THE Q-UNIVERSE

A Set Theoretic Construction

"This ourth of years is naught save brickdust and being humus the same roturns . . . for on the bunk of our breadwinning lies the cropse of our Seedfather." – James Joyce, Finnegan's Wake [JJ][RW], pages 18 and 55 in [JJ].

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Preface

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|————|————|————|————|————|————|————|————|

Let R_Q be the set of all q-reals and let $S = \{a\_b \mid (a\_b \in R_Q) \wedge (a = 0)\}$, then, for any $a\_b \in R_Q - S$, $(a\_b)^{1/2}$ has four distinct roots:

1. $a^{1/2}\_[-a^{1/2} + (a + b)^{1/2}]$;
2. $a^{1/2}\_[-a^{1/2} - (a + b)^{1/2}]$;
3. $-a^{1/2}\_[a^{1/2} + (a + b)^{1/2}]$;
4. $-a^{1/2}\_[a^{1/2} - (a + b)^{1/2}]$.

Let R be the set of standard real numbers and let $R_Q{}^R = \{a\_b \mid (a\_b \in R_Q) \wedge (b = 0)\}$, then there is a ring isomorphism on R onto $R_Q{}^R$ defined by f(a) = a_0, for all $a \in R$, yet R_Q properly contains $R_Q{}^R$, hence, for any $a\_0 \in R_Q{}^R - \{0\_0\}$, $(a\_0)^{1/2}$ has four distinct roots:

1. $a^{1/2}\_0$;
2. $a^{1/2}\_-2a^{1/2}$;
3. $-a^{1/2}\_0$;
4. $-a^{1/2}\_2a^{1/2}$.

Let $S = \{a\_b \mid (a\_b \in Z_Q{}^-) \wedge (0 < b) \wedge (|a| < b)\}$, let $T = \{a\_b \mid (a\_b \in Z_Q{}^-) \wedge (0 < b) \wedge (|a| = b)\}$, let $V = \{a\_b \mid (a\_b \in Z_Q{}^+) \wedge (b < 0) \wedge (a < |b|)\}$, let $W = \{a\_b \mid (a\_b \in Z_Q{}^+) \wedge (b < 0) \wedge (a = |b|)\}$, let $X = \{a\_b \mid (a\_b \in Z_Q{}^-) \wedge (a = 0)\}$, and let $Y = \{a\_b \mid (a\_b \in Z_Q{}^+) \wedge (a = 0)\}$, then:

1. Multiplication takes S x X to X;
2. Multiplication takes V x Y to X;
3. Multiplication takes (T U W) x (X U Y) to {0_0}.

In other words, in the Q-Universe there does not exist a q-naturally lattice complete ordered field (Definition 5.1.20) and there does exist non-zero q-integers (and q-reals in general) which, when multiplied together, yield the zero q-number, 0_0. The q-natural lattice completion does, however, conform to an analog of the Archimedean Property, although it doesn't seem very useful for proving theorems, at least not in any way we have discovered.

The Q-Universe: A Set Theoretic Construction

By Wes Hansen

Abstract. In an earlier paper, "The Q-Naturals: A Recursive Arithmetic Which Extends the 'Standard' Model," we developed a set of non-standard naturals called q-naturals and demonstrated, by construction, the existence of a recursive arithmetical structure which extends the "standard" model. In this paper we extend the q-naturals out to algebraic closure and explore the properties of the various sub-structures along the way. In the process of this development, we realize that the "standard" model of arithmetic and the Q-Natural model are simply the zeroth-order and first-order recursive arithmetics, respectively, in a countable subsumption hierarchy of recursive arithmetics; there exist countably many recursive arithmetical structures.

1. Introduction. Inspired by a blog post, "Computing the Uncomputable,"[JB] by John Baez, and the subsequent commentary, we developed a novel set of numbers we call q-naturals. In the resulting paper, "The Q-Naturals: A Recursive Arithmetic Which Extends the 'Standard' Model,"[WH] we directed attention to the possibility of extending the q-naturals to the q-integers, the q-integers to the q-rationals, the q-rationals to the q-reals, and the q-reals to the q-complex; in the current work we undertake this endeavor and explore some of the properties of the various q-numbers along the way. We adhere rather closely to the format demonstrated by Clayton Dodge in his book, "Numbers and Mathematics,"[CD] and utilize a number of results contained therein, together with results from, "Introduction to Set Theory,"[HJ] by Karel Hrbracek and Thomas Jech, many of which appeared in the earlier work.

The Q-Universe distinguishes itself early on with the construction of the q-integers. Due to the definitions of addition and subtraction on the q-naturals, when we extend the q-naturals into the negative we also extend the q-components into the negative – i.e. for any q-integer a_b, the standard component "a" can be either positive or negative and the q-component "b" can also be either positive or negative. This, of course, carries through to the q-reals and we end up with what amounts to a copy of Euclidean two-space embedded in the q-reals; this has interesting implications as indicated by the preface. Also indicated by the preface, we see that, in actuality, the q-components of the q-reals must be allowed to take complex values, hence, there is, in essence, a copy of R x C embedded in the q-reals. This, of course, would seem to re-introduce the historical difficulties associated with the imaginary numbers in that, in the present context, they cannot be interpreted as rotations in the plane.

An additional work we found helpful is the excellent paper by Lothar Sebastian Krapp, "Constructing the Real Numbers: A Set Theoretical Approach."[LK] Motivated by the existence of non-standard models, there is implicit in [LK] the question, "What makes a complete ordered field truly complete?" Krapp answers this question formally by stating that an ordered field is complete relative to some completeness axiom. Intuitively, he suggests, in a rather compelling way, that it is completeness, or, more precisely, degree of completeness, which separates standard models from non-standard ones: non-standard models are complete to a higher degree in that they introduce new elements to a fundamentally complete field. Krapp doesn't make it explicit but implicit in his paper, in section four, are three philosophical propositions:

Proposition 1. A standard model is complete iff transfinite concepts are required for the introduction of additional elements.

Proposition 2. A model is standard iff its completion is.

Proposition 3. A model is non-standard iff it assumes the transfinite.

These seem rather reasonable propositions and we will, of course, have more to say about them in the closing remarks.

In the present work we include ω in our notations but this is utilized solely as a limit ordinal in our development of foundations and should not be mistaken as assuming the transfinite; the use of this limit ordinal is constrained to q-components and essentially acts as a place-holder allowing the consistent definition of von Neumann ordinals in the present context.

Notation. We use the standard notation together with:

@	|	a one-place non-logical symbol called the hyperloop
ω	|	a constant symbol representing the largest natural number
I_H	|	a hyper-inductive set
I_Q	|	a q-inductive set
N_Q	|	the set of all q-naturals
Z_Q	|	the set of all q-integers
Q_Q	|	the set of all q-rationals
R_Q	|	the set of all q-reals
C_Q	|	the set of all q-complex

Additionally, due to our use of Dedekind cuts to develop R_Q, to mitigate against confusion we designate regular subtraction with "-" and set difference with "−". Then, for any q-number a_b, the additive inverse of a_b can be represented, depending on set membership, by an equivalence class or Dedekind cut; it can be represented in general in three equivalent ways:

1. - 1_0 * a_b;
2. - (a_b);
3. - a_- b.

In general, there is no identity between - a_b and - (a_b).

In our previous work, we distinguish between natural relations/operations and q-natural relations/operations with a "q" subscript; this practice is not continued in the present work for the simple reason that doing so across 90+ pages of proofs would be cumbersome and which relation/operation is being referenced in any given case should be readily apparent from the context.

Acknowledgments. We would like to express sincere gratitude to Dr. Dave Morris, Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, Alberta, Canada, for pointing out an error in our earlier paper, and to His Holiness the Dalai Lama, for radiating the four immeasurables from his boundless heart, Om Mani Padme Hum. This work is dedicated, in memoriam, to Richard Dedekind and John Bell.

2. Q-Naturals. The q-naturals were defined and shown to exist in reference [WH]. For convenience, we reproduce those definitions and arguments here and use those definitions and arguments to demonstrate a number of properties exhibited by the q-naturals.

2.1. Definitions. We define our mathematical entities using standard terminology:

Definition 2.1.01. A set is reflexive if $X = \{X\}$; a reflexive set is called a hyperset (reference [BE], Chapter 3).

Definition 2.1.02. A one-place operation, @, when applied to any set X, generates a reflexive set; this operation, called a hyperloop, can be applied recursively.

Definition 2.1.03. Let X be an arbitrary von Neumann ordinal, then $@^nX$ designates the recursive application of @ to X "n" times, where $n \in N$; specifically, $@^0X$, the zeroth-order application, is identical to no application, i.e. $@^0X = X$.

Definition 2.1.04. Hypersets have two distinct successor functions; let $@^nX$ be an arbitrary hyperset, then $S(@^nX) = @(@^nX) = @^{(n+1)}X$, while $@^nS(X) = @^n(X \cup \{X\}) = @^n(X + 1)$.

Definition2.1.05. $\phi = 0\_0$, $@\phi = 0\_1$, $@^2\phi = 0\_2$, … , $@^{(\omega-2)}\phi = 0\_(\omega-2)$, $@^{(\omega-1)}\phi = 0\_(\omega-1)$, $@^\omega\phi = 0\_\omega$, $\{@^\omega\phi\} = 1\_0$, $@\{@^\omega\phi\} = 1\_1$, $@^2\{@^\omega\phi\} = 1\_2$, … , $@^{(\omega-2)}\{@^\omega\phi\} = 1\_(\omega-2)$, $@^{(\omega-1)}\{@^\omega\phi\} = 1\_(\omega-1)$, $@^\omega\{@^\omega\phi\} = 1\_\omega$, … , $\{@^\omega\phi, \{@^\omega\phi\}\} = 2\_0$, $@\{@^\omega\phi, \{@^\omega\phi\}\} = 2\_1$, $@^2\{@^\omega\phi, \{@^\omega\phi\}\} = 2\_2$, …

Definition2.1.06. ϕ and any number $@^\omega X$ are examples of base elements; any number $@^nX$, where $n < \omega$, is an example of a hyper-element; for example, ϕ is the only base element of $@^\omega\phi$ and for any $m \in N_Q$, $m > @^\omega\phi$, $m = @^k\{0\_\omega, 1\_\omega, \ldots, n\_\omega\}$, where n is some standard von Neumann ordinal, and every x_ω is a base element of m, while every $@^p\{0\_\omega, 1\_\omega, \ldots, n\_\omega\}$, $p \in [0, k]$, is a hyper-element of m.

Definition 2.1.07. A set, I_H, is hyper-inductive if:

1. $\phi \in I_H$;
2. if $X \in I_H$, then $S(X) \in I_H$;
3. if $X \in I_H$, then $@X \in I_H$;
4. if $@X \in I_H$, then $S(@X) \in I_H$.

Definition 2.1.08. Consistent with Definition 2.1.05, a q-natural number is an ordered pair of natural numbers, (a, b), such that (a, b) = a_b.

Definition 2.1.09. Consistent with Definition 2.1.04, any q-natural number, a_b, has two distinct successor functions which can be applied independently or in conjunction; specifically, $S(a)\_b = (a \cup \{a\})\_b = (a + 1)\_b$ and $a\_S(b) = a\_(b \cup \{b\}) = a\_(b + 1)$ (reference [HJ], Chapter 3, page 52).

Definition 2.1.10. A set, I_Q, is q-inductive if:

1. $0\_0 \in I_Q$;
2. if $a\_b \in I_Q$, then, $S(a)\_b \in I_Q$;
3. if $a\_b \in I_Q$, then, $a\_S(b) \in I_Q$.

Definition 2.1.11. The set of all q-natural numbers is the set

$N_Q = \{ x \mid x \in I_Q \text{ for every q-inductive set } I_Q\}$

Definition 2.1.12. The relation "<" (strict order) on N_Q is defined by:

for all $a\_b, c\_d \in N_Q$, $a\_b < c\_d$ iff $(a < c) \vee [(a = c) \wedge (b < d)]$, where <(a, b) is the natural order and <(a_b, c_d) is the q-natural or lexicographic order.

Definition 2.1.13. The operation "+" (addition) on N_Q is defined by:

for all $a\_b, c\_d \in N_Q$, $a\_b + c\_d = (a + c)\_(b + d)$, where +(a, c) is as defined on the set of natural numbers.

Definition 2.1.14. The operation "*" (multiplication) on N_Q is defined by:

for all a_b, c_d ϵ N_Q, a_b * c_d = (a_b * c)_(a_b * d)

= (a * c)_(b * c)_(a * d)_(b * d)

= (a * c)_(b * c) + (a * d) + (b * d),

where *(a,c) and +(b, d) are both as defined on the set of natural numbers.

Definition 2.1.15. The operation "-" (subtraction) on N_Q is defined by:

for all k_p, m_q, n_r ϵ N_Q, (k_p - m_q = n_r) iff (k_p = n_r + m_q).

Definition 2.1.16. The operation "÷" (division) on N_Q is defined by:

for all k_p, m_q, n_r ϵ N_Q, m_q ≠ 0_0, (k_p ÷ m_q = n_r) iff (k_p = n_r * m_q).

Definition 2.1.17. Let k_p, m_q, n_r ϵ N_Q be arbitrary but such that k_p = m_q * n_r, then we write m_q| k_p and say that m_q divides k_p, m_q is a factor of k_p, or k_p is a multiple of m_q. If k_p ≠ m_q * n_r, then we are entitled to state the negation of all of the above.

Definition 2.1.18. For all a_b ϵ N_Q, k ϵ N, [$(a\_b)^0$ = 1_0] ^ [$(a\_b)^1$ = a_b] ^ [$(a\_b)^{k+1}$ = $(a\_b)^k$ * a_b].

2.2.Arguments. We demonstrate our arguments using the standard methods and terminology of mathematical logic and ZFC/AFA or generalizations thereof. Specific to the current work, we generalize the Principle of Induction to the Principle of Q-Induction, we reproduce certain arguments, verbatim, from reference [HJ], and utilize results from references [HJ] and [CD].

