We now include the $\beta_0 = -1$ in the negative sum $\sum_{\beta_k=-1} \beta_k 2^{k-\xi}$ and we obtain

$$\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{\beta_j=+1} \beta_j 2^j - (-\sum_{\beta_k=-1} \beta_k 2^k + 1)$$

and expressing by D_2 the odd number

$$D_2 = -\sum_{\beta_k=-1} \beta_k 2^k + 1 \tag{4.26}$$

we take

$$\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{\beta_j=+1} \beta_j 2^j - D_2 . \tag{4.27}$$

Expressing by 2^x the lowest power of the sum $\sum_{\beta_j=+1} \beta_j 2^j$ in equation (4.27) we take

$$\Pi = 2^x (2^{\nu+1-x} + 2^{\nu-x} + \sum_{\beta_j=+1} \beta_j 2^{j-x}) - D_2 . \tag{4.28}$$

In equation (4.28) exactly one of the differences $j - x$ is equal to zero. Therefore, the expression $2^{\nu+1-x} + 2^{\nu-x} + \sum_{\beta_j=+1} \beta_j 2^{j-x}$ is a sum of powers of 2 with the lowest being the $2^0 = 1$; thus it is an odd number and expressing by D_1,

$$D_1 = 2^{\nu+1-x} + 2^{\nu-x} + \sum_{\beta_j=+1} \beta_j 2^{j-x} \tag{4.29}$$

we take

$$\Pi = 2^x D_1 - D_2$$

which is the second of equations (4.12).

In the case that equation (4.20) is $\beta_i = +1 \forall i = 1, 2, 3, \ldots\ldots, \nu - 1$ we obtain from equation (4.21)

$$\Pi = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + \ldots\ldots + 2^1 - 1 = 2(2^{\nu} + 2^{\nu-1} + 2^{\nu-2} + \ldots\ldots + 1) - 1 = 2(2^{\nu+1} - 1) - 1$$

and equivalently

$$\Pi = 2^{\nu+2} - 3, \nu \in \mathbb{N}$$

which is equation (4.13).

For $\beta_0 = +1$ in equation (4.1) we take

$$\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{i=1}^{\nu-1} \beta_i 2^i + 1. \qquad (4.30)$$

In the case there exists at least one $\beta_s = -1, s \in \{1, 2, 3, ..., \nu - 1\}$ in equation (4.30) we separate the positive and negative $\beta_i = \pm 1, i = 1, 2, 3, ..., \nu - 1$, we repeat the previous proof process and we obtain the odd number Π in the form

$$\Pi = 2^x D_1 - D_2 = \Phi_1 - 2^{\xi} \Phi_2$$

thus, in the form of equation (4.12). In the case that $\beta_i = +1 \forall i = 1, 2, 3, ..., \nu - 1$ in equation (4.30) we get

$$\Pi = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... + 2^1 + 1$$

and equivalently

$$\Pi = 2^{\nu+2} - 1, \nu \in \mathbb{N}$$

which is equation (4.14).

2. For even numbers that are not powers of 2 we combine equation (4.2) with equations (4.12), (4.13) and (4.14) and we obtain equations (4.16), (4.17) and (4.18), respectively.

3. We prove inequality

$$\Phi_1 \leq 2^{\nu+2} - 3$$

and similarly the rest of equations (4.19) are proved. For $\beta_0 = +1$, $\beta_1 = -1$ and $\beta_i = +1 \forall i = 2, 3, ..., \nu - 1$ in equation (4.1) we obtain

$$\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... + 2^2 - 2^1 + 1$$

and equivalently

$$\Pi = \Pi(\nu, \beta_i) = (2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... + 2^2 + 1) - 2^1 = \Phi_1 - 2^1 \cdot 1$$

from where we get that for $\beta_0 = +1$ it is

$$\Phi_{1,\max} = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... + 2^2 + 1$$

and equivalently

$$\Phi_{1,\max} = 2^2 (2^{\nu-1} + 2^{\nu-2} + 2^{\nu-3} + ... + 2^1 + 1) + 1$$
$$\Phi_{1,\max} = 2^2 (2^{\nu} - 1) + 1$$

$$\begin{aligned}\Phi_{1,\max} &= 2^{\nu+2}-3 \\ \beta_0 &= +1\end{aligned} \quad . \qquad (4.31)$$

