Uncertainty and the lonely runner conjecture

The conjecture of [1] is that for k points (ideal runners) starting at one point and moving at distinct constant speeds around an ideal circle, for each runner there is a time when the arc distance between that runner and any other is at least C/k where C is the circumference of the circle. Only the cases when $k \leq 7$ have been solved in the 45 years since the conjecture was posed.

When some of the a_i may be rational, the conditions for each runner to be distance at least C/k from the chosen runner are not in general mutually statistically independent. (And the mutual expectation can actually be zero.)

Despite this, there are cases when the speeds are all rational, while the proportion of times when all runners are distance at least C/k from the chosen runner is just the continuous extension of the transcendental case.

In examples, for the sequence of rational speeds $(a_1, ..., a_{k-1}, 0)$, the proportion of time when the k'th runner is distance $\geq C/k$ from all others seems to be the same as if a_j were transcendental, provided for each prime divisor p of k $|v_p(a_j) - v_p(a_s)| \geq v_p(k)$ for all $s \neq j, k$.

We'll explain a bit later that if $(a_1, ..., a_{k-1}, 0)$ were a counterexample and some a_i is prime to k then some other must be a multiple of k. The observation above, if it is always true, implies when k is a prime p that at least two speeds must be divisible by p, and if we rescale so they are not all divisible by p then either all but one are divisible by p, or at least two must be divisible by exactly the same highest power of p larger than zero.

To begin considering the phenomenon relating primes with statistical independence, let's think of it this way. Instead of attempting to approximate how the expected proportion of time becomes discontinuous at certain rational points, let's introduce some uncertainty into the starting time of just one of the runners to remove the discontinuity.

Let $b_j = \frac{1}{a_j} lcm(a_1, ..., .a_{k-1})$ for $j = 1, ..., k-1$. The involution which interchanges integer sequences $(a_1, ..., a_{k-1})$ and $(b_1, ..., b_{k-1})$ arises by the relation between two approaches to the problem, fixing a point in space and considering how speed and time are reciprocals.

1. Theorem. A sequence of integer speeds $(a_1, ..., a_{k-1}, 0)$ is a counterexample to the lonely runner conjecture if and only if the conjugate sequence $(b_1, ..., b_{k-1})$ has the property that the union of the discrete arithmetic progressions $kb_i\mathbb{Z} + j$ for $-b_i < j < b_i$ is the whole of $\mathbb{Z}$.

Proof. Choosing units of time so that $C = k\ lcm(a_1, ..., a_{k-1})$. The probability distributions of times when each runner is within C/k of zero is a step function which changes values at integers.

In the next few paragraphs we're really just calculating unions of arithmetic progressions but we let the formalization by real valued step functions persist so that we can talk about the proportion of time when a runner is lonely without making new definitions. We will assume $C = k\ lcm(a_1, ..., a_{k-1})$.

Choose one of the runners, the i'th runner for $i \neq k$. Define integers

$$y = gcd(b_i, k\ gcd(b_i, lcm(b_1, b_2, ..., \widehat{b_i}, ..., b_{k-1})))$$

where the hat denotes a deleted entry, and

$$x = \frac{k\ gcd(b_i, lcm(b_1, ..., \widehat{b_i}, ..., b_{k-1}))}{y}.$$

Let ψ be the step function which takes the value $1/x$ at x steps of width 1, alternating with $x - 1$ steps of zero, which are of width y.

Now we alter the statistical behaviour of the i'th runner by convolving its probability distribution with ψ.

2. Theorem. When the probability distribution of the i'th runner is altered by convolving with ψ, the resulting joint probability of all the runners being of distance larger than C/k from the k'th reverts to the joint probability which is the continuous extension of the transcendental (for that runner) case and is therefore nonzero assuming LRC for $k-1$.[1]

Proof. We are considering a product of periodic functions with different periods. We think of this as the product of two functions, the one the i'th runner, on the one hand, and the product of all the others, on the other hand. The function for the i'th runner is a step function with a single high step and a single low step; and as the period replicates in the longer least common multiple the step function walks through the product function. In the transcendental case the corresponding sum of translates of the step function can be considered as if it were translationally invariant. Here there is the possibility that the greatest common divisor of the two periods may not be an integer divisor of the greatest common divisor of the size of the upstep and the downstep. According to the relevant cyclic group extension the convolution replaces the step function with a finite sum of translates to repair this. QED

Our observation about primes and valuations is partly proven by this theorem. If k is a power p^e of a prime p and the valuation of b_i is higher than the others by as much as e. Then $x = 1$, and the step function is the one which acts by the identity under convolution.