Theorem 2.2.01. *The Axiom of Anti-Foundation implies that there exists a unique reflexive set.*

Proof. This theorem is reproduced verbatim from [HJ] (Chapter 14, page 263) and the proof can be found therein, as desired. □

Theorem 2.2.02. *A hyper-inductive set, I_H, defined by Definition 2.1.07, exists.*

Proof. Let I be an arbitrary set satisfying properties "1" and "2" of Definition 2.1.07, then I is an inductive set and, by the Axiom of Infinity, I exists. Let K be a family of intervals, [n, n + 1), such that n ϵ I and [n, n + 1) satisfies properties "3" and "4" of Definition 2.1.07. Let [n, n + 1) be an arbitrary element of K, then, by the Axiom of Infinity and Theorem 2.2.01, [n, n + 1) exists. Since [n, n + 1) was arbitrary, every [n, n + 1) ϵ K exists, hence, K exists. Finally, by the Axiom of Union, UK = I_H exists, as desired. □

Theorem 2.2.03. *For any hyper-inductive set, I_H, and any X ϵ I_H, X can be represented as an ordered pair of natural numbers, (a, b), such that (a, b) = a_b.*

Proof. This follows immediately from Definition 2.1.03, 2.1.04, 2.1.05, and the properties of natural numbers, (reference [HJ], Chapter3), as desired. □

Theorem 2.2.04. *A q-inductive set, I_Q, defined by Definition 2.1.10, exists.*

Proof. This is a direct consequence of a number of facts about the set of natural numbers,N:

1. N exists and is inductive (reference [HJ], Chapter 3, page 41);

2. By the Axiom of Power Set, the power set of any set exists;

3. By the definition of ordered pair and the definition of Cartesian product, N x N exists;

together with Definition 2.1.08 and 2.1.09, as desired. □

Theorem 2.2.05. *The set, N_Q, defined by Definition 2.1.11 exists and is q-inductive.*

Proof. Let X be the family of all q-inductive sets I_Q, then, by the Axiom of Union, the set UX exists and, by Definition 2.1.10, UX is q-inductive. By Definition 2.1.11, UX contains N_Q, hence, N_Q exists and is q-inductive, as desired. □

Theorem 2.2.06. *(The Principle of Q-Induction) Let P(x) be a property and assume that:*

1. *P(0_0) is true;*
2. *for all n_k ϵ N_Q, P(n_k) → (P[(n + 1)_k] ^ P[n_(k + 1)]).*

Then P holds for all q-natural numbers n_k.

Proof. By Definition 2.1.09 and 2.1.10, "1" and "2" above define a q-inductive set I_Q. By Definition 2.1.11, that set, I_Q, contains N_Q, as desired. □

Lemma 2.2.07. *For all a_b ϵ N_Q, a, b ϵ N.*

Proof. This follows immediately from Definition 2.1.10, Theorem 2.2.05, and the fact that N is inductive (reference [HJ], Chapter 3, page 41), as desired. □

Theorem 2.2.08. *There is a unique function, +:N_Q x N_Q → N_Q, such that:*

1. *+(m_p, 0_0) = m_p, for all m_p ϵ N_Q;*
2. *+(m_p, n_q + 1_0) = +(m_p, n_q) + 1_0, for all m_p, n_q ϵ N_Q.*

Proof. In the parametric version of the Recursion Theorem (reference [HJ], Chapter 3, page 51), let a:N_Q → N_Q be the identity function, and let g:N_Q x N_Q x N_Q → N_Q be defined by g(k_p, m_q, n_r) = m_q + 1_0, for all k_p, m_q, n_r ϵ N_Q. Then, by the Recursion Theorem, there exists a unique function, f:N_Q x N_Q → N_Q, such that:

1. f(k_p, 0_0) = a(k_p) = k_p, for all k_p ϵ N_Q;
2. f(k_p, m_q + 1_0) = g(k_p, f(k_p, m_q), m_q) = f(k_p, m_q) + 1_0, for all k_p, m_q ϵ N_Q.

Let + = f, as desired. □

Theorem 2.2.09. *There is a unique function, *:N_Q x N_Q → N_Q, such that:*

1. **(m_p, 0_0) = 0_0, for all m_p ϵ N_Q;*
2. **(m_p, n_q + 1_0) = *(m_p, n_q) + m_p, for all m_p, n_q ϵ N_Q.*

Proof. In the parametric version of the Recursion Theorem (reference [HJ], Chapter 3, page 51), let a:N_Q → N_Q be the constant function defined by a(m_p) = 0_0, for all m_p ϵ N_Q, and let g:N_Q x N_Q x N_Q → N_Q be defined by g(k_p, m_q, n_r) = m_q + n_r, for all k_p, m_q, n_r ϵ N_Q. Then, by the Recursion Theorem, there exists a unique function, f: N_Q x N_Q → N_Q, such that:

1. f(k_p, 0_0) = a(k_p) = 0_0, , for all k_p ϵ N_Q;
2. f(k_p, m_q + 1_0) = g(k_p, f(k_p, m_q), m_q) = f(k_p, m_q) + m_q, for all k_p, m_q ϵ N_Q.

Let * = f, as desired. □

Theorem 2.2.10. *N_Q is closed under the arithmetical operations "+" (addition) and "*" (multiplication).*

Proof. This is a immediate consequence of Definition 2.1.13 and 2.1.14, Lemma 2.2.07, and Theorem 2.2.08 and 2.2.09, together with the fact that N is closed under addition and multiplication (reference [HJ], Chapter 4, page 108), as desired. □

Theorem 2.2.11. *For all a_b, c_d ϵ N_Q, a_b + c_d = c_d + a_b.*

Proof. By Definition 2.1.13 and Lemma 2.2.07:

a_b + c_d = (a + c)_(b + d);

= (c + a)_(d + b);

= c_d + a_b.

Therefore, addition on N_Q is commutative, as desired. □

Theorem 2.2.12. *For all a_b, c_d ϵ N_Q, a_b * c_d = c_d * a_b.*

Proof. By Definition 2.1.14 and Lemma 2.2.07:

a_b * c_d = (a * c)_(b * c + a * d + b * d);

= (a * c)_(a * d + b * c + b * d);

= (c * a)_(d * a + c * b + d * b);

= c_d * a_b.

Therefore, multiplication on N_Q is commutative, as desired. □

Theorem 2.2.13. *For all k_p, m_q, n_r ϵ N_Q, (k_p + m_q) + n_r = k_p + (m_q + n_r).*

Proof. By Definition 2.1.13 and Lemma 2.2.07:

(k_p + m_q) + n_r = (k + m)_(p + q) + n_r;

= [(k + m) + n]_[(p + q) + r];

= [k +(m + n)]_[p + (q + r)];

= k_p + (m + n)_(q + r);

= k_p + (m_q + n_r).

Therefore, addition on N_Q is associative, as desired. □

Theorem 2.2.14. *For all k_p, m_q, n_r ϵ N_Q, (k_p * m_q) * n_r = k_p * (m_q * n_r).*

Proof. By Definition 2.2.14 and Lemma 2.2.07:

(k_p * m_q) * n_r = [(k * m)_(p * m + k * q + p * q)] * n_r;

= [(k * m) * n]_ [(p * m + k * q + p * q) *n + (k * m) * r + (p * m + k * q + p * q) * r];

= [k * (m * n)]_[(p * m) * n + (k * q) * n + (p * q) *n + (k * m) * r + (p * m) * r + (k * q) * r + (p * q) * r];

= [k * (m * n)]_[p *(m * n) + k * (q * n) + p * (q *n) + k * (m * r) + p * (m * r) + k * (q * r) + p * (q * r)];

= [k * (m * n)]_ [p *(m * n) + k * (q * n) + k * (m * r) + k * (q * r) + p * (q *n) + p * (m * r) + p * (q * r)];

= [k * (m * n)]_[p *(m * n) + k * (q * n + m * r + q * r) + p * (q *n + m * r + q * r)];

= k_p * [(m * n)_(q * n + m * r + q * r)];

= k_p * (m_q * n_r).

Therefore, multiplication on N_Q is associative, as desired. □

Theorem 2.2.15. For all k_p, m_q, n_r ϵ N_Q, k_p * (m_q + n_r) = (k_p * m_q) + (k_p * n_r).

Proof. By Definition 2.2.13 and 2.2.14 and Lemma 2.2.07:

k_p * (m_q + n_r) = k_p * (m + n)_(q + r);

= [k * (m + n)]_[p * (m + n) + k * (q + r) + p * (q + r)];

= (k * m + k * n)_[p * m + (p * n + k * q) + (k * r + p * q) + p * r];

= (k * m + k * n)_[p * m + k * q + (p * n + p * q) + k * r + p * r];

= (k * m + k * n)_[p * m + k * q + (p * q + p * n) + k * r + p * r];

= [(k * m) + (k * n)]_[(p * m + k * q + p * q) + (p * n + k * r + p * r)];

= [(k * m)_(p * m + k * q + p * q)] + [(k * n)_(p * n + k * r + p * r)];

= (k_p * m_q) + (k_p * n_r).

Therefore, multiplication is left distributive over addition, as desired. □

Theorem 2.2.16. For all k_p, m_q, n_r ϵ N_Q, (m_q + n_r) * k_p = (m_q * k_p) + (n_r * k_p).

Proof. By Definition 2.1.13 and 2.1.14 and Lemma 2.2.07:

(m_q + n_r) * k_p = (m + n)_(q + r) * k_p;

= [(m + n) * k]_[(q + r) * k + (m + n) * p + (q + r) * p];

= (m * k + n * k)_[q * k + (r * k + m * p) + (n * p + q * p) + r * p];

= (m * k + n * k)_(q * k + m * p + (r * k + q * p) + n * p + r * p);

= (m * k + n * k)_(q * k + m * p + (q * p + r * k) + n * p + r * p);

= [(m * k) + (n * k)]_[(q * k + m * p + q * p) + (r * k + n * p + r * p)];

= [(m * k)_(q * k + m * p + q * p)] + [(n * k)_(r * k + n * p + r * p)];

= (m_q * k_p) + (n_r * k_p).

Therefore, multiplication is right distributive over addition, as desired. □

Corollary 2.2.17. For all k_p, m_q, n_r , o_s ϵ N_Q,(k_p + m_q) * (n_r + o_s) = (k_p * n_r + k_p * o_s) + (m_q * n_r + m_q * o_s).

Proof. By Theorem 2.2.11, 2.2.13, 2.2.15, and 2.2.16:

(k_p + m_q) * (n_r + o_s) = (k_p + m_q) * n_r + (k_p + m_q) * o_s;

= (k_p * n_r + m_q * n_r) + (k_p * o_s + m_q * o_s);

= k_p * n_r + [m_q * n_r + (k_p * o_s + m_q * o_s)];

= k_p * n_r + [(m_q * n_r + k_p * o_s) + m_q * o_s];

= k_p * n_r + [(k_p * o_s + m_q * n_r) + m_q * o_s];

= k_p * n_r + [k_p * o_s + (m_q * n_r + m_q * o_s)];

= (k_p * n_r + k_p * o_s) + (m_q * n_r + m_q * o_s), as desired. □

Theorem 2.2.18 *For all a_b ϵ N_Q, a_b * 1_0 = a_b.*

Proof. By Definition 2.1.14 and Lemma 2.2.07:

a_b * 1_0 = (a * 1)_(b * 1 + a * 0 + b * 0);

= a_b.

Therefore, 1_0 is a multiplicative identity for N_Q, as desired. □

Corollary 2.2.19. *For all a_b, c_d ϵ N_Q, (a_b * c_d = a_b) → (c_d = 1_0).*

Proof. This is an immediate consequence of Theorem 2.2.18 and the fact that "1" is the unique multiplicative identity on N (reference [HJ], Chapter 4, page 110) and that n * 0 = 0 for all n ϵ N (reference [HJ], Chapter 4, page 54). Therefore, 1_0 is unique, as desired. □

Corollary 2.2.20. *For all k_p, m_q, n_r ϵ N_Q, (k_p + m_q = k_p + n_r) iff (m_q = n_r).*

Proof. Suppose k_p + m_q = k_p + n_r, then, by Definition 2.1.13, (k + m)_(p + q) = (k + n)_(p + r), hence, (k + m = k + n) ^ (p + q = P + r). But then, by Lemma 2.2.07, (m = n) ^ (q = r) (reference [CD], Chapter 4, page 109), hence, m_q = n_r. Therefore, (k_p + m_q = k_p + n_r) → (m_q = n_r).

Suppose m_q = n_r, then, by Definition 2.1.13 and Lemma 2.2.07, k_p + m_q = (k + m)_(p + q) = (k + n)_(p + r) = k_p + n_r (reference [CD], Chapter 4, page 109). Therefore, (m_q = n_r) → (k_p + m_q = k_p + n_r).

Therefore, (k_p + m_q = k_p + n_r) iff (m_q = n_r), as desired. □

Corollary 2.2.21. *For all k_p, m_q, n_r ϵ N_Q, (k_p * m_q = k_p * n_r) iff (m_q = n_r).*

Proof. Suppose k_p * m_q = k_p * n_r, then, by Definition 2.1.14, (k * m)_(p * m + k * q + p * q) = (k * n)_(p * n + k * r + p * r), hence, (k * m = k * n) ^ [(p * m + k * q + p * q) = (p * n + k * r + p * r)]. But then, by Lemma 2.2.07, (m = n) ^ (q = r) (reference [CD], Chapter 4, pages 109, 110, 116, and 117), hence, m_q = n_r. Therefore, (k_p * m_q = k_p * n_r) → (m_q = n_r).

Suppose m_q = n_r, then, by Definition 2.1.14 and Lemma 2.2.07, k_p * m_q = (k *m)_(p * m + k *q + p * q) = (k * n)_ (p * n + k * r + p * r) = k_p * n_r (reference [CD], Chapter 4, pages 109, 110, 116, and 117). Therefore, (m_q = n_r) → (k_p * m_q = k_p * n_r).

Therefore, (k_p * m_q = k_p * n_r) iff (m_q = n_r), as desired. □

Theorem 2.2.22. *(N, <) is a linearly ordered set.*

Proof. This theorem is reproduced verbatim from reference [HJ] (Chapter 3, page 43) and the proof can be found therein, as desired. □

Lemma 2.2.23. *For all a_b, $c\_d \in N_Q$:*

1. *$0\_0 \leq c\_d$;*
2. *$[a\_b < c\_(d + 1)]$ iff $(a\_b \leq c\_d)$.*

Proof. The proof is in two parts:

1) We proceed by q-induction. Let $P(x\_y)$ be the property, "$0\_0 \leq x\_y$," then:

P(0_0). $0\_0 = 0\_0$, hence, $0\_0 \leq 0\_0$.

Suppose $P(n\_k)$ is true, then $(0\_0 < n\_k) \vee (0\_0 = n\_k)$ and:

P[(n + 1)_k] ^ P[n_(k + 1)]. In both cases, by Lemma 2.2.07 and Theorem 2.2.22, $[0\_0 < (n + 1)\_k] \wedge [0\_0 < n\_(k + 1)]$.