For $\beta_0 = -1$, $\beta_1 = -1$ and $\beta_i = +1 \forall i = 2,3,........,\nu-1$ in equation (4.1) we obtain

$$\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... + 2^2 - 2^1 - 1$$

and equivalently

$$\Pi = \Pi(\nu, \beta_i) = (2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... + 2^2 - 1) - 2^1 = \Phi_1 - 2^1 \cdot 1$$

from where we get

$$\Phi_{1,\max} = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... + 2^2 - 1$$

and equivalently

$$\Phi_{1,\max} = 2^2(2^{\nu-1} + 2^{\nu-2} + 2^{\nu-3} + ... + 2^1 + 1) - 1$$

$$\Phi_{1,\max} = 2^2(2^{\nu} - 1) - 1$$

$$\begin{aligned}\Phi_{1,\max} &= 2^{\nu+2}-5 \\ \beta_0 &= -1\end{aligned} \quad . \qquad (4.32)$$

From equations (4.31) and (4.32) we obtain

$$\Phi_{1,\max} = 2^{\nu+2} - 3$$

and equivalently

$\Phi_1 \le 2^{\nu+2} - 3$. □

Example 4.3. For the odd number $\Pi = 293$ we take $\nu = 7$ from inequality (4.4) and by conducting the $\nu = 7$ steps it is written in the form of

$$293 = 256 + 128 - 64 - 32 + 16 - 8 - 4 + 2 - 1$$

thus, $\beta_0 = -1$. By including the $\beta_0 = -1$ in the positive sum of the powers of 2 we get

$$\begin{aligned}293 &= (256 + 128 + 16 + 2 - 1) - (64 + 32 + 8 + 4) \\ 293 &= 401 - 2^2 \cdot 27\end{aligned} \quad . \qquad (4.33)$$

By including the $\beta_0 = -1$ in the negative sum of the powers of 2 we take

$$\begin{aligned}293 &= (256 + 128 + 16 + 2) - (64 + 32 + 8 + 4 + 1) \\ 293 &= 2^1 \cdot 201 - 109\end{aligned} \quad . \qquad (4.34)$$

Thus, we have

$293 = 401 - 2^2 \cdot 27 = 2^1 \cdot 201 - 109$.

Example 4.4. For the odd number $\Pi = 72899$ we take $\nu = 15$ from inequality (4.4) and by conducting the $\nu = 15$ steps it is written in the form of

$$72899 = 65536 + 32768 - 16384 - 8192 - 4096 + 2048$$
$$+1024 + 512 - 256 - 128 + 64 + 32 - 16 - 8 - 4 - 2 + 1$$

thus, $\beta_0 = +1$. By including the $\beta_0 = +1$ in the positive sum of the powers of 2 we get

$$\begin{aligned} 72899 &= (65536 + 32768 + 2048 + 1024 + 512 + 64 + 32 + 1) \\ &-(16384 + 8192 + 4096 + 256 + 128 + 16 + 8 + 4 + 2) \end{aligned} . \qquad (4.35)$$

$$72899 = 101985 - 2^1 \cdot 14543.$$

By including the $\beta_0 = +1$ in the negative sum of the powers of 2 we get

$$\begin{aligned} 2899 &= (65536 + 32768 + 2048 + 1024 + 512 + 64 + 32) \\ &-(16384 + 8192 + 4096 + 256 + 128 + 16 + 8 + 4 + 2 - 1) \end{aligned} \qquad (4.36)$$

$$72899 = 2^5 \cdot 3187 - 29085.$$

Thus, we have

$$72899 = 101985 - 2^1 \cdot 14543 = 2^5 \cdot 3187 - 29085.$$

Mersenne numbers are of the form (4.14). Thus, they cannot be written in the form of equation (4.12).