Note that the condition about prime valuations is invariant under the involution interchanging $(a_1, ..., a_{k-1})$ with $(b_1, ..., b_{k-1})$.

The change in the joint expectation from modifying the i'th distribution can be derived from the same calculation but instead we convolve with the difference $\phi - \delta$ between the step function and the 'delta function' which has a single step of height 1. The difference function is reminiscent of $sin(x)/x$ which by its convolution performs low bandpass filtering.

[1]because decreasing k has the same effect as increasing the arc from $-C/k$ to C/k by the same ratio.

On the other side of the picture, we can think of a Cartesian plane where one axis is speed and the other is position. Lines through the origin correspond to choices of time, and so belong to the real projective line.

Our assumption that no other runner has the same speed as the k'th provides nonzero denominators which allow us to ignore the point at infinity and pass to the interval of ratios in the real line $(\frac{-1+mk}{a_i}, \frac{1+mk}{a_i})$ for $i = 1, ..., k-1$. The conjecture is equivalent to saying that these open sets are not an open cover of the real line. Nestedness considerations lead us to considering that the condition

$$\frac{-1+mk}{a_i} < \frac{1+nk}{a_j} < \frac{1+mk}{a_i}$$

for the limiting point of one open interval to be contained in another, is equivalent to the determinantal condition

$$0 < det \begin{pmatrix} a_j & 1+nk \\ a_i & 1+mk \end{pmatrix} < 2a_j.$$

If $(a_1, ..., a_{k-1}, 0)$ is a counterexample, for any integer n and choice of a_j with $j \leq k-1$ there is an integer m and an a_i with $i \leq k-1$ so that the determinantal condition holds. The second row of the matrix can play the role of the first row of a new matrix.

3. Theorem. If $(a_1, ..., a_{k-1}, 0)$ is a lonely runner counterexample the algorithm runs forever from every starting point. If not, it terminates in finite time from every starting point.

In the second case the algorithm then constructs an infinite fan for a toric surface with the given upper bound on the indices of its cyclic quotient singular points. The condition is periodic up to adding a multiple of a_i to m and one can mod out the surface by an analytic automorphism to create a compact surface. Thus

4. Theorem. Such a surface exists if and only if the conjecture is false.

A topological aspect is to consider the configurations of the $k-1$ points as a point on a $k-1$-torus. One considers the closed cube of side $k-2$. The Radon transform of the the cube is a function whose domain is the projectivized tangent bundle, and while it usually has a nonempty zero locus, the lonely runner conjecture refers to the $\mathbb{P}^{k-2}$ fiber over the unique furthest point from the cube. The points in the $\mathbb{P}^{k-2}$ fiber are exactly the speed ratios $[a_1 : ... : a_{k-1}]$. Because lengths of closed geodesics change discontinuously we should try to define the Radon transform to return an average. We could multiply the characteristic function of the cube by the normal distribution on the line with standard deviation δ and take the limit of the integral as δ tends to ∞. Technical issues remain in trying to understand what discontinuities remain. One can sample the normal distribution instead of integrating the product with the step function. The limit will be the same as the standard deviation tends to infinity. While the standard deviation is finite, the sum approximating the integral is a theta series of a union of arithmetic progressions; a counterexample to the lonely runner conjecture would be a case when without passing to the limit the sum is a full Jacobi theta function with no missing terms. For small standard deviations the sum does not approach the integral, but it can be represented by an integral by using the functional equation instead.

The conjecture that every tangent line though the unique furthest point from the cube meets the closed cube is equivalent to saying that if a spanning set $p_1, ..., p_{k-1}$ of a free abelian group satsify the single relation $0 = a_1p_1 + ... + a_{k-1}p_{k-1}$ and none besides its logical consequences, the element

$$-p_1 - p_2 ... - p_{k-1}$$

is congruent in the corresponding rational vector space modulo that relation to some $c_1p_1 + ... + c_{k-1}p_{k-1}$ with $0 \leq c_i \leq k$ rational numbers.

This allows us to visualize the determinantal condition in terms of a three dimensional axonometric projection.