Therefore, $P(n\_k) \rightarrow (P[(n + 1)\_k] \wedge P[n\_(k + 1)])$ and, by the Principle of Q-Induction, for all $n\_k \in N_Q$, $0\_0 \leq n\_k$, as desired. □

2) Suppose $a\_b < c\_(d + 1)$, then, by Definition 2.1.12, $a < c \vee [a = c \wedge b < (d +1)]$. If $a < c$, then, by Definition 2.1.12 again, $a\_b < c\_d$; otherwise, if $(a = c) \wedge [b < (d +1)]$, then, by Lemma 2.2.07 and Theorem 2.2.22, $a\_b \leq c\_d$.

In both cases $a\_b \leq c\_d$, hence, $[a\_b < c\_(d + 1)] \rightarrow (a\_b \leq c\_d)$.

Suppose $a\_b \leq c\_d$, then, by Definition 2.1.12, $\{(a < c) \vee [(a = c) \wedge (b < d)]\} \vee [(a = c) \wedge (b = d)]$ and three cases arise:

Case 1. Suppose $a < c$, then, by Definition 2.1.12, $a\_b < c\_(d + 1)$.

Case 2. Suppose $(a = c) \wedge (b < d)$, then, by Lemma 2.2.07 and Theorem 2.2.22, $a\_b < c\_(d + 1)$.

Case 3. Suppose $(a = c) \wedge (b = d)$, then , by Lemma 2.2.07 and Theorem 2.2.22, $a\_b < c\_(d + 1)$.

In all three cases $a\_b < c\_(d + 1)$, hence, $(a\_b \leq c\_d) \rightarrow [a\_b < c\_(d + 1)]$.

Therefore, $[a\_b < c\_(d + 1)]$ iff $(a\_b \leq c\_d)$, as desired. □

Theorem 2.2.24. *$(N_Q, <)$ is a linearly ordered set.*

Proof. The proof is in three parts:

1) *Transitivity.* Let k_p, m_q, $n\_r \in N_Q$ be arbitrary but such that $(k\_p < m\_q) \wedge (m\_q < n\_r)$. Then, by Definition 2.1.12, $\{(k < m) \vee [(k = m) \wedge (p < q)]\} \wedge \{(m < n) \vee [(m = n) \wedge (q < r)]\}$ and four cases arise:

Case 1. Suppose $(k < m) \wedge (m < n)$, then, by Lemma 2.2.07 and Theorem 2.2.22, $k < n$, and, by Definition 2.1.12 again, $k\_p < n\_r$.

Case 2. Suppose $(k < m) \wedge (m = n) \wedge (q < r)$, then, by Lemma 2.2.07 and Theorem 2.2.22, $k < n$, and, by Definition 2.1.12 again, $k\_p < n\_r$.

Case 3. Suppose $(k = m) \wedge (p < q) \wedge (m < n)$, then, by Lemma 2.2.07 and Theorem 2.2.22, $k < n$, and, by Definition 2.1.12 again, $k\_p < n\_r$.

Case 4. Suppose $(k = m) \wedge (p < q) \wedge (m = n) \wedge (q < r)$, then, by Lemma 2.2.07 and Theorem 2.2.22, $k = n \wedge p < r$, and, by Definition 2.1.12 again, $k\_p < n\_r$.

In all four cases, $k\_p < n\_r$, hence, $[(k\_p < m\_q) \wedge (m\_q < n\_r)] \rightarrow (k\_p < n\_r)$.

2) *Asymmetry.* Let $k_p, m_q \in N_Q$ be arbitrary and suppose, for contradiction, that $(k_p < m_q) \wedge (m_q < k_p)$, then, by transitivity, $k_p < k_p$, contradicting Definition 2.1.12.

3) *Linearity.* We proceed by q-induction. Let $P(x_y)$ be the property, "for all $m_p \in N_Q$, $(m_p < x_y) \vee (m_p = x_y) \vee (x_y < m_p)$," then:

P(0.0). This is an immediate consequence of Lemma 2.2.23.

Suppose $P(n_k)$ is true, then for all $m_p \in N_Q$, $(m_p < n_k) \vee (m_p = n_k) \vee (n_k < m_p)$ and:

$P[(n + 1)_k] \wedge P[n_{(k + 1)}]$. There are three cases to consider:

Case 1. Suppose $m_p < n_k$, then, by Definition 2.1.12, Lemma 2.2.07, and Theorem 2.2.08, $[n_k < (n + 1)_k] \wedge [n_k < n_{(k + 1)}]$, hence, by transitivity, $[m_p < (n + 1)_k] \wedge [m_p < n_{(k + 1)}]$.

Case 2. Suppose $m_p = n_k$, then, by Definition 2.1.12, Lemma 2.2.07, and Theorem 2.2.08, $[m_p < (n + 1)_k] \wedge [m_p < n_{(k + 1)}]$.

Case 3. Suppose $n_k < m_p$, then, by Definition 2.1.12, $(n < m) \vee [(n = m) \wedge (k < p)]$ and two cases arise:

Case 3.a. Suppose $n < m$, then, by Lemma 2.2.07 and Theorem 2.2.22, $\{[(n + 1) < m] \vee [(n + 1) = m]\} \wedge \{[(k + 1) < p] \vee [(k + 1) = p] \vee [p < (k + 1)]\}$ and four cases arise:

Case 3.a.1. Suppose $(n + 1) < m$, then, by Definition 2.1.12, Lemma 2.2.07, and Theorem 2.2.22, $[(n + 1)_k < m_p] \wedge [n_{(k + 1)} < m_p]$.

Case 3.a.2. Suppose $[(n + 1) = m] \wedge [(k + 1) < p]$, then, by Definition 2.1.12, Lemma 2.2.07, and Theorem 2.2.08, $[(n + 1)_k < m_p] \wedge [n_{(k + 1)} < m_p]$.

Case 3.a.3. Suppose $[(n + 1) = m] \wedge [(k + 1) = p]$, then, by Definition 2.1.12, Lemma 2.2.07, and Theorem 2.2.22, $[(n + 1)_k < m_p] \wedge [n_{(k + 1)} < m_p]$.

Case 3.a.4. Suppose $[(n + 1) = m] \wedge [p < (k + 1)]$, then, by Definition 2.1.12, Lemma 2.2.07, and Theorem 2.2.22, $[m_p \leq (n + 1)_k] \wedge [n_{(k + 1)} < m_p]$.

In all four cases, $\{[(n + 1)_k < m_p] \wedge [n_{(k + 1)} < m_p]\} \vee \{[m_p \leq (n + 1)_k] \wedge [n_{(k + 1)} < m_p]\}$, hence, $(n < m) \rightarrow \{[(n + 1)_k < m_p] \vee [(n + 1)_k = m_p] \vee [m_p < (n + 1)_k]\} \wedge \{[n_{(k + 1)} < m_p] \vee [n_{(k + 1)} = m_p] \vee [m_p < n_{(k + 1)}]\}$.

Case 3.b. Suppose $(n = m) \wedge (k < p)$, then, by Lemma 2.2.07 and Theorem 2.2.22, $[m < (n + 1)] \wedge [(k + 1) \leq p]$, and, by Definition 2.1.12, $[m_p < (n + 1)_k] \wedge \{[n_{(k + 1)} < m_p] \vee [n_{(k + 1)} = m_p]\}$.

In both cases, $\{[(n + 1)_k < m_p] \vee [(n + 1)_k = m_p] \vee [m_p < (n + 1)_k]\} \wedge \{[n_{(k + 1)} < m_p] \vee [n_{(k + 1)} = m_p] \vee [m_p < n_{(k + 1)}]\}$, hence, $(n_k < m_p) \rightarrow \{[(n + 1)_k < m_p] \vee [(n + 1)_k = m_p] \vee [m_p < (n + 1)_k)]\} \wedge \{[n_{(k + 1)} < m_p] \vee [n_{(k + 1)} = m_p] \vee [m_p < n_{(k + 1)}]\}$.

Therefore, $P(n_k) \rightarrow (P[(n + 1)_k] \wedge P[n_{(k + 1)}])$ and, by the Principle of Q-Induction, linearity.

Therefore, $(N_Q, <)$ is a linearly ordered set, as desired. □

Theorem 2.2.25. *$(N_Q, <)$ is a well-ordered set.*

Proof. This is an immediate consequence of Lemma 2.2.07, Theorem 2.2.22, and Lemma 2.2.23, as desired. □

Theorem 2.2.26. *$(N_Q, <)$ is isomorphic to ω^2.*

Proof. Let Y = {S_i | i ϵ N} = ran S for some index function S, where each S_i is the set of natural numbers. Then ω^2 = { a_i | (a_i ϵ S_i ϵ Y) ^ [for all i,j,a,b ϵ N, (a_i < b_j) iff [(i < j) ˅ [(i = j) ^ (a < b)]]]} (reference [CD], Chapter 13, page 470) and there is an obvious isomorphism, f:ω^2 → (N_Q, <), defined by f(a_i) = i_a, as desired. □

Theorem 2.2.27. *For all k_p, m_q, n_r , o_s ϵ N_Q,(k_p < m_q) iff [(k_p + n_r) < (m_q + n_r)].*

Proof. Suppose k_p < m_q, then, by Definition 2.1.12, (k < m) ˅ [(k = m) ^ (p < q)]. Suppose k < m, then (k + n) < (m + n) (reference [CD], Chapter 4, page 113) and, by Definition 2.1.13, (k_p + n_r) < (m_q + n_r). Otherwise, suppose (k = m) ^ (p < q), then, by Lemma 2.2.07, [(k + n) = (m + n)] ^ [(p + r) < (q + r)] (reference [CD], Chapter 4, page 113) and, by Definition 2.1.13, (k_p + n_r) < (m_q + n_r). Therefore, (k_p < m_q) → [(k_p + n_r) < (m_q + n_r)].

Suppose (k_p + n_r) < (m_q + n_r), then, by Definition 2.1.13, (k + n)_(p + r) < (m + n)_(q + r) and, by Definition 2.1.12, [(k + n) < (m + n)] ˅ {[(k + n) = (m + n)] ^ [(p + r) < (q + r)]}. Suppose (k + n) < (m + n), then, by Lemma 2.2.07, k < m (reference [CD], Chapter 4, page 114) and, by Definition 2.1.12, k_p < m_q. Otherwise, suppose [(k + n) = (m + n)] ^ [(p + r) < (q + r)], then, by Lemma 2.2.07, (m = n) ^ (p < q) (reference [CD], Chapter 4, page 114) and, by Definition 2.1.12, k_p < m_q. Therefore, [(k_p + n_r) < (m_q + n_r)] → (k_p < m_q).

Therefore, (k_p < m_q) iff [(k_p + n_r) < (m_q + n_r)], as desired. □

Theorem 2.2.28. *For all k_p, m_q, n_r, o_s ϵ N_Q, [(k_p < m_q) ^ (n_r < o_s)] → [(k_p + n_r) < (m_q + o_s)].*

Proof. Suppose that (k_p < m_q) ^ (n_r < o_s), then, by Definition 2.1.12, {(k < m) ˅ [(k = m) ^ (p < q)]} ^ {[(n < o) ˅ [(n = o) ^ (r < s)]] and four cases arise:

Case 1. Suppose (k < m) ^ (n < o), then, by Lemma 2.2.07, (k + n) < (m + o) (reference [CD], Chapter 4, page 114) and, by Definition 2.1.13, (k_p + n_r) < (m_q + o_s).

Case 2. Suppose (k < m) ^ (n = o) ^ (r < s), then, by Lemma 2.2.07, (k + n) < (m + n) = (m + o) (reference [CD], Chapter 4, page113) and, by Definition 2.1.13, (k_p + n_r) < (m_q + o_s).

Case 3. Suppose (k = m) ^ (p < q) ^ (n < o), then, by Lemma 2.2.07, (k + n) < (k + o) = (m + o) (reference [CD], Chapter 4, page 113) and, by Definition 2.1.13, (k_p + n_r) < (m_q + o_s).

Case 4. Suppose (k = m) ^ (p < q) ^ (n = o) ^ (r < s), then, by Lemma 2.2.07, [(k + n) = (m + o)] ^ [(p + r) < (q + s)] (reference [CD], Chapter 4, page 114) and, by Definition 2.1.13, (k_p + n_r) < (m_q + o_s).

In all four cases, (k_p + n_r) < (m_q + o_s).

Therefore, [(k_p < m_q) ^ (n_r < o_s)] → [(k_p + n_r) < (m_q + o_s)], as desired.□

Theorem 2.2.29. *For all k_p, m_q, n_r ϵ N_Q, (k_p < m_q) iff [(k_p * n_r) < (m_q * n_r)].*

Proof. Suppose k_p < m_q, then, by Definition 2.1.12, (k < m) ˅ [(k = m) ^ (p < q)]. Suppose k < m, then, by Lemma 2.2.07, (k * n) < (m * n) (reference [CD], Chapter 4, page 114) and, by Definition 2.1.14, (k_p * n_r) < (m_q * n_r). Otherwise, suppose (k = m) ^ (p < q), then, by Lemma 2.2.07, (k * n = m * n) ^ (k * r = m * r) ^ (p * n < q * n) ^ (p * r < q * r) (reference [CD], Chapter 4, page 114) and, by Definition 2.1.14, (k_p * n_r) < (m_q * n_r). Therefore, (k_p < m_q) → [(k_p * n_r) < (m_q * n_r)].

Suppose (k.p * n.r) < (m.q * n.r), then, by Definition 2.1.14, (k * n)_(p * n + k * r + p * r) < (m * n)_(q * n + m * r + q * r) and, by Definition 2.1.12, (k * n < m * n) ˅ {(k * n = m * n) ^ [(p * n + k * r + p * r) < (q * n + m * r + q * r)]}. Suppose k * n < m * n, then, by Lemma 2.2.07, k < m (reference [CD], Chapter 4, page 115) and, by Definition 2.1.12, k_p < m_q. Otherwise, suppose (k * n = m * n) ^ [(p * n + k * r + p * r) < (q * n + m * r + q * r)], then, by Lemma 2.2.07, (k = m) ^ (k * r = m * r) (reference [CD], Chapter 4, page 110) and [p * (n + r)] + k * r = [p * (n + r)] + m * r < [q * (n + r)] + m * r, hence, (k = m) ^ (p < q) (reference [CD], Chapter 4, pages 114 and 115) and, by Definition 2.1.12, k_p < m_q. Therefore, [(k_p * n_r) < (m_q * n_r)] → (k_p < m_q).