Example 4.5. The odd number $\Pi = 131071 = 2^{17} - 1$ is derived from equation (4.14) for $\nu = 15$. Thus, it cannot be written in the form of equation (4.12). From equation (4.13) it is also derived that the odd number $\Pi = 131069 = 2^{17} - 3$ cannot be written in the form of equation (4.12).

We now prove the following corollary:

Corollary 4.2. ''1. The number of possible ways with which an odd number Π,

$\Pi \neq 1$
$\Pi \neq 3$
$\Pi \neq 5$
$\Pi \neq 2^{\nu+2} - 3, \nu \in \mathbb{N}$
$\Pi \neq 2^{\nu+2} - 1, \nu \in \mathbb{N}$

can be written either in the form of equation

$$\Pi = \Phi_1 - 2^{\xi} \cdot \Phi_2 \qquad (4.37)$$

or in the form of equation

$$\Pi = 2^x \cdot D_1 - D_2 \tag{4.38}$$

is given from the number $t = t(\Pi)$,

$$t = t(\Pi) = \sum_{k=1}^{T} \binom{T}{k} \tag{4.39}$$

where $T = T(\Pi)$ is the number of $\beta_i = -1, i \in \{1, 2, 3, ..., \nu - 1\}$ of the odd number $\Pi = \Pi(\nu, \beta_i)$.

2. For every odd number Π the following inequality holds

$$0 \leq T(\Pi) \leq \nu - 1 .'' \tag{4.40}$$

Proof.1. We will present the proof for the odd numbers $\Pi = \Pi(\nu, \beta_i)$ for which $\beta_0 = -1$ and, similarly, the proof for the odd numbers $\Pi = \Pi(\nu, \beta_i)$ for which $\beta_0 = +1$ can be derived. From equations (4.22) and (4.25) we obtain

$$\Pi = \Phi_1 - 2^{\xi} \cdot \Phi_2 = (2^{\nu+1} + 2^{\nu} + \sum_{\beta_j = +1} \beta_j 2^j - 1) - 2^{\xi} \cdot (-\sum_{\beta_k = -1} \beta_k 2^{k-\xi}) . \tag{4.41}$$

Equation (4.41) remains in the form $\Pi = \Phi_1 - 2^{\xi} \cdot \Phi_2$ if and only if we move the powers of 2 from the sum

$$2^{\xi} \cdot (-\sum_{\beta_k = -1} \beta_k 2^{k-\xi}) = -\sum_{\beta_k = -1} \beta_k 2^k$$

to the sum

$$2^{\nu+1} + 2^{\nu} + \sum_{\beta_j = +1} \beta_j 2^j - 1$$

with all possible ways, except from moving them all; if we move all of the powers of 2 then the $\beta_0 = -1$ is moved to the right sum of the equation (4.37) which takes the form of

$$\Pi = (2^{\nu+1} + 2^{\nu} + \sum_{i=1}^{\nu-1} \beta_i 2^i) - 1$$

which is not in the form of

$$\Pi = \Phi_1 - 2^{\xi} \cdot \Phi_2 .$$

Exactly this number of powers of 2 in the sum

$$2^{\xi} \cdot (-\sum_{\beta_k = -1} \beta_k 2^{k-\xi}) = -\sum_{\beta_k = -1} \beta_k 2^k$$

is the number $T = T(\Pi)$. Considering the original form in which the equation (4.37) is written, in total we can derive $t = t(\Pi)$,

$$t = t(\Pi) = 1 + \left(\binom{T}{1} + \binom{T}{2} + \binom{T}{3} + ... + \binom{T}{T-1} \right)$$

and equivalently

$$t = t(\Pi) = \binom{T}{T} + \left(\binom{T}{1} + \binom{T}{2} + \binom{T}{3} + ... + \binom{T}{T-1} \right)$$

and equivalently

$$t = t(\Pi) = \binom{T}{1} + \binom{T}{2} + \binom{T}{3} + ... + \binom{T}{T} = \sum_{k=1}^{T} \binom{T}{k}$$

ways in which the odd number $\Pi = \Pi(\nu, \beta_i)$ can be written in the form of equation (4.37). The proof for equation (4.38) is similarly derived through equations (4.26) and (4.29).