For example, for the tuple of speeds $(1, 3, 5, 0)$ we consider the cube in three space $\{(x, y, z) : 0 \leq x, y, z \leq 2\}$ translated by all elements of $4\mathbb{Z}^3$ projected along the line of slope $[1 : 3 : 5]$. The conjecture asserts that a point of any cube projects to $(-1, -1, -1)$. Modding out by the direction $[1 : 3 : 5]$ gives the relation among the standard basic translation vectors

$$p_1 + 2p_2 + 3p_3 = 0$$

Now p_3 projects to $-\frac{1}{5}p_1 - \frac{3}{5}p_2$ and $-p_1 - p_2 - p_3$ projects to $(\frac{1}{5} - 1)p_1 + (\frac{3}{5} - 1)p_2 = \frac{-4}{5}p_1 + \frac{-2}{5}p_2$. For projections of points congruent to this modulo 4 we must allow translations by multiples of $4p_1$ and $4p2$ but also by the projection of $4p3$ or equivalently of $4(-p_1 - p_2 - p_3)$ which is four times the translation we just calculated already.

Thus we're considering the projection of $(1 + mk)(-p_1 - p_2 - p_3)$ for integers m and if one of these projected points lies in the standard square with coefficients of p_1, p_2 between 0 and 2 this means the projection of $(-1, -1, -1)$ is also the projection of the bottom face of a cube. If this were to fail we'd repeat using p_2 and p_1 in place of p_3. A line hits a cube if and only if it hits one of the faces of course!

Thus such a sequence $(a_1, ..., a_{k-1}, 0)$ is a counterexample if and only if for all i and all numbers m there is a j such that

$$(a_i - a_j)(1 + mk) \ mod \ ka_i \in \{1, ..., 2a_i - 1\} \ mod \ a_ik$$

Subtracting a_i from both side and negating this is equivalent to

$$a_j(1 + mk) \ mod \ ka_i \in \{-a_i + 1, ..., a_i - 1\}.$$

Dividing by common denominators to make the $1+mk$ be units and dividing through by the units we see

5. Theorem. Let $(a_1, ...a_{k-1})$ be distinct nonzero natural numbers with no common prime divisor. The speeds $(a_1, ..., a_{k-1}, 0)$ is a lonely runner counterexample if and only if for every i, for every divisor g of a_i such that a_i/g is coprime to k, and for every unit u in the integers modulo kg which reduces modulo k to a_i/g there is an entry a_j in the sequence and a number $m \in \{-g+1, ..., g-1\}$ such that $a_j \equiv \ mu \ mod \ a_i k$.

In the discussion about uncertainty we promised to prove that in a counterexample when some a_i is prime to k then some other a_i must be a multiple of k. We can deduce this now. If some a_i is prime to k we can take the divisor g in the theorem to be 1 (since a_i/g is prime to k). Now $m = 0$ is forced and the theorem says that there is an entry a_j congruent to 0 modulo k.

Finally, it is possible of course to replace the integrals in the considerations about sums of arithmetic progressions with sums. Applying our involution to $(a_1, ..., a_{k-1})$ to arrive at $(b_1, ..., b_{k-1})$ we form the finite abelian group $A = \oplus \mathbb{Z}/(kb_i\mathbb{Z})$ and the exact sequence

$$0 \to C \to \Lambda^1(A) \to \Lambda^2(A)$$

The kernel C is the cyclic group $\mathbb{Z}/(k \ lcm(b_1, ..., b_{k-1})\mathbb{Z})$ and the union of the arithmetic progressions is an inverse image of a subset of C. A slight subtlety is that when we were calculating statistical expectation we considered an interval like $(-b_i, b_i)$ to have measure $2b_i$, but now we represent it as $\{-b_i+1, ..., b_i-1\}$ which has counting measure $2b_i - 1$. This is inessential since we are now considering just unions of arithmetic progressions rather than measures. With this understood,

6. Theorem. Elements of C which represent an element *not* in the union of the arithmetic progressions correspond bijectively with $(k-1)$ cliques in the graph with vertices the disjoint union of the sets of integers $\{b_i, b_i + 1, ..., (k-1)b_i\}$ and we say a vertex in the i'th set is connected by an edge with a vertex in the j'th set if and only if the integers are congruent modulo $k \ gcd(b_i, b_j)$.

This makes a $(k-1)$ partite graph and of course all $(k-1)$ cliques are $k-1$ partite. A $(k-1)$ clique in the graph describes a cocycle whose components belong to the various arithmetic progressions. The graph is a disjoint union along residue classes modulo k. The conjecture is equivalent to the statement that such a $k-1$ clique always exists. In cases when the convolving function ψ is the delta function it does exist since the expectation is nonzero.

References

1. Betke, U.; Wills, J. M. (1972). Untere Schranken fr zwei diophantische Approximations-Funktionen. Monatshefte fr Mathematik. 76 (3)

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