Therefore, (k_p < m_q) iff [(k_p * n_r) < (m_q * n_r)], as desired. □

Theorem 2.2.30. *For all k_p, m_q, n_r, o_s ϵ N_Q, [(k_p < m_q) ^ (n_r < o_s)] → (k_p * n_r < m_q * o_s).*

Proof. Suppose (k_p < m_q) ^ (n_r < o_s), then, by Definition 2.1.12, {(k < m) ˅ [(k = m) ^ (p < q)]} ^ {(n < o) ˅ [(n = o) ^ (r < s)]} and four cases arise:

Case 1. Suppose (k < m) ^ (n < o), then, by Lemma 2.2.07, k * n < m * o (reference [CD], Chapter 4, page 115) and, by Definition 2.1.14, k_p * n_r < m_q * o_s.

Case 2. Suppose (k < m) ^ (n = o) ^ (r < s), then, by Lemma 2.2.07, k * n < m * n = m * o and, by Definition 2.1.14, k_p * n_r < m_q * o_s.

Case 3. Suppose (k = m) ^ (p < q) ^ (n < o), then, by Lemma 2.2.07, k * n < k * o = m * o and, by Definition 2.1.14, k_p * n_r < m_q * o_s.

Case 4. Suppose (k = m) ^ (p < q) ^ (n = o) ^ (r < s), then, by Lemma 2.2.07, (k * n = k * o = m * o) ^ (p * n = p * o = q * o) ^ (p * r < q * s) (reference [CD], Chapter 4, pages 114 and 115), hence, (k * n = m * o) ^ [(p * n + k * r + p * r) < (q * o + m * s + q * s) (reference [CD], Chapter 4, page 114) and, by Definition 2.1.14, k_p * n_r < m_q * o_s.

In all four cases, k_p * n_r < m_q * o_s.

Therefore, [(k_p < m_q) ^ (n_r < o_s)] → (k_p * n_r < m_q * o_s), as desired. □

Theorem 2.2.31. *For all k_p, m_q, n_r, o_s ϵ N_Q, [(k_p + m_q < n_r + o_s) ^ (k_p ≥ n_r)] → (m_q < o_s).*

Proof. Suppose (k_p + m_q < n_r + o_s) ^ (k_p ≥ n_r), then, by Definition 2.1.12 and 2.1.13, {(k + m < n + o) ˅ [(k + m = n + o) ^ (p + q < r + s)]} ^ {[k > n ˅ (k = n ^ p > r) ˅ (k = n ^ p = r)]} and six cases arise:

Case 1. Suppose (k + m < n + o) ^ (k > n), then, by Lemma 2.2.07, m < o (reference [CD], chapter 4, page 115) and, by Definition 2.1.12, m_q < o_s.

Case 2. Suppose (k + m < n + o) ^ (k = n) ^ (p > r), then, by Lemma 2.2.07, m < o (reference [CD], chapter 4, page 115) and, by Definition 2.1.12, m_q < o_s.

Case 3. Suppose (k + m < n + o) ^ (k = n) ^ (p = r), then, by Lemma 2.2.07, m < o (reference [CD], chapter 4, page 115) and, by Definition 2.1.12, m_q < o_s.

Case 4. Suppose (k + m = n + o) ^ (p + q < r + s) ^ (k > n), then, by Lemma 2.2.07, (n + m < n + o) ^ (m < o) (reference [CD], Chapter 4, page 114), hence, by Definition 2.1.12, m_q < o_s.

Case 5. Suppose (k + m = n + o) ^ (p + q < r + s) ^ (k = n) ^ (p > r), then, by Lemma 2.2.07, (m = o) ^ (r + q < r + s) ^ (q < s) (reference [CD], Chapter 4, page 113), hence, by Definition 2.1.12, m_q < o_s.

Case 6. Suppose (k + m = n + o) ^ (p + q < r + s) ^ (k = n) ^ (p = r), then (m = o) ^ (r + q < r + s) ^ (q < s) (reference [CD], Chapter 4, page 113), hence, by Definition 2.1.12, m_q < o_s.

In all six cases, m_q < o_s.

Therefore, [(k_p + m_q < n_r + o_s) ^ (k_p ≥ n_r)] → (m_q < o_s), as desired. □

Theorem 2.2.32. *For all k_p, m_q, n_r, o_s ϵ N_Q, [(k_p * m_q < n_r * o_s) ^ (k_p ≥ n_r)] → (m_q < o_s).*

Proof. Suppose (k_p * m_q < n_r * o_s) ^ (k_p ≥ n_r), then, by Definition 2.1.12 and 2.1.14, {(k * m < n * o) ˅ [(k * m = n * o) ^ [(p * m + k * q + p * q) < (r * o + n * s + r * s)]} ^ {(k > n) ˅ [(k = n) ^ (p > r)] ˅ [(k = n) ^ (p = r)]} and six cases arise:

Case 1. Suppose (k * m < n * o) ^ (k > n), then, by Lemma 2.2.07, m < o (reference [CD], chapter 4, page 115) and, by Definition 2.1.12, m_q < o_s.

Case 2. Suppose (k * m < n * o) ^ (k = n) ^ (p > r), then, by Lemma 2.2.07, m < o (reference [CD], chapter 4, page 115) and, by Definition 2.1.12, m_q < o_s.

Case 3. Suppose (k * m < n * o) ^ (k = n) ^ (p = r), then, by Lemma 2.2.07, m < o (reference [CD], chapter 4, page 115) and, by Definition 2.1.12, m_q < o_s.

Case 4. Suppose (k * m = n * o) ^ [(p * m + k * q + p * q) < (r * o + n * s + r * s)] ^ (k > n), then, by Lemma 2.2.07, (n * m < k * m = n * o) (reference [CD], Chapter 4, page 114), hence, m < o (reference [CD], Chapter 4, page 115) and, by Definition 2.1.12, m_q < o_s.

Case 5. Suppose (k * m = n * o) ^ [(p * m + k * q + p * q) < (r * o + n * s + r * s)] ^ (k = n) ^ (p > r), then, by Lemma 2.2.07, (m = o) ^ [(r *m + n * q + r * q) < (r * o + n * s + r * s)] (reference [CD], Chapter 4, pages 110, 114, and 115), hence, by commutativity and associativitiy of addition on N (reference [CD], Chapter 4, page 138), q * (n + r) + r * m = q * (n + r) + r * o < s * (n + r) + r * o, hence, q * (n + r) < s * (n + r) and q < s (reference [CD], Chapter 4, pages 114 and 115). Then (m = o) ^ (q < s) and, by Definition 2.1.12, m_q < o_s.

Case 6. Suppose (k * m = n * o) ^ [(p * m + k * q + p * q) < (r * o + n * s + r * s)] ^ (k = n) ^ (p = r), then, by Lemma 2.2.07, (m = o) ^ [(r *m + n * q + r * q) <(r * o + n * s + r * s)] (reference [CD], Chapter 4, pages 110, 114, and 115), hence, by commutativity and associativitiy of addition on N (reference [CD], Chapter 4, page 138), q * (n + r) + r * m = q * (n + r) + r * o < s * (n + r) + r * o, hence, q * (n + r) < s * (n + r) and q < s (reference [CD], Chapter 4, pages 114 and 115). Then (m = o) ^ (q < s) and, by Definition 2.1.12, m_q < o_s.

In all six cases, m_q < o_s.

Therefore, [(k_p * m_q < n_r * o_s) ^ (k_p ≥ n_r)] → (m_q < o_s), as desired. □

Theorem 2.2.33. *For all a_b, c_d, x_y ϵ N_Q, (a_b + x_y ≤ c_d) → (a_b < c_d).*

Proof. Suppose a_b + x_y < c_d, then, by Definition 2.1.12 and 2.1.13, (a + x < c) ˅ [(a + x = c) ^ (b + y < d)] and two cases arise:

Case 1. Suppose a + x < c, then, by Lemma 2.2.07, a < c (reference [CD], Chapter 4, page 115) and, by Defintion 2.1.12, a_b < c_d.

Case 2. Suppose (a + x = c) ^ (b + y < d), then, by Lemma 2.2.07, if x = 0, then (a = c) ^ (b < d) (reference [CD], Chapter 4, page115) and, by Definition 2.1.12, a.b < c.d. Otherwise, if x ≠ 0, then a < c (reference [CD], Chapter 4, page113) and, by Definition 2.1.12, a_b < c_d.

In both cases, a_b < c_d.

Therefore, (a_b + x_y < c_d) → (a_b < c_d).

Suppose a_b + x_y = c_d, then, by Definition 2.1.13, (a + x = c) ^ (b + y = d), hence, by Lemma 2.2.07, a < c (reference [CD], Chapter 4, page113) and, by Definition 2.1.12, a_b < c_d.

Therefore, (a_b + x_y = c_d) → (a_b < c_d).

Therefore, (a_b + x_y ≤ c_d) → (a_b < c_d), as desired. □

Theorem 2.2.34. *N_Q is not closed under the operation " -" (subtraction) of Definition 2.1.15.*

Proof. Let a_b ϵ N_Q – {0_0} be arbitrary, then, by Definition 2.1.13 and 2.1.17, (0_0 - a_b = c_d) iff [0_0 = c_d + a_b = (c + a)_(d + b)]. Suppose, for contradiction, that 0_0 = (c + a)_(d + b), then 0 = c + a and, by Lemma 2.2.07 and the definition of "<" (strict order) on N – {0} (reference [CD], Chapter 4, page 113), c < 0, hence, by Definition 2.1.12, c_d < 0_0, contradicting Lemma 2.2.23. Therefore, N_Q is not closed under subtraction, as desired. □

Theorem 2.2.35. *For all a_b, c_d ϵ N_Q, (a_b - c_d ϵ N_Q) → [(a_b - c_d) + c_d = a_b].*

Proof. Suppose a_b - c_d ϵ N_Q, then, by Definition 2.1.15, there exists some x_y ϵ N_Q such that a_b - c_d = x_y and a_b = x_y + c_d. But then, by substitution, a_b = (a_b - c_d) + c_d, as desired. □

Theorem 2.2.36. *For all a_b, c_d, k_p ϵ N_Q, (a_b - c_d ϵ N_Q) → [k_p * (a_b - c_d) = k_p *a_b - k_p * c_d].*

Proof. Suppose a_b - c_d ϵ N_Q, then, by Definition 2.1.15, there exists some x_y ϵ N_Q such that a_b - c_d = x_y and a_b = x_y + c_d. But then, by Theorem 2.2.15, k_p * (a_b - c_d) = k_p * x_y and k_p * a_b = k_p * (x_y + c_d) = k_p * x_y + k_p * c_d. By Definition 2.1.17 again, k_p * a_b - k_p * c_d = k_p * x_y, hence, by substitution, k_p * (a_b - c_d) = k_p * a_b - k_p * c_d.

Therefore, multiplication is left distributive over subtraction, as desired. □

Theorem 2.2.37. *For all a_b, c_d, k_p ϵ N_Q, (a_b - c_d ϵ N_Q) → [(a_b - c_d) * k_p = a_b * k_p - c_d * k_p].*

Proof. Suppose a_b - c_d ϵ N_Q, then, by Definition 2.1.15, there exists some x_y ϵ N_Q such that a_b - c_d = x_y and a_b = x_y + c_d. But then, by Theorem 2.2.16, (a_b - c_d) * k_p = x_y * k_p and a_b * k_p = (x_y + c_d) * k_p = x_y * k_p + c_d * k_p. By Definition 2.1.15 again, a_b * k_p - c_d * k_p = x_y * k_p, hence, by substitution, (a_b - c_d) * k_p = a_b * k_p - c_d * k_p.

Therefore, multiplication is right distributive over subtraction, as desired. □

Theorem 2.2.38. *For all a_b, c_d ϵ N_Q, (a_b - c_d ϵ N_Q)iff [(c ≤ a) ^ (d ≤ b)].*

Proof. Suppose a_b - c_d ϵ N_Q, then, by Definition 2.1.15, there exists some x_y ϵ N_Q such that a_b - c_d = x_y and a_b = x_y + c_d and, by Definition 2.1.13, a_b = (x + c)_(y + d). By Lemma 2.2.07 and Theorem 2.2.22, (0 ≤ x) ^ (0 ≤ y) and four cases arise:

Case 1. Suppose (0 < x) ^ (0 < y), then, by the definition of "<" (strict order) on N (reference [CD], Chapter 4, page 113), (c < a) ^ (d < b).

Case 2. Suppose (0 < x) ^ (0 = y), then, by the definition of "<" (strict order) on N (reference [CD], Chapter 4, page 113), (c < a) ^ (d = b).

Case 3. Suppose (0 = x) ^ (0 < y), then, by the definition of "<" (strict order) on N (reference [CD], Chapter 4, page 113), (c = a) ^ (d < b).

Case 4. Suppose (0 = x) ^ (0 = y), then (c = a) ^ (d = b).

In all four cases, (c ≤ a) ^ (d ≤ b).

Therefore, (a_b - c_d ϵ N_Q) → [(c ≤ a) ^ (d ≤ b)].

Suppose (c ≤ a) ^ (d ≤ b), then four cases arise:

Case 1. Suppose (c < a) ^ (d < b), then, by Lemma 2.2.07 and the definition of "<" (strict order) on N (reference [CD], Chapter 4, page 113), there exists x,y ϵ N such that [a = (x + c)] ^ [b = (y + d)] and, by Definition 2.1.13 and 2.1.15, a_b - c_d ϵ N_Q.

Case 2. Suppose (c < a) ^ (d = b), then, by Lemma 2.2.07 and the definition of "<" (strict order) on N (reference [CD], Chapter 4, page 113), there exists x,y ϵ N, y = 0, such that [a = (x + c)] ^ [b = (0 + d)] and, by Definition 2.1.13 and 2.1.15, a_b - c_d ϵ N_Q.

Case 3. Suppose (c = a) ^ (d < b), then, by Lemma 2.2.07 and the definition of "<" (strict order) on N (reference [CD], Chapter 4, page 113), there exists x,y ϵ N, x = 0, such that [a = (0 + c)] ^ [b = (y + d)] and, by Definition 2.1.13 and 2.1.15, a_b - c_d ϵ N_Q.

Case 4. Suppose (c = a) ^ (d = b), then a_b = c_d and, by Definition 2.1.10, 2.1.13, and 2.1.15 and Theorem 2.2.05, a_b - c_d = 0_0 ϵ N_Q.

In all four cases, a_b - c_d ϵ N_Q.

Therefore, [(c ≤ a) ^ (d ≤ b)] → (a_b - c_d ϵ N_Q).