2. For the odd numbers of equation (4.13) we get

$$\Pi = 2^{\nu+2} - 3 = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... 2^1 - 1$$

and thus, $T = T(\Pi) = 0$. For the odd numbers of equation (4.14) we obtain

$$\Pi = 2^{\nu+2} - 1 = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... 2^1 + 1$$

and thus, $T = T(\Pi) = 0$. For the odd numbers of equation (4.12) it is $1 \le T(\Pi) \le \nu - 1$. Thus, inequality (4.40) holds.□

Example 4.6. In Example 4.3 it is $T(273) = 4$ and $t(4) = \sum_{k=1}^{4} \binom{4}{k} = 9$. The same conclusion follows from the equations (4.33) and (4.34) taking into account the initial equation $293 = 401 - 2^2 \cdot 27 = 2^1 \cdot 201 - 109$. In equation (4.33) the powers of 2 which are exchanged are included in the initial $2^{\xi}\Phi_2 = (64 + 32 + 8 + 4)$. In equation (4.34) are included in the initial $D_2 = (64 + 32 + 8 + 4 + 1)$. This holds for all odd numbers $\Pi = \Pi(\nu, \beta_i)$ for which $\beta_0 = -1$.

Example 4.7. In example 4.4 it is $T(72899) = 9$ and $t(9) = \sum_{k=1}^{9} \binom{9}{k} = 349$. The same conclusion follows from the equations (4.35) and (4.36) taking into account the initial equation $72899 = 101985 - 2^1 \cdot 14543 = 2^5 \cdot 3187 - 29085$. In equation (4.35) the powers of 2 which are exchanged are included in the initial $2^x D_2 = (16384 + 8192 + 4096 + 256 + 128 + 16 + 8 + 4 + 2)$. In equation (4.36) are included in the initial $D_1 = \left(16384 + 8192 + 4096 + 256 + 128 + 16 + 8 + 4 + 2 - 1\right)$. $D_2 = \left(64 + 32 + 8 + 4 + 1\right)$. This holds for all odd numbers $\Pi = \Pi\left(\nu, \beta_i\right)$ for which $\beta_0 = +1$.

Corollary 4.2 provides the way in which we can change the pairs $\left(\Phi_1, \Phi_2\right)$ and $\left(D_1, D_2\right)$, independently of one another, for the odd number Π. In addition, it is derived that for different odd numbers $\Pi_1 \neq \Pi_2$ we have $t\left(\Pi_1\right) = t\left(\Pi_2\right)$ if and only if $T\left(\Pi_1\right) = T\left(\Pi_2\right)$.

In order to write an odd number in the form of equation (4.12) it must first be written in the form of equation (4.1). In the following paragraph, we will directly write the odd numbers either in the form of equation (4.1) or in the form of equation (4.12).

5.THE SYMMETRY OF PRIME ODD-COMPOSITE ODD NUMBERS AND A METHOD FOR DEFINING LARGE PRIME NUMBERS

Corollaries 4.1 and 4.2 of the previous paragraph provide a large set of information about the internal structure of natural numbers. One of these concerns the symmetry (SPCΠ) between the distribution of the odd prime numbers and the distribution of the composite odd numbers. This symmetry will be studied in this paragraph.