Therefore, (a_b - c_d ϵ N_Q) iff [(c ≤ a) ^ (d ≤ b)], as desired. □

Theorem 2.2.39. *N_Q is not closed under the operation "÷" (division) of Definition 2.1.16.*

Proof. Let a_b, c_d ϵ N_Q – {0_0} be arbitrary and suppose c_d|a_b, then, by Definition 2.1.16, there exists x_y ϵ N_Q such that a_b = x_y * c_d = (x * c)_(y * c + x * d + y * d). Then [a = (x * c)] ^ [b = (y * c + x * d + y * d)], hence, (c = a ÷ x) ^ (d = {[b - (y * c)] ÷ (x + y)}). By Lemma 2.2.07, (a ÷ x), {[b - (y * c)] ÷ (x + y)} ϵ N, which is not closed under subtraction nor division (reference [CD], Chapter 4, pages 116 thru 118), hence, N_Q is not closed under division, as desired. □

Theorem 2.2.40. *For all a_b, c_d ϵ N_Q, c_d ≠ 0_0, (a_b ÷ c_d ϵ N_Q) → [(a_b ÷ c_d) * c_d = a_b = (a_b * c_d) ÷ c_d].*

Proof. Suppose a_b ÷ c_d ϵ N_Q, then, by Definition 2.1.16, there exists x_y ϵ N_Q such that (a_b ÷ c_d = x_y) ^ (a_b = x_y * c_d) and, by substitution, a_b = (a_b ÷ c_d) * c_d. Furthermore, suppose, for contradiction, that a_b ≠ (a_b * c_d) ÷ c_d, then, by Definition 2.1.16, a_b * c_d ≠ a_b * c_d, a contradiction. Therefore, (a_b ÷ c_d) * c_d = a_b = (a_b * c_d) ÷ c_d, as desired. □

Theorem 2.2.41. *For all a_b, c_d, m_q ϵ N_Q, m_q ≠ 0_0, [(a_b ÷ m_q ϵ N_Q) ^ (c_d ÷ m_q ϵ N_Q)] → [(a_b + c_d) ÷ m_q = (a_b ÷ m_q) + (c_d ÷ m_q)].*

Proof. Suppose (a_b ÷ m_q ϵ N_Q) ^ (c_d ÷ m_q ϵ N_Q), then, by Definition 2.2.16, there exists e_f, g_h ϵ N_Q such that (a_b ÷ m_q = e_f) ^ (a_b = e_f * m_q) ^ (c_d ÷ m_q = g_h) ^ (c_d = g_h * m_q), and, by Theorem 2.2.16 and 2.2.40 and Definition 2.1.16 again:

(a_b + c_d) ÷ m_q = (e_f * m_q + g_h * m_q) ÷ m_q;

= [(e_f + g_h) * m_q] ÷ m_q;

= e_f + g_h;

= (a_b ÷ m_q) + (c_d ÷ m_q).

Therefore, division is right distributive over addition, as desired. □

Theorem 2.2.42. *For all k_p, m_q, n_r ϵ N_Q, m_q, n_r ≠ 0_0, {[(k_p ÷ m_q) ϵ N_Q] ^ [(k_p ÷ n_r) ϵ N_Q] ^ [[k_p ÷ (m_q + n_r)] ϵ N_Q]} → [k_p ÷ (m_q + n_r) ≠ (k_p ÷ m_q) + (k_p ÷ n_r)].*

Proof. Suppose [(k_p ÷ m_q) ϵ N_Q] ^ [(k_p ÷ n_r) ϵ N_Q] ^ {[k_p ÷ (m_q + n_r)] ϵ N_Q} and suppose, for contradiction, that k_p ÷ (m_q + n_r) = (k_p ÷ m_q) + (k_p ÷ n_r). Then, by Definition 2.1.16 and Theorem 2.2.15, 2.2.16, and 2.2.40:

k_p = [(k_p ÷ m_q) + (k_p ÷ n_r)] * (m_q + n_r);

= [(k_p ÷ m_q) * (m_q + n_r)] + [(k_p ÷ n_r) * (m_q + n_r)];

= [(k_p ÷ m_q) * m_q] + [(k_p ÷ m_q) * n_r] + [(k_p ÷ n_r) * m_q] + [(k_p ÷ n_r) * n_r];

= k_p + [(k_p ÷ m_q) * n_r] + [(k_p ÷ n_r) * m_q] + k_p, a contradiction.

Therefore, division is not left distributive over addition, as desired. □

Theorem 2.2.43. *For all k_p, m_q, n_r ϵ N_Q, m_q ≠ 0_0, {[(k_p ÷ m_q) ϵ N_Q] ^ [(n_r ÷ m_q) ϵ N_Q] ^ [(k_p - n_r) ϵ N_Q]} → [(k_p - n_r) ÷ m_q = (k_p ÷ m_q) - (n_r ÷ m_q)].*

Proof. Suppose [(k_p ÷ m_q) ϵ N_Q] ^ [(n_r ÷ m_q) ϵ N_Q] ^ [(*k_p - n_r*) ϵ N_Q], then, by Definition 2.1.16, there exists a_b, c_d ϵ N_Q such that (k_p ÷ m_q = a_b) ^ (k_p = a_b * m_q) ^ (n_r ÷ m_q = c_d) ^ (n_r = c_d * m_q) and, by Theorem 2.2.40 and 2.2.41 and Definition 2.1.16 again:

(*k_p - n_r*) ÷ m_q = (a_b * m_q - c_d * m_q) ÷ m_q;

= [(a_b - c_d) * m_q] ÷ m_q;

= a_b - c_d;

= (k_p ÷ m_q) - (n_r ÷ m_q).

Therefore, division is right distributive over subtraction, as desired. □

Lemma 2.2.44. *For all a_b, c_d, m_q, n_r ϵ N_Q, [(a_b * m_q = c_d * n_r) ^ (n_r < m_q)] → (a_b < c_d).*

Proof. Suppose (a_b * m_q = c_d * n_r) ^ (n_r < m_q), then, by Definition 2.1.12 and 2.1.14, {(n < m) ᵛ [(n = m) ^ (r < q)]} ^ (a *m = c *n) ^ [(b * m + a * q + b * q) = (d * n + c * r + d * r)] and two cases arise:

Case 1. Suppose n < m, then, by Lemma 2.2.07 and the definition of "<" (strict order) on N (reference [CD], Chapter 4, page 113), there exists x ϵ N such that n + x = m. Then, by substitution and additive closure on N (reference [CD], Chapter 4, page 108), a * n + a * x = c * n, hence, a * n < c * n and a < c (reference [CD], Chapter 4, page 115). But then, by Definition 2.1.12, a_b < c_d.

Case 2. Suppose (n = m) ^ (r < q), then, by Lemma 2.2.07 and the definition of "<" (strict order) on N (reference [CD], Chapter 4, page 113), there exists x ϵ N such that r + x = q. Clearly a = c and, by substitution and commutativity and distributivity on N (reference [CD], Chapter 4, page 113):

d *n + c *r + d * r = d * n + d * r + c * r;

= d * (m + r) + a * r;

= b * m + [a * (r + x) + b * (r + x)];

= b * m + b * (r + x) + a * (r + x);

= b * m + b * r + (b * x + a * r) + a * x;

= b * (m + r) + a * r + (b * x + a * x);

= b * (m + r) + a * r + (b + a) * x.

Hence, (a = c) ^ (b < d) (reference [CD], Chapter 4, pages 113 – 115) and, by Definition 2.1.12, a_b < c_d.

In both case, a_b < c_d.

Therefore, [(a_b * m_q = c_d * n_r) ^ (n_r < m_q)] → (a_b < c_d), as desired. □

Theorem 2.2.44. *For all k_p, m_q, n_r ϵ N_Q, (m_q, n_r ≠ 0_0) ^ (m_q ≠ n_r), {[(k_p ÷ m_q) ϵ N_Q] ^ [(k_p ÷ n_r) ϵ N_Q] ^ [(m_q - n_r) ϵ N_Q] ^ [k_p ÷ (m_q - n_r) ϵ N_Q]} → [k_p ÷ (m_q - n_r) ≠ (k_p ÷ m_q) - (k_p ÷ n_r)].*

Proof. Suppose [(k_p ÷ m_q) ϵ N_Q] ^ [(k_p ÷ n_r) ϵ N_Q] ^ [(m_q - n_r) ϵ N_Q] ^ [k_p ÷ (m_q - n_r) ϵ N_Q], then, by Definition 2.1.16, there exists a_b, c_d, e_f ϵ N_Q such that (k_p ÷ m_q = a_b) ^ (k_p = a_b * m_q) ^ (k_p ÷ n_r = c_d) ^ (k_p = c_d * n_r) ^ [k_p ÷ (m_q - n_r) = e_f] ^ [k_p = e_f * (m_q - n_r)]. Suppose, for contradiction, that e_f = a_b - c_d, then, since [(m_q - n_r) ϵ N_Q] ^ [k_p ÷ (m_q - n_r) ϵ N_Q], by Definition 2.1.16 and Theorem 2.2.38, (n < m) ^ (r < q) and, by Lemma 2.2.24, a_b < c_d. But then, by Definition 2.1.12, (a < c) ˅ [(a = c) ^ (b <d)], contradicting Theorem 2.2.38 in both cases. Therefore, k_p ÷ (m_q - n_r) ≠ (k_p ÷ m_q) - (k_p ÷ n_r) and division is not left distributive over subtraction, as desired. □

Theorem 2.2.45. *For all a_b ϵ N_Q, a_b ÷ 1_0 = a_b.*

Proof. This is an immediate consequence of Definition 2.1.16 and Theorem 2.2.18, therefore, 1_0 is a right identity for division, as desired. □

Theorem 2.2.46. *For all a_b ϵ N_Q, (a_b ≠ 0_0) ^ (a_b ≠ 1_0), 1_0 ÷ a_b ≠ a_b.*

Proof. Suppose, for contradiction, that *1_0 ÷ a_b = a_b,* then, by Definition 2.1.16 and 2.1.14, 1_0 = a_b * a_b = (a * a)_(2 * a * b + b * b), hence, a = 1 and b = 0 (reference [CD], Chapter 3, page 52, and Chapter 5, page 163), a contradiction. Therefore, 1_0 is not a left identity for division, as desired. □

Theorem 2.2.47. *For all a_b ϵ N_Q, a_b ≠ 0_0, a_b ÷ a_b = 1_0.*

Proof. This is an immediate consequence of Definition 2.1.16 and Theorem 2.2.12 and 2.2.18; therefore, each q-natural number is its own inverse under division, as desired. □

Theorem 2.2.48. *For all a_b ϵ N_Q, b ≠ 0, a_b has a q-prime factor, p_0, iff "a" and "b" share the prime factor p.*

Proof. Suppose a_b has a q-prime factor, then, by Definition 2.1.14, 2.1.16 and 2.1.17, there is some m_q, p_0 ϵ N_Q, p prime, such that a_b = m_q * p_0 = (m * p)_(q * p + m * 0 + q * 0) = (m * p)_(q * p) and p is a factor of both a and b.

Suppose "a" and "b" share a prime factor p, then there exist m, q ϵ N such that (m * p = a) ^ (q * p = b). But then, by Lemma 2.2.07, a_b = (m * p)_(q * p) = (m * p)_(q * p + m * 0 + q * 0), hence, by Definition 2.1.16, a_b ÷ p_0 = m_q and a_b has a q-prime factor.

Therefore, a_b has a q-prime factor iff "a" and "b" share a prime factor, as desired. □

Theorem 2.2.49. *For all n ϵ N, a_b ϵ N_Q, (a_b)^n is defined.*

Proof. We proceed by induction on n. Let P(x) be the property, "(a_b)^x is defined," then:

P(0). By Definition 2.1.18, (a_b)^0 = 1_0.

Suppose P(n) is true, then (a_b)^n is defined and:

P(n + 1). By Definition 2.1.14 and 2.1.18, (a_b)$^{(n+1)}$ = (a_b)^n * a_b and, by Theorem 2.2.10, (a_b)$^{(n+1)}$ is defined.

Therefore, P(n) → P(n + 1) and, by the Principle of Induction (reference [HJ], Chapter 3, page 42), for all n ϵ N, a_b ϵ N_Q, (a_b)^n is defined, as desired. □

Theorem 2.2.50. *For all n ϵ N, (1_0)^n = 1_0.*

Proof. We proceed by induction on n. Let P(x) be the property, "(1_0)^x = 1_0," then:

P(0). By Definition 2.1.18, $(1\_0)^0 = 1\_0$.

Suppose P(n) is true, then $(1\_0)^n = 1\_0$ and:

P(n + 1). By Definition 2.1.18 and Theorem 2.2.18, $(1\_0)^{(n+1)} = (1\_0)^n * 1\_0 = 1\_0 * 1\_0 = 1\_0$.

Therefore, P(n) → P(n + 1) and, by the Principle of Induction (reference [HJ], Chapter 3, page 42), for all n ϵ N, $(1\_0)^n = 1\_0$, as desired. □

Theorem 2.2.51. *For all n ϵ N, a_b, c_d ϵ* N_Q*,* $(a\_b * c\_d)^n = (a\_b)^n * (c\_d)^n$*.*

Proof. We proceed by induction on n. Let P(x) be the property, "$(a\_b * c\_d)^x = (a\_b)^x * (c\_d)^x$," then:

P(0). By Definition 2.1.18 and Theorem 2.2.18, $(a\_b * c\_d)^0 = 1\_0 = 1\_0 * 1\_0 = (a\_b)^0 * (c\_d)^0$.

Suppose P(n) is true, then $(a\_b * c\_d)^n = (a\_b)^n * (c\_d)^n$ and:

P(n + 1). By Definition 2.1.18 and Theorem 2.2.12 and 2.2.14:

$$\begin{aligned}(a\_b * c\_d)^{(n+1)} &= (a\_b * c\_d)^n * (a\_b * c\_d);\\ &= (a\_b)^n * (c\_d)^n * (a\_b * c\_d);\\ &= (a\_b)^n * [(c\_d)^n * a\_b] * c\_d;\\ &= (a\_b)^n * [a\_b * (c\_d)^n] * c\_d;\\ &= [(a\_b)^n * a\_b] * [(c\_d)^n * c\_d];\\ &= (a\_b)^{(n+1)} * (c\_d)^{(n+1)}.\end{aligned}$$

Therefore, P(n) → P(n + 1) and, by the Principle of Induction (reference [HJ], Chapter 3, page 42), for all n ϵ N, a_b, c_d ϵ N_Q, $(a\_b * c\_d)^n = (a\_b)^n * (c\_d)^n$, as desired. □

Theorem 2.2.52. *For all m, n ϵ N, a_b ϵ* N_Q*,* $(a\_b)^m * (a\_b)^n = (a\_b)^{(m+n)}$*.*

Proof. We proceed by induction on n. Let P(x) be the property, "$(a\_b)^m * (a\_b)^x = (a\_b)^{(m+x)}$," then:

P(0). By Definition 2.1.18 and Theorem 2.2.18, $(a\_b)^m * (a\_b)^0 = (a\_b)^m * 1\_0 = (a\_b)^m = (a\_b)^{(m+0)}$.

Suppose P(n) is true, then $(a\_b)^m * (a\_b)^n = (a\_b)^{(m+n)}$ and:

P(n + 1). By Definition 2.1.18 and Theorem 2.2.14:

$$\begin{aligned}(a\_b)^m * (a\_b)^{(n+1)} &= (a\_b)^m * [(a\_b)^n * a\_b];\\ &= [(a\_b)^m * (a\_b)^n] * a\_b;\\ &= (a\_b)^{(m+n)} * a\_b;\\ &= (a\_b)^{[(m+n)+1]};\\ &= (a\_b)^{[m+(n+1)]}.\end{aligned}$$

Therefore, P(n) → P(n + 1) and, by the Principle of Induction (reference [HJ], Chapter 3, page 42), for all m, n ϵ N, a_b ϵ N_Q, $(a\_b)^m * (a\_b)^n = (a\_b)^{(m+n)}$, as desired. □

Theorem 2.2.53. *For all m, n ϵ N, a_b ϵ* N_Q*,* $[(a\_b)^m]^n = (a\_b)^{(m * n)}$*.*

Proof. We proceed by induction on n. Let P(x) be the property, "$[(a\_b)^m]^x = (a\_b)^{(m * x)}$," then:

P(0). By Definition 2.1.18, $[(a\_b)^m]^0 = 1\_0 = (a\_b)^0 = (a\_b)^{(m * 0)}$.

Suppose P(n) is true, then $[(a\_b)^m]^n = (a\_b)^{(m * n)}$ and:

P(n + 1). By Definition 2.1.18 and Theorem 2.2.52:

$[(a\_b)^m]^{(n + 1)} = [(a\_b)^m]^n * (a\_b)^m$;

$= (a\_b)^{(m * n)} * (a\_b)^m$;

$= (a\_b)^{[(m * n) + m]}$;

$= (a\_b)^{[m * (n + 1)]}$.

Therefore, P(n) → P(n + 1) and, by the Principle of Induction (reference [HJ], Chapter 3, page 42), for all m, n ϵ N, a_b ϵ N_Q, $[(a\_b)^m]^n = (a\_b)^{(m * n)}$, as desired. □

Theorem 2.2.54. *N_Q is countable.*

Proof. By Theorem 2.2.26 (reference [CD], Chapter 13, page 472), as desired. □

Theorem 2.2.55. *N_Q has no greatest element.*

Proof. By Definition 2.1.10 and Theorem 2.2.05, as desired. □

3. Q-Integers. We develop the q-integers as equivalence classes of ordered pairs of q-naturals, where, for all (a_b, c_d) ϵ N_Q x N_Q, (a_b, c_d) is to be considered equivalent to a_b - c_d. Here, - (a_b, c_d) is as defined by Definition 2.1.15 but we extend subtraction to allow a_b < 0_0; in doing so, we also extend the q-components into the negative, since, by Definition 2.1.13 and 2.1.16, (0_0 - a_b = x_y) iff [x_y + a_b = (x + a)_(y + b) = 0_0], hence, (x = 0 - a) ^ (y = 0 - b).

3.1. Definitions. We define our mathematical entities using standard terminology.

Definition 3.1.01. Let Z_Q’ = N_Q x N_Q = { (a_b, c_d) | a_b, c_d ϵ N_Q}.

Definition 3.1.02. The relation E on Z_Q’ is defined by:

for all (k_p, m_q), (n_r, o_s) ϵ N_Q, [(k_p, m_q) E (n_r, o_s)] iff [(k_p + o_s) = (m_q + n_r)].

Definition 3.1.03. The operation “+” (addition) on Z_Q’ is defined by:

for all (k_p, m_q), (n_r, o_s) ϵ N_Q, + [(k_p, m_q), (n_r, o_s)] = [(k_p + n_r), (m_q + o_s)].

Definition 3.1.04. The operation “*” (multiplication) on Z_Q’ is defined by:

for all (k_p, m_q), (n_r, o_s) ϵ N_Q,* [(k_p, m_q), (n_r, o_s)] = [(k_p * n_r + m_q * o_s), (k_p * o_s + m_q * n_r)]

Definition 3.1.05. Let (a_b, c_d) ϵ Z_Q’ be arbitrary, then the equivalence class of (a_b, c_d) modulo E, $[(a\_b, c\_d)]_E$, which is subject to Definition 3.1.03 and 3.1.04, will be called q-integers and the set of all such q-integers will be designated Z_Q.

Definition 3.1.06. Let $[(k\_p, m\_q)]_E$, $[(n\_r, o\_s)]_E$ ϵ Z_Q be arbitrary, then:

$+\{[(k\_p, m\_q)]_E, [(n\_r, o\_s)]_E\} = [\{(k\_p + n\_r), (m\_q + o\_s)\}]_E$; and, *{[(k_p, m_q)]$_E$, [(n_r, o_s)]$_E$} = [(k_p * n_r + m_q * o_s), (k_p * o_s + m_q * n_r)]$_E$.

Definition 3.1.07. The "zero" q-integer, 0_0, is the equivalence class [(a_b, a_b)]$_E$ ϵ Z_Q.

Definition 3.1.08. Let [(a_b, c_d)]$_E$ ϵ Z_Q be arbitrary, then [(a_b, c_d)]$_E$ will be called "negative" if a_b < c_d and [(a_b, c_d)]$_E$ will be called "positive" if a_b > c.d; the set of all positive q-integers will be designated Z_Q^+ and the set of all negative q-integers by Z_Q^-.

Definition 3.1.09. Positive q-integers are of the form, [(k_p + m_q), k_p}]$_E$, and will be signified by m_q.

Definition 3.1.10. Let [(a_b, c_d)]$_E$ ϵ Z_Q be arbitrary, then the "additive inverse" of [(a_b, c_d)]$_E$ is signified by - [(a_b, c_d)]$_E$ = [(c_d, a_b)]$_E$.

Definition 3.1.11. The operation "-" (subtraction) on Z_Q is defined by:

> for all [(k_p, m_q)]$_E$, [(n_r, o_s)]$_E$ ϵ Z_Q, (- {[(k_p, m_q)]$_E$, [(n_r, o_s)]$_E$} = [(a_b, c_d)]$_E$) iff ([(k_p, m_q)]$_E$ = + {[(n_r, o_s)]$_E$, [(a_b, c_d)]$_E$}).

Definition 3.1.12. Let [(k_p, m_q)]$_E$, [(n_r, o_s)]$_E$ ϵ Z_Q be arbitrary, then the operation "÷" (division) on Z_Q is defined by:

> for all [(a_b, c_d)]$_E$, [(k_p, m_q)]$_E$, [(n_r, o_s)]$_E$ ϵ Z_Q, ÷ {[(k_p, m_q]$_E$, [(n_r, o_s)]$_E$} = [(a_b, c_d)]$_E$ iff [(k_p, m_q)]$_E$ = *{[(n_r, o_s)]$_E$, [(a_b, c_d)]$_E$}, where [(n.r, o.s)]$_E$ ≠ 0_0.

Definition 3.1.13. The relation "<" (stict order) on Z_Q is defined by:

> for all [(k_p, m_q)]$_E$, [(n_r, o_s)]$_E$ ϵ Z_Q, < {[(k_p, m_q)]$_E$, [(n_r, o_s)]$_E$} iff there exists [(a_b, c_d)]$_E$ ϵ Z_Q^+ such that +{{[(k_p, m_q)]$_E$, [(a_b, c_d)]$_E$} = [(n_r, o_s)]$_E$.

Definition 3.1.14. The operation "| |" (absolute value) on Z_Q is defined by:

> for all [(k_p, m_q)]$_E$, [(n_r, o_s)]$_E$ ϵ Z_Q, |[(k_p, m_q)]$_E$| = [(k_p, m_q)]$_E$, if 0_0 ≤ [(k_p, m_q)]$_E$;
>
> = - [(k_p, m_q)]$_E$ = [(m_q, k_p)]$_E$, otherwise.

Definition 3.1.15. For all a_b ϵ Z_Q, k ϵ N, [(a_b)^0 = 1_0] ^ [(a_b)^1 = a_b] ^ [(a_b)$^{k+1}$ = (a_b)^k * a_b].

3.2. Arguments. We demonstrate our arguments using the standard methods and terminology of mathematical logic and ZFC/AFA or generalizations thereof. Specific to the current work, we generalize the Principle of Induction to the Principle of Q-Induction and we utilize results from reference [HJ] and [CD].

Theorem 3.2.01. *The set Z_Q' of Definition 3.1.01 exists.*

Proof. By Theorem 2.2.05, N_Q exists, hence, by the Axiom of Power Set, the definition of ordered pair, and the definition of Cartesian product, N_Q x N_Q = Z_Q' exists, as desired. □

Theorem 3.2.02. *The relation E on Z_Q, from Definition 3.1.02, is an equivalence relation.*

Proof. The proof is in three parts:

1) *Reflexivity.* Let (a_b, c_d) ϵ Z_Q' be arbitrary, then, by Theorem 2.2.11, a_b + c_d = c_d + a_b, hence, by Definition 3.1.02, (a_b, c_d) E (a_b, c_d).
2) *Symmetry.* Let (k_p, m_q), (n_r, o_s) ϵ Z_Q' be arbitrary but such that (k_p, m_q) E (n_r, o_s), then, by Definition 3.1.02, k_p + o_s = m_q + n_r and, by Theorem 2.2.11, n_r + m_q = o_s + k_p, hence, by Definition 3.1.02 again, (n_r, o_s) E (k_p, m_q).

3) *Transitivity.* Let (k_p, m_q), (n_r, o_s), (v_t, w_u) ϵ Z_Q’ be arbitrary but such that [(k_p, m_q) E (n_r, o_s)] ^ [(n_r, o_s) E (v_t, w_u)], then, by Definition 3.1.02, (k_p + o_s = m_q + n_r) ^ (n_r + w_u = o_s + v_t). By Theorem 2.2.33 and 2.2.38, k_p = (m_q + n_r) - o_s ϵ N_Q and w_u = (o_s + v_t) - n_r ϵ N_Q and, by Theorem 2.2.13 and 2.2.35:

k_p + w_u = [(m_q + n_r) - o_s] + [(o_s + v_t) - n_r];
= {[(m_q + n_r) - o_s] + o_s} + v_t - n_r;
= (m_q + n_r) + v_t - n_r;
= m_q + (n_r + v_t) - n_r;
= m_q + (v_t + n_r) - n_r;
= m_q + v_t.

Hence, (k_p, m_q) E (v_t, w_u) and E is transitive.

Therefore, E is an equivalence relation on Z_Q’, as desired. □

Theorem 3.2.03. *For all k_p, m_q, n_r ϵ N_Q, (k_p, m_q) E [(k_p + n_r), (m_q + n_r)].*

Proof. By Theorem 2.2.11 and 2.2.13:

k_p + (m_q + n_r) = (k_p + m_q) + n_r;
= (m_q + k_p) + n_r;
= m_q + (k_p + n_r).

Therefore, by Definition 3.1.02, (k_p, m_q) E [(k_p + n_r), (m_q + n_r)], as desired. □

Theorem 3.2.04. *Addition on Z_Q', as defined by Definition 3.1.03, is well-defined relative to the relation E of Definition 3.2.02.*

Proof. Let (a_b, c_d), (e_f, g_h), (k_p, m_q), (n_r, o_s) ϵ Z_Q’ be arbitrary but such that [(a_b, c_d) E (e_f, g_h)] ^ [(k_p, m_q) E (n_r, o_s)], then, by Definition 3.1.02, [(a_b + g_h) = (c_d + e_f)] ^ [(k_p + o_s) = (m_q + n_r)]. Then, by Definition 3.1.03, {(a_b, c_d) + (k_p, m_q) = [(a_b + k_p), (c_d + m_q)]} ^ {(e_f, g_h) + (n_r, o_s) = [(e_f + n_r), (g_h + o_s)]} and, by Theorem 2.2.11 and 2.2.13:

(a_b + k_p) + (g_h + o_s) = [(a_b + k_p) + g_h] + o_s;
= [a_b + (k_p + g_h)] + o_s;
= [a_b + (g_h + k_p)] + o_s;
= [(a_b + g_h) + k_p] + o_s;
= (a_b + g_h) + (k_p + o_s);
= (c_d + e_f) + (m_q + n_r);
= [(c_d + e_f) + m_q] + n_r;
= [c_d + (e_f + m_q)] + n_r;
= [c_d + (m_q + e_f)] + n_r;
= [(c_d + m_q) + e_f] + n_r;
= (c_d + m_q) + (e_f + n_r).