We consider the odd numbers Π which are derived from the product of two odd prime numbers q and Q, excluding 3 and 5, and we write them in the form of equation (4.12),

$$\begin{aligned} &\Pi = qQ = \Phi_1 - 2^{\xi}\Phi_2 = 2^x D_1 - D_2 \\ &q, Q = prime \\ &q < Q \\ &q \neq 3,5 \\ &Q \neq 3,5 \end{aligned} \qquad . \tag{5.1}$$

Equation (5.1) expresses SPCΠ: Using corollary 4.2 we can write each of the equations

$$\Pi = qQ = \Phi_1 - 2^{\xi} \cdot \Phi_2 \tag{5.2}$$

$$\Pi = qQ = 2^x \cdot D_1 - D_2 \tag{5.3}$$

in $t = t(\Pi)$ mathematical expressions. In these equations, a high proportion of Φ_1, Φ_2, D_1, D_2 is divided either by 3 or by 5 or it is $\Phi_2 = 1$. This is the first property of SPCΠ.

The second property of the SPCΠ concerns the Φ_1 of equation (5.2) and D_2 of equation (5.3): As the difference $Q - q$ decreases, the proportion of Φ_1, D_2 that are primes increases relatively to the proportion of Φ_1, D_2 that are composite (with the Φ_2, D_1 being divided by 3 or 5, or $\Phi_2 = 1$). As the difference $Q - q$ increases, the proportion of Φ_1, D_2 π that are primes decreases relatively to the proportion of Φ_1, D_2 that are composite.

These two characteristic properties of SPCП provide a method for defining large prime numbers . This method is completely different from the methods developed in previous centuries [1-11] and it is not associated with the sieve of Eratosthenes and relevant methods: We chose a pair of (q,Q) prime numbers, $q \neq 3,5$ and $Q \neq 3,5$, with a small difference $Q-q$ (we can choose two consecutive primes but this is not mandatory). We define the pairs (Φ_1,Φ_2) of equation (5.2) in which Φ_2 is divided by 3 or by 5 or it is $\Phi_2=1$. To a very high proportion the Φ_1 in these pairs are primes. Moreover, the primes Φ_1 defined by this method are extremely larger than q and Q due to the first of the inequalities (4.19). The efficiency of the method is given by the quotient of the pairs (Φ_1,Φ_2) in which at least one of the Φ_1, Φ_2 is either divided by 3 or by 5 or it is $\Phi_2=1$, to the total number $t=t(q,Q)=t(q\cdot Q)$ of the pairs (Φ_1,Φ_2).

Example 5.1. For $q=11$ and $Q=13$ in equation (5.2) we obtain

$$11\cdot 13=143=(128+64+4+2+1)-(32+16+8) \tag{5.4}$$

and using corollary 4.2 we get

$$\begin{aligned}143 &= 199-2^3\cdot 7\\ &=191-2^4\cdot 3\\ &=183-2^3\cdot 5\\ &=167-2^1\cdot 3\\ &=175-2^5\cdot 1\\ &=159-2^4\cdot 1\\ &=151-2^3\cdot 1\end{aligned}$$

From equation (5.4) we obtain $T(11\cdot 13)=3$ and $t(11,13)=t(3)=\sum_{k=1}^{3}\binom{3}{k}=7$. The only pair (Φ_1,Φ_2) in which none of the Φ_1, Φ_2 is divided by 3 or by 5 and it does not hold that $\Phi_2=1$ is the pair of $(199,7)$. Thus, method's efficiency in this example is $\alpha=\frac{6}{7}$. The pairs of the SPCП are the following:

$$(\Phi_1,\Phi_2)=(191,3),(183,5),(167,3),(175,1),(159,1),(151,1)$$

Among these pairs, in the pair $(\Phi_1,\Phi_2)=(183,5)$ $\Phi_1=183$ is divided by 3 (since the sum of its digits is divided by 3), in the pair $(\Phi_1,\Phi_2)=(175,1)$, $\Phi_1=175$ is divided by 5 (since its last digit is 5) and in the pair $(\Phi_1,\Phi_2)=(159,1)$, $\Phi_1=159$ is divided by 3. Thus, $\Phi_1=191,167,151$ in the rest of the pairs of the SPCП have a great chance of being primes, since we have chosen q and Q with a small difference $Q-q$. By conducting the necessary test (for relatively small odd numbers existing prime number tables can be used ,however for large odd numbers Φ_1 a mathematical algorithm should be applied in order to determine if the a number is prime) we find that all $\Phi_1 = 191,167,151$ are primes. The largest prime Φ_1 that the method gives in this example is the $\Phi_1 = 191$ (and not $\Phi_1 = 199$).