Therefore, by Definition 3.1.02, addition is well-defined relative to E, as desired. □

Theorem 3.2.05. *Multiplication on Z_Q', as defined by Definition 3.1.04, is well-defined relative to the relation E of Definition 3.2.02.*

Proof. Let (a_b, c_d), (e_f, g_h), (k_p, m_q), (n_r, o_s) ϵ Z_Q' be arbitrary but such that [(a_b, c_d) E (e_f, g_h)] ^ [(k_p, m_q) E (n_r, o_s)], then, by Definition 3.1.02, [(a_b + g_h) = (c_d + e_f)] ^ [(k_p + o_s) = (m_q + n_r)]. Then, by Definition 3.2.04, {(a_b, c_d) * (k_p, m_q) = [(a_b * k_p + c_d * m_q), (a_b * m_q + c_d * k_p)]} ^ {(e_f, g_h) * (n_r, o_s) = [(e_f * n_r + g_h * o_s), (e_f * o_s + g_h * n_r)]} and, letting A = (a_b * k_p + c_d * m_q) + (e_f * o_s + g_h * n_r) and B = (g_h * k_p + e_f * m_q + e_f * k_p + g_h * m_q), by Definition 2.1.16 and Theorem 2.2.11, 2.2.13, 2.2.15, and 2.2.16:

A = (A + B) – B;

= {[(a_b * k_p + c_d * m_q) + (e_f * o_s + g_h * n_r)] + (g_h * k_p + e_f * m_q + e_f * k_p + g_h * m_q)} - B;

= {[a_b * k_p + (c_d * m_q + e_f * o_s + g_h * n_r) + g_h * k_p] + (e_f * m_q + e_f * k_p + g_h * m_q)} - B;

= {[a_b * k_p + g_h * k_p + (c_d * m_q + e_f * o_s + g_h * n_r)] + (e_f * m_q + e_f * k_p + g_h * m_q)} - B;

= {[(a_b * k_p + g_h * k_p) + c_d * m_q + (e_f * o_s + g_h * n_r) + e_f * m_q] + (e_f * k_p + g_h * m_q)} - B;

= {[(a_b * k_p + g_h * k_p) + c_d * m_q + e_f * m_q + (e_f * o_s + g_h * n_r)] + (e_f * k_p + g_h * m_q)} - B;

= {(a_b * k_p + g_h * k_p) + (c_d * m_q + e_f * m_q) + (e_f * o_s + (g_h * n_r + e_f * k_p) + g_h * m_q)} - B;

= {(a_b * k_p + g_h * k_p) + (c_d * m_q + e_f * m_q) + (e_f * o_s + (e_f * k_p + g_h * n_r) + g_h * m_q)} - B;

= {(a_b * k_p + g_h * k_p) + (c_d * m_q + e_f * m_q) + (e_f * o_s + e_f * k_p) + (g_h * n_r + g_h * m_q)} - B;

= {(a_b * k_p + g_h * k_p) + (c_d * m_q + e_f * m_q) + (e_f * k_p + e_f * o_s) + (g_h * m_q + g_h * n_r)} - B;

= {(a_b + g_h) * k_p + (c_d + e_f) * m_q + e_f * (k_p + o_s) + g_h * (m_q + n_r)} - B;

= {(c_d + e_f) * k_p + (a_b + g_h) * m_q + e_f *(m_q + n_r) + g_h * (k_p + o_s)} - B;

= {(c_d * k_p + e_f * k_p) + (a_b * m_q + g_h * m_q) + (e_f *m_q + e_f * n_r) + (g_h * k_p + g_h * o_s)} - B;

= {c_d * k_p + (e_f * k_p + a_b * m_q) + g_h * m_q + (e_f * n_r + e_f *m_q) + (g_h * o_s + g_h * k_p)} - B;

= {(c_d * k_p + a_b * m_q) + e_f * k_p + (g_h * m_q + e_f * n_r) + (e_f *m_q + g_h * o_s) + g_h * k_p} - B;

= {(a_b * m_q + c_d * k_p) + (e_f * n_r + g_h * m_q) + e_f * k_p + (g_h * o_s + e_f *m_q) + g_h * k_p} - B;

= {(a_b * m_q + c_d * k_p + e_f * n_r) + (g_h * m_q + e_f * k_p) + g_h * o_s + (e_f *m_q + g_h * k_p)} - B;

= {(a_b * m_q + c_d * k_p + e_f * n_r + g_h * o_s) + (e_f * k_p + g_h * m_q) + (g_h * k_p + e_f *m_q)} - B;

= {(a_b * m_q + c_d * k_p + e_f * n_r + g_h * o_s) + (g_h * k_p + e_f *m_q) + (e_f * k_p + g_h * m_q)} - B;

= (a_b * m_q + c_d * k_p) + (e_f * n_r + g_h * o_s).

Therefore, by Definition 3.1.02, multiplication is well-defined relative to E, as desired. □

Theorem 3.2.06. *The set of all q-integers, Z_Q, is closed under the arithmetical operations "+" (addition) and "*" (multiplication).*

Proof. This is an immediate consequence of Definition 3.1.01, 3.1.05 and 3.1.06 and Theorem 2.2.10, as desired. □

Theorem 3.2.07. *For all $[(a\_b, c\_d)]_E$, $[(e\_f, g\_h)]_E \in Z_Q$, $[(a\_b, c\_d)]_E + [(e\_f, g\_h)]_E = [(e\_f, g\_h)]_E + [(a\_b, c\_d)]_E$.*

Proof. By Definition 3.1.06 and Theorem 2.2.11:

$$\begin{aligned}[(a\_b, c\_d)]_E + [(e\_f, g\_h)]_E &= [(a\_b + e\_f), (c\_d + g\_h)]_E; \\ &= [(e\_f + a\_b), (g\_h + c\_d)]_E; \\ &= [(e\_f, g\_h)]_E + [(a\_b, c\_d)]_E.\end{aligned}$$

Therefore, addition on Z_Q is commutative, as desired. □

Theorem 3.2.08. *For all $[(a\_b, c\_d)]_E$, $[(e\_f, g\_h)]_E \in Z_Q$, $[(a\_b, c\_d)]_E * [(e\_f, g\_h)]_E = [(e\_f, g\_h)]_E * [(a\_b, c\_d)]_E$.*

Proof. By Definition 3.1.06 and Theorem 2.2.11 and 2.2.12:

$$\begin{aligned}[(a\_b, c\_d)]_E * [(e\_f, g\_h)]_E &= [(a\_b * e\_f + c\_d * g\_h), (a\_b * g\_h + c\_d * e\_f)]_E; \\ &= [(e\_f * a\_b + g\_h * c\_d), (g\_h * a\_b + e\_f * c\_d)]_E; \\ &= [(e\_f * a\_b + g\_h * c\_d), (e\_f * c\_d + g\_h * a\_b)]_E; \\ &= [(e\_f, g\_h)]_E * [(a\_b, c\_d)]_E.\end{aligned}$$

Therefore, multiplication on Z_Q is commutative, as desired. □

Theorem 3.2.09. *For all $[(a\_b, c\_d)]_E$, $[(e\_f, g\_h)]_E$, $[(k\_l, m\_n)]_E \in Z_Q$, $\{[(a\_b, c\_d)]_E + [(e\_f, g\_h)]_E\} + [(k\_l, m\_n)]_E = [(a\_b, c\_d)]_E + \{[(e\_f, g\_h)]_E + [(k\_l, m\_n)]_E\}$.*

Proof. By Definition 3.1.06 and Theorem 2.2.11:

$$\begin{aligned}\{[(a\_b, c\_d)]_E + [(e\_f, g\_h)]_E\} + [(k\_l, m\_n)]_E &= [\{(a\_b + e\_f), (c\_d + g\_h)\}]_E + [(k\_l, m\_n)]_E; \\ &= [\{[(a\_b + e\_f) + k\_l], [(c\_d + g\_h) + m\_n]\}]_E; \\ &= [\{[a\_b + (e\_f + k\_l)], [c\_d + (g\_h + m\_n)]\}]_E; \\ &= [(a\_b, c\_d)]_E + [\{(e\_f + k\_l), (g\_h + m\_n)\}]_E; \\ &= [(a\_b, c\_d)]_E + \{[(e\_f, g\_h)]_E + [(k\_l, m\_n)]_E\}.\end{aligned}$$

Therefore, addition on Z_Q is associative, as desired. □

Theorem 3.2.10. *For all $[(a\_b, c\_d)]_E$, $[(e\_f, g\_h)]_E$, $[(k\_l, m\_n)]_E \in Z_Q$, $\{[(a\_b, c\_d)]_E * [(e\_f, g\_h)]_E\} * [(k\_l, m\_n)]_E = [(a\_b, c\_d)]_E * \{[(e\_f, g\_h)]_E * [(k\_l, m\_n)]_E\}$.*

Proof. Letting $A = \{[(a\_b, c\_d)]_E * [(e\_f, g\_h)]_E\} * [(k\_l, m\_n)]_E$, by Definition 3.1.06 and Theorem 2.2.11, 2.2.12, 2.2.15, and 2.2.16:

$A = [(a\_b * e\_f + c\_d * g\_h), (a\_b * g\_h + c\_d * e\_f)]_E * [(k\_l, m\_n)]_E;$

$= [\{[(a\_b * e\_f + c\_d * g\_h) * k\_l + (a\_b * g\_h + c\_d * e\_f) * m\_n], [(a\_b * e\_f + c\_d * g\_h) * m\_n + (a\_b * g\_h + c\_d * e\_f) * k\_l]\}]_E;$

$= [\{(a\_b * e\_f * k\_l + [(c\_d * g\_h * k\_l) + (a\_b * g\_h * m\_n + c\_d * e\_f * m\_n)]), [(a\_b * e\_f * m\_n + c\_d * g\_h * m\_n) + (a\_b * g\_h * k\_l + c\_d * e\_f * k\_l)]\}]_E;$

$= [\{(a\_b * e\_f * k\_l + a\_b * g\_h * m\_n + c\_d * e\_f * m\_n + c\_d * g\_h * k\_l), [c\_d * g\_h * m\_n + (a\_b * e\_f * m\_n + c\_d * e\_f * k\_l) + a\_b * g\_h * k\_l)]\}]_E;$

= [{(a_b * e_f * k_l + a_b * g_h * m_n + c_d * e_f * m_n + c_d * g_h * k_l), (c_d * e_f * k_l + c_d * g_h * m_n + a_b * e_f * m_n + a_b * g_h * k_l)}]$_E$;

= [{[a_b * (e_f * k_l + g_h * m_n) + c_d * (e_f * m_n + g_h * k_l)], [c_d * (e_f * k_l + g_h * m_n) + a_b * (e_f * m_n + g_h * k_l)]}]$_E$;

= [(a_b, c_d)]$_E$ * [{(e_f * k_l + g_h * m_n), (e_f * m_n + g_h * k_l)}]$_E$;

= [(a_b, c_d)]$_E$ * {[(e_f, g_h)]$_E$ * [(k_l, m_n)]$_E$}.

Therefore, multiplication on Z$_Q$ is associative, as desired. □

Theorem 3.2.11. *For all [(a_b, c_d)]$_E$, [(e_f, g_h)]$_E$, [(k_l, m_n)]$_E$ ϵ Z$_Q$, [(a_b, c_d)]$_E$ * {[(e_f, g_h)]$_E$} + [(k_l, m_n)]$_E$} = [(a_b, c_d)]$_E$ * [(e_f, g_h)]$_E$ + [(a_b, c_d)]$_E$ * [(k_l, m_n)]$_E$.*

Proof. Letting A = [(a_b, c_d)]$_E$ * {[(e_f, g_h)]$_E$} + [(k_l, m_n)]$_E$}, by Definition 3.1.06 and Theorem 2.2.11 and 2.2.15:

A = [(a_b, c_d)]$_E$ * [{(e_f + k_l), (g_h + m_n)}]$_E$;

= [{[a_b * (e_f + k_l) + c_d * (g_h + m_n)], [a_b * (g_h + m_n) + c_d * (e_f + k_l)]}]$_E$;

= [(a_b * e_f + a_b * k_l + c_d * g_h + c_d * m_n), (a_b * g_h + a_b * m_n + c_d * e_f + c_d * k_l)]$_E$;

= [{[(a_b * e_f + c_d * g_h) + (a_b * k_l + c_d * m_n)], [(a_b * g_h + c_d * e_f) + (a_b * m_n + c_d * k_l)]}]$_E$;

= [(a_b * e_f + c_d * g_h), (a_b * g_h + c_d * e_f)]$_E$ + [(a_b * k_l + c_d * m_n), (a_b * m_n + c_d * k_l)]$_E$;

= [(a_b, c_d)]$_E$ * [(e_f, g_h)]$_E$ + [(a_b, c_d)]$_E$ * [(k_l, m_n)]$_E$.

Therefore, multiplication is left distributive over addition on Z$_Q$, as desired. □

Theorem 3.2.12. *For all [(a_b, c_d)]$_E$, [(e_f, g_h)]$_E$, [(k_l, m_n)]$_E$ ϵ Z$_Q$, {[(e_f, g_h)]$_E$} + [(k_l, m_n)]$_E$} * [(a_b, c_d)]$_E$ = [(e_f, g_h)]$_E$ * [(a_b, c_d)]$_E$ + [(k_l, m_n)]$_E$ * [(a_b, c_d)]$_E$.*

Proof. Letting A = {[(e_f, g_h)]$_E$} + [(k_l, m_n)]$_E$} * [(a_b, c_d)]$_E$, by Definition 3.1.06 and Theorem 2.2.11 and 2.2.16:

A = [{(e_f + k_l), (g_h + m_n)}]$_E$ * [(a_b, c_d)]$_E$;

= [{[(e_f + k_l) * a_b + (g_h + m_n) * c_d], [(g_h + m_n) * a_b + (e_f + k_l) * c_d]}]$_E$;

= [{(e_f * a_b + (k_l * a_b + g_h * c_d) + m_n * c_d), (g_h * a_b + m_n * a_b + e_f * c_d + k_l * c_d)}]$_E$;

= [{(e_f * a_b + (g_h * c_d + k_l * a_b) + m_n * c_d), (e_f * c_d + (k_l * c_d + g_h * a_b) + m_n * a_b)}]$_E$;

= [{(e_f * a_b + g_h * c_d + k_l * a_b + m_n * c_d), (e_f * c_d + g_h * a_b + k_l * c_d + m_n * a_b}}]$_E$;

= [{[(e_f * a_b + g_h * c_d) + (k_l * a_b + m_n * c_d)], [(e_f * c_d + g_h * a_b) + (k_l * c_d + m_n * a_b)]}]$_E$;

= [{(e_f * a_b + g_h * c_d), (e_f * c_d + g_h * a_b)}]$_E$ + [{(k_l * a_b + m_n * c_d), (k_l * c_d + m_n * a_b)}]$_E$;

= [(e_f, g_h)]$_E$ * [(a_b, c_d)]$_E$ + [(k_l, m_n)]$_E$ * [(a_b, c_d)]$_E$.

Therefore, multiplication is right distributive over addition, as desired. □

Theorem 3.2.13. *For all [(c_d, e_f)]$_E$ ϵ Z$_Q$, [(c_d, e_f)]$_E$ + [(a_b, a_b)]$_E$ = [(c_d, e_f)]$_E$.*

Proof. This is an immediate consequence of Definition 3.1.01, 3.1.05, 3.1.06 and 3.1.07 and Theorem 3.2.03, hence, the zero q-integer, 0_0, is an additive identity for Z_Q, as desired. □

Corollary 3.2.14. *For all [(c_d, e_f)]$_E$, [(v_w, x_y)]$_E$ ϵ Z_Q, {[(c_d, e_f)]$_E$ + [(v_w, x_y)]$_E$ = [(c_d, e_f)]$_E$} → {[(v_w, x_y)]$_E$ = [(a_b, a_b)]$_E$}.*

Proof. Suppose [(c_d, e_f)]$_E$ + [(v_w, x_y)]$_E$ = [(c_d, e_f)]$_E$, then, by Definition 3.1.01, 3.1.05, and 3.1.06, (c_d, e_f) E {(c_d + v_w), (e_f + x_y)} and, by Definition 3.1.02, c_d + (e_f + x_y) = e_f + (c_d + v_w). But then, by Theorem 2.2.11 and 2.2.13, (c_d + e_f) + x_y = (c_d + e_f) + v_w and, by Corollary 2.2.20, x_y = v_w. Therefore, the additivie identity for Z_Q, is unique, as desired. □

Theorem 3.2.15. *For all [(c_d, e_f)]$_E$ ϵ Z_Q, [(c_d, e_f)]$_E$ * [(a_b, a_b)]$_E$ = [(a_b, a_b)]$_E$.*

Proof. By Definition 3.1.06:

[(c_d, e_f)]$_E$ * [(a_b, a_b)]$_E$ = [{(c_d * a_b + e_f * a_b), (c_d * a_b + e_f * a_b)}]$_E$;

= [(a_b, a_b)]$_E$.