In case we want to identify only the largest Φ_1 for a specific pair (q,Q) we follow a different approach. We start searching for prime numbers from the original Φ_1 if Φ_2 is divided by 3 or by 5 or it is $\Phi_2 = 1$, and if the initial Φ_1 is not a prime we proceed by applying corollary 4.2.

Example 5.2. For $q = 271$ and $Q = 277$ in equation (5.2) we obtain

$$271 \cdot 277 = 75067 = (65536+32768+4096+512+128+16+8+4+1)$$
$$-(16384+8192+2048+1024+256+64+32+2)$$

from where we obtain

$$75067 = 103069 - 2^1 \cdot 14001.$$

The $\Phi_2 = 14001$ is divided by 3 and thus the $(103069,14001)$ is a pair of the SPCΠ. By making the necessary test we find that the $\Phi_1 = 103069$ is prime.

Example 5.3. For $q = 263$ and $Q = 269$ in equation (5.2) we get

$$\begin{aligned} 263 \cdot 269 = 70747 = (65536+32768+2048+512+32+8+4+1) \\ -(16384+8192+4096+1024+256+128+64+16+2) \end{aligned} \qquad (5.5)$$

from where we obtain

$$70747 = 100909 - 2^1 \cdot 15081.$$

The $\Phi_2 = 15081$ is divide by 3 and thus the $(100909,15081)$ is a pair of the SPCΠ. By making the necessary test we find that $\Phi_1 = 100909$ is not a prime .Thus, we proceed in the application of corollary 4.2. In equation (5.5) we move the $2 = 2^1$ from the second parenthesis to the first and we get $70747 = 100907 - 2^4 \cdot 1885$.

The $\Phi_2 = 1885$ is divide by 5 and thus the $(100907,1885)$ is a pair of the SPCΠ. By making the necessary test we find that $\Phi_1 = 100907$ is prime.

The density of prime numbers decreases in the set $\mathbb{N}$ as we move on to larger prime numbers. For this reason, we search for a second prime number at a larger distance from the $\Phi_1 = 100907$: In equation (5.5) we move the sum $(16384+8192+4096+1024+256)$ from the second parenthesis to the first and we get $70747 = 70957 - 2^1 \cdot 105$.

The $\Phi_2 = 105$ is divided by 3 and thus the $(70957,105)$ is a pair of the SPCΠ. By making the necessary test we find that $\Phi_1 = 70957$ is prime.

Example 5.4. We choose the primes $q=7$ and $Q=1033$ so that the difference $Q-q$ is great. We expect that the Φ_1 of the SPCΠ which are primes will be extremely fewer compared to the Φ_1 which are composite. For $q=7$ and $Q=1033$ in equation (5.2) we get

$$\begin{aligned}7\cdot 1033 = 7231 = &(4096+2048+1024+512+16+8+4+2+1)\\ &-(256+128+64+32)\end{aligned} \tag{5.6}$$

and using corollary 4.2 we obtain

$$\begin{aligned}7231 &= 7711 - 2^5\cdot 15\\ &= 7679 - 2^6\cdot 7\\ &= 7647 - 2^5\cdot 13\\ &= 7583 - 2^5\cdot 11\\ &= 7455 - 2^5\cdot 7\\ &= 7615 - 2^7\cdot 3\\ &= 7551 - 2^6\cdot 5\\ &= 7423 - 2^6\cdot 3\\ &= 7519 - 2^5\cdot 9\\ &= 7391 - 2^5\cdot 5\\ &= 7327 - 2^5\cdot 3\\ &= 7487 - 2^8\cdot 1\\ &= 7295 - 2^6\cdot 1\\ &= 7263 - 2^5\cdot 1\\ &= 7359 - 2^7\cdot 1\end{aligned} \quad . \tag{5.7}$$

From (5.6) we get $T(7\cdot 1033)=4$ and $t(7,1033)=t(4)=\sum_{k=1}^{4}\binom{4}{k}=15$. In the pairs $(\Phi_1,\Phi_2)=(7679,7),(7455,7)$ of equation (5.7) we know that $\Phi_1 = 7679, 7455$ are divided by 7 since $7231 = 7\cdot 1033$ and should not be taken into account in calculating the efficiency of the method. From the rest of the 13 pairs, the 12

$$\begin{aligned}(\Phi_1,\Phi_2) = &(7711,15),(7647,13),(7615,3),(7551,5),(7423,3),(7519,9),\\ &(7391,5),(7327,3),(7487,1),(7595,1),(7263,1),(7359,1)\end{aligned}$$

are pairs of the SPCΠ. Thus, the efficiency of the method in this example is $\alpha = \frac{12}{13}$. By doing the necessary testing we find that only in the pair $(\Phi_1, \Phi_2) = (7487, 1)$ the $\Phi_1 = 7487$ is prime. In the rest of the 11 pairs of the SPCΠ the
$\Phi_1 = 7711, 7647, 7615, 7551, 7423, 7519, 7391, 7327, 7595, 7263, 7359$ is composite.

In such cases for a small difference the number of primes Φ_1 in the pairs (Φ_1, Φ_2) of the SPCΠ is not larger than the number of composites Φ_1. A criterion for such cases is given by inequality (4.40). For

$$T(\Pi) \to \nu - 1 \qquad (5.8)$$

the number of primes Φ_1 in pairs (Φ_1, Φ_2) of the SPCΠ is larger than the number of composites Φ_1, while for

$$T(\Pi) \to 1 \qquad (5.9)$$

this primes/composites proportion is reversed; the number of primes Φ_1 is lower than the number of composites Φ_1. This criterion gives the third property of the SPCΠ.

When applying corollary 4.2 methods efficiency depends on the power of 2 or the pair of powers or the triad of powers etc. that we move from $2^{\xi}\Phi_2$ to Φ_1. A criterion is required so that that the first pairs of the SPCP in which Φ_1 is prime can be derived. The use of the aforementioned criterion would enhance the efficiency of the method in means of computational resources.

There are $T(\Pi)$ pairs of the SPCΠ in which $\Phi_2 = 1$. These pairs can be easily determined by leaving a power of 2 in $2^{\xi}\Phi_2$. Thus, in the case that in the initial pair (Φ_1, Φ_2) of SPCΠ Φ_1 is not a prime (for a small $Q - q$ difference) we apply corollary 4.2, starting with the $T(\Pi)$ pairs of the SPCΠ in which $\Phi_2 = 1$.

Example 5.5. For $q = 1667$ and $Q = 1669$ in equation (5.2) we obtain

$\Pi = 1667 \cdot 1669 = 2782223 = 3455495 - 673272 = 3455495 - 2^3 \cdot 84159$ and the first pair of the SPCΠ is $(\Phi_1, \Phi_2) = (3455495, 84159)$ in which $\Phi_1 = 3455495$ is divided by 5. By choosing the $T(1667 \cdot 1669) = 10$ pairs of the SPCΠ in which $\Phi_2 = 1$ we get

$$(\Phi_1, \Phi_2) = (2782215, 2^3 \cdot 1), (2782207, 2^4 \cdot 1), (2782191, 2^5 \cdot 1), (2782159, 2^6), (2782095, 2^7 \cdot 1),$$
$$(2781967, 2^8 \cdot 1), (2781199, 2^{10} \cdot 1), (2765839, 2^{14} \cdot 1), (2651151, 2^{17} \cdot 1), (2257935, 2^{19} \cdot 1).$$

In pairs $\left(2782215, 2^3 \cdot 1\right), \left(2782095, 2^7 \cdot 1\right), \left(2257935, 2^{19} \cdot 1\right)$ Φ_1 is divided by 5. In pairs $\left(2782191, 2^5 \cdot 1\right), \left(2651151, 2^{17} \cdot 1\right)$ Φ_1 is divided by 3. From the remaining pairs $\Phi_1 = 2782159, 2781199, 2765839$ are primes and $\Phi_1 = 2782207, 2781967$ are composites.

In some cases Φ_1, Φ_2 in equation (5.2) are both prims. Thus, form the (small) prime Φ_2 we can determine the extremely large prime Φ_1. Firstly, however, we need a criterion that identifies these pairs in the equation (5.2).

The calculations in the previous examples were carried out with a calculator. To identify large prime numbers Φ_1 from equation (5.1) an electronic system device and appropriate software development is required.

6. THE β_0 SIGN SYMMETRY

We write the powers of an odd number Π in the form of equation (4.1),

$$\begin{aligned} &\Pi^{\lambda} = \Pi^{\lambda}\left(\nu, \beta_i\right) = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i, \lambda \in \mathbb{N}^* \\ &\beta_i = \pm 1, i = 0,1,2,........,\nu-1 \\ &\nu \in \mathbb{N} \end{aligned} \qquad . \qquad (6.1)$$

Equation (6.1) separates odd numbers in two categories. In the first category the sign of $\beta_0 = \pm 1$ alternates in the consecutive powers of the odd number Π. In the second, it remains constant.

Example 6.1. In the odd powers of 3 it holds that $\beta_0 = +1$, while in the even powers $\beta_0 = -1$.

Example 6.2. In the powers of 5 it holds that $\beta_0 = -1$.

Example 6.3. In the power of 15 it holds that $\beta_0 = +1$, while in the even powers $\beta_0 = -1$.

Example 6.4. In the powers of 77 it holds that $\beta_0 = -1$.

Theorem 4.1 reveals other symmetries in the internal structure of natural numbers. An investigation is needed, similar to the one of the fundamental theorem of arithmetic of the previous centuries.

References

1. Apostol, Tom M. *Introduction to analytic number theory*. Springer Science & Business Media, 2013.

2. Manin, Yu I., and Alexei A. Panchishkin. *Number theory I: fundamental problems, ideas and theories*. Vol. 49. Springer Science & Business Media, 2013.

3. Diamond, Harold G. "Elementary methods in the study of the distribution of prime numbers." *Bulletin of the American Mathematical Society* 7.3 (1982): 553-589.

4. Newman, David J. "Simple analytic proof of the prime number theorem." *The American Mathematical Monthly* 87.9 (1980): 693-696.

5. Titchmarsh, Edward Charles, and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford University Press, 1986

6. Poussin, Charles Jean de La Vallee. *Sur la fonction [zeta](s) de Riemann et le nombre des nombres premiers inferieurs à une limite donnée*. Vol. 59. Hayez, 1899.

7. Pereira, N. Costa. "A Short Proof of Chebyshev's Theorem." *The American Mathematical Monthly* 92.7 (1985): 494-495.

8. Bateman, P. T., J. L. Selfridge, and S. Samuel Wagstaff. "The Editor's Corner: The New Mersenne Conjecture." *The American Mathematical Monthly* 96.2 (1989): 125-128.

9. Deléglise, Marc, and Joël Rivat. "Computing the summation of the Möbius function." *Experimental Mathematics* 5.4 (1996): 291-295.

10. Cashwell, Edmond, and C. J. Everett. "The ring of number-theoretic functions." *Pacific Journal of Mathematics* 9.4 (1959): 975-985.

11. Abramowitz, Milton, Irene A. Stegun, and Robert H. Romer. "Handbook of mathematical functions with formulas, graphs, and mathematical tables." (1988): 958-958.