Therefore, [(c_d, e_f)]$_E$ * [(a_b, a_b)]$_E$ = [(a_b, a_b)]$_E$, as desired. □

Theorem 3.2.16. *For all [(a_b, c_d)]$_E$ ϵ Z_Q, [(a_b, c_d)]$_E$ * [{(m_q + 1_0), m_q}]$_E$ = [(a_b, c_d)]$_E$.*

Proof. By Theorem 2.2.11 and 2.2.13, for all a_b, c_d, m_q ϵ N_Q:

(a_b * m_q + a_b + c_d * m_q) + c_d = (a_b * m_q + c_d * m_q + a_b) + c_d;

= a_b * m_q + c_d * m_q + (a_b + c_d);

= a_b * m_q + c_d * m_q + (c_d + a_b);

= (a_b * m_q + c_d * m_q + c_d) + a_b.

By Definition 3.1.06 and Theorem 2.2.15 and 2.2.18:

[(a_b, c_d)]$_E$ * [{(m_q + 1_0), m_q}]$_E$ = [{[a_b * (m_q + 1_0) + c_d * m_q], [a_b * m_q + c_d * (m_q + 1_0)]}]$_E$;

= [(a_b * m_q + a_b + c_d * m_q), (a_b * m_q + c_d * m_q + c_d)]$_E$.

Hence, by Definition 3.1.02, [(a_b * m_q + a_b + c_d * m_q), (a_b * m_q + c_d * m_q + c_d)]$_E$ = [(a_b, c_d)]$_E$.

Therefore, the unity q-integer, 1_0, is a multiplicative identity for Z_Q, as desired. □

Corollary 3.2.17. *For all [(a_b, c_d)]$_E$, [(e_f, g_h)]$_E$ ϵ Z_Q, {[(a_b, c_d)]$_E$ * [(e_f, g_h)]$_E$ = [(a_b, c_d)]$_E$} → {[(e_f, g_h)]$_E$ = [{(m_q + 1_0), m_q}]$_E$}.*

Proof. Suppose [(a_b, c_d)]$_E$ * [(e_f, g_h)]$_E$ = [(a_b, c_d)]$_E$ and suppose, for contradiction, that e_f ≠ (g_h + 1_0), then, by Definition 3.1.02 and 3.1.06, a_b * e_f + c_d * g_h + c_d = a_b * g_h + c_d * e_f + a_b and, by Theorem 2.2.11, 2.2.13, 2.2.15, and 2.2.18:

a_b * e_f + c_d * g_h + c_d ≠ a_b * (g_h + 1_0) + c_d * g_h + c_d;

≠ a_b * g_h + a_b + c_d * g_h + c_d;

≠ a_b * g_h + [a_b + (c_d * g_h + c_d)];

≠ a_b * g_h + [(c_d * g_h + c_d) + a_b];

≠ a_b * g_h + [c_d * (g_h + 1_0) + a_b];

$\neq$ a_b * g_h + c_d * e_f + a_b, a contradiction.

Therefore, the multiplicative identity for Z_Q, 1_0, is unique, as desired. □

Theorem 3.2.18. *For all $[(a\_b, c\_d)]_E \in Z_Q$, $[(a\_b, c\_d)]_E + [(c\_d, a\_b)]_E = 0\_0$.*

Proof. By Definition 3.1.06 and 3.1.07 and Theorem 2.2.11:

$[(a\_b, c\_d)]_E + [(c\_d, a\_b)]_E = [\{(a\_b + c\_d), (c\_d + a\_b)\}]_E;$

$= [\{(a\_b + c\_d), (a\_b + c\_d)\}]_E;$

= 0_0, as desired. □

Theorem 3.2.19. *For all $[(a\_b, c\_d)]_E \in Z_Q$, $[\{m\_q, (m\_q + 1\_0)\}]_E * [(a\_b, c\_d)]_E = [(c\_d, a\_b)]_E$.*

Proof. Letting $A = [\{m\_q, (m\_q + 1\_0)\}]_E * [(a\_b, c\_d)]_E$, by Definition 3.1.06 and Theorem 2.2.10, 2.2.11, 2.2.13, 2.2.16, and 3.2.03:

$A = [\{[m\_q * a\_b + (m\_q + 1\_0) * c\_d], [m\_q * c\_d + (m\_q + 1\_0) * a\_b]\}]_E;$

$= [\{(m\_q * a\_b + m\_q * c\_d + c\_d), (m\_q * c\_d + m\_q * a\_b + a\_b)\}]_E;$

$= [\{[(m\_q * a\_b + m\_q * c\_d) + c\_d], [(m\_q * a\_b + m\_q * c\_d) + a\_b]\}]_E;$

$= [\{[c\_d + (m\_q * a\_b + m\_q * c\_d)], [a\_b + (m\_q * a\_b + m\_q * c\_d)]\}]_E;$

$= [(c\_d, a\_b)]_E$, as desired. □

Theorem 3.2.20. *For all $(a\_b, c\_d) \in Z_Q{}'$, $\{(a\_b, c\_d) \in [(e\_f, g\_h)]_E\} \rightarrow \{[(a\_b < c\_d)$ iff $(e\_f < g\_h)] \wedge [(a\_b = c\_d)$ iff $(e\_f = g\_h)] \wedge [(a\_b > c\_d)$ iff $(e\_f > g\_h)]\}$.*

Proof. Suppose $(a\_b, c\_d) \in [(e\_f, g\_h)]_E$, then, by Definition 3.1.05, (a_b, c_d) E (e_f, g_h) and, by Definition 3.1.02, a_b + g_h = e_f + c_d and the proof is in three parts:

1) Suppose a_b < c_d, then, by Definition 2.1.12, $(a < c) \vee [(a = c) \wedge (b < d)]$ and two cases arise:

Case 1. Suppose a < c, then, by Lemma 2.2.07, there exists $x \in N$ such that a + x = c (reference [CD], Chapter 4, page 113) and x = c - a. Then, by Definition 2.1.13, a + g = e + c and g = e + (c - a) = e + x, hence, e < g and, by Definition 2.1.12, e_f < g_h.

Case 2. Suppose $(a = c) \wedge (b < d)$, then, by Lemma 2.2.07, there exists $x \in N$ such that b + x = d (reference [CD], Chapter 4, page 113) and x = d - b. Then, by Definition 2.1.13, $(a + g = e + c) \wedge (b + h = f + d)$, hence, $(g = e) \wedge [h = f + (d - b) = f + x]$. But then, $(e = g) \wedge (f < h)$ and, by Definition 2.1.12, e_f < g_h.

In both cases, e_f < g_h.

Therefore, $(a\_b < c\_d) \rightarrow (e\_f < g\_h)$.

Suppose e_f < g_h, then, by Definition 2.1.12, $(e < g) \vee [(e = g) \wedge (f < h)]$ and two cases arise:

Case 1. Suppose e < g, then, by Lemma 2.2.07, there exists $x \in N$ such that e + x = g (reference [CD], Chapter 4, page 113) and x = g - e. Then, by Definition 2.1.13, a + g = e + c and c = a + (g - e) = a + x, hence, a < c and, by Definition 2.1.12, a_b < c_d.

Case 2. Suppose $(e = g) \wedge (f < h)$, then, by Lemma 2.2.07, there exists $x \in N$ such that f + x = h (reference [CD], Chapter 4, page 113) and x = h - f. Then, by Definition 2.1.13, $(a + g = e + c) \wedge (b + h = f + d)$, hence, $(a = c) \wedge [d = b + (h - f) = b + x]$. But then, $(a = c) \wedge (b < d)$ and, by Definition 2.1.12, a_b < c_d.

In both cases, a_b < c_d.

Therefore, (e_f < g_h) → (a_b < c_d).

Therefore, (a_b < c_d) iff (e_f < g_h).

2) Suppose a_b = c_d, then a_b + g_h = a_b + e_f and, by Theorem 2.2.20, e_f = g_h, hence, (a_b = c_d) → (e_f = g_h). Suppose e_f = g_h, then a_b + e_f = c_d + e_f and, by Theorem 2.2.20, a_b = c_d, hence, (e_f = g_h) → (a_b = c_d). Therefore, (a_b = c_d) iff (e_f = g_h).
3) Suppose c_d < a_b, then, by Definition 2.1.12, (c < a) ᐯ [(c = a) ^ (d < b)] and two cases arise:

Case 1. Suppose c < a, then, by Lemma 2.2.07, there exists x ϵ N such that c + x = a (reference [CD], Chapter 4, page 113) and x = a - c. Then, by Definition 2.1.13, a + g = e + c and e = g + (a - c) = g + x, hence, g < e and, by Definition 2.1.12, g_h < e_f.

Case 2. Suppose (c = a) ^ (d < b), then, by Lemma 2.2.07, there exists x ϵ N such that d + x = b (reference [CD], Chapter 4, page 113) and x = b - d. Then, by Definition 2.1.13, (a + g = e + c) ^ (b + h = f + d), hence, (e = g) ^ [f = h + (b - d) = h + x]. But then, (g = e) ^ (h < f) and, by Definition 2.1.12, g_h < e_f.

In both cases, g_h < e_f.

Therefore, (c_d < a_b) → (g_h < e_f).

Suppose g_h < e_f, then, by Definition 2.1.12, (g < e) ᐯ [(g = e) ^ (h < f)] and two cases arise:

Case 1. Suppose g < e, then, by Lemma 2.2.07, there exists x ϵ N such that g + x = e (reference [CD], Chapter 4, page 113) and x = e - g. Then, by Definition 2.1.13, a + g = e + c and a = c + (e - g) = c + x, hence, c < a and, by Definition 2.1.12, c_d < a_b.

Case 2. Suppose (g = e) ^ (h < f), then, by Lemma 2.2.07, there exists x ϵ N such that h + x = f (reference [CD], Chapter 4, page 113) and x = f - h. Then, by Definition 2.1.13, (a + g = e + c) ^ (b + h = f + d), hence, (c = a) ^ [b = d + (f - h) = d + x]. But then, (c = a) ^ (d < b) and, by Definition 2.1.12, c_d < a_b.

In both cases, c_d < a_b.

Therefore, (g_h < e_f) → (c_d < a_b).

Therefore, (c_d < a_b) iff (g_h < e_f).

Therefore, {(a_b, c_d) ϵ [(e_f, g_h)]$_E$} → {[(a_b < c_d) iff (e_f < g_h)] ^ [(a_b = c_d) iff (e_f = g_h)] ^ [(a_b > c_d) iff (e_f > g_h)]}, as desired. □

Lemma 3.2.21. *Let K = {[(a_b, c_d)]$_E$ | ([(a_b, c_d)]$_E$ ϵ Z_Q^+) ^ (d ≤ b)}, then, for all [(a_b, c_d)]$_E$ ϵ K, there exists a unique x_y ϵ N_Q such that, for any m_q ϵ N_Q, [(a_b, c_d)]$_E$ = [{(m_q + x_y), m_q}]$_E$.*

Proof. By Definition 2.1.12 and 3.1.08, there are three cases to consider:

Case 1. Suppose (c < a) ^ (d < b), then, by Lemma 2.2.07, there exists x,y ϵ N such that (c + x = a) ^ (d + y = b) (reference [CD], Chapter 4, page 113), hence, by Definition 2.1.13, a_b = c_d + x_y. Then, by Theorem 2.2.11 and 2.2.13 and Corollary 2.2.20:

a_b + m_q = (c_d + x_y) + m_q;

= c_d + (x_y + m_q);

= c_d + (m_q + x_y).

Hence, by Definition 3.1.02, $(a\_b, c\_d)\ E\ [(m\_q + x\_y), m\_q]$.

Case 2. Suppose $(c < a) \wedge (d = b)$, then, by Lemma 2.2.07, there exists $x \in N$ such that $(c + x = a) \wedge (d + 0 = b)$ (reference [CD], Chapter 4, page 113), hence, by Definition 2.1.13, a_b = c_d + x_0. Then, by Theorem 2.2.11 and 2.2.13 and Corollary 2.2.20:

$$a\_b + m\_q = (c\_d + x\_0) + m\_q;$$

$$= c\_d + (x\_0 + m\_q);$$

$$= c\_d + (m\_q + x\_0).$$

Hence, by Definition 3.1.02, $(a\_b, c\_d)\ E\ [(m\_q + x\_0), m\_q]$.

Case 3. Suppose $(c = a) \wedge (d < b)$, then, by Lemma 2.2.07, there exists $y \in N$ such that $(c + 0 = a) \wedge (d + y = b)$ (reference [CD], Chapter 4, page 113), hence, by Definition 2.1.13, a_b = c_d + 0_y. Then, by Theorem 2.2.11 and 2.2.13 and Corollary 2.2.20:

$$a\_b + m\_q = (c\_d + 0\_y) + m\_q;$$

$$= c\_d + (0\_y + m\_q);$$

$$= c\_d + (m\_q + 0\_y).$$

Hence, by Definition 3.1.02, $(a\_b, c\_d)\ E\ [(m\_q + 0\_y), m\_q]$.

In all three cases, there exists an $x\_y \in N_Q$, such that $(a\_b, c\_d)\ E\ [(m\_q + x\_y), m\_q]$.

Let $s\_t \in N_Q$ be such that $(a\_b, c\_d)\ E\ [(m\_q + s\_t), m\_q]$, then, by Theorem 3.2.02, $[(m\_q + s\_t), m\_q]\ E\ [(m\_q + x\_y), m\_q]$ and, by Theorem 2.2.11 and 2.2.13:

$$(m\_q + s\_t) + m\_q = m\_q + (m\_q + s\_t);$$

$$= (m\_q + m\_q) + s\_t;$$

$$= m\_q + (m\_q + x\_y);$$

$$= (m\_q + m\_q) + x\_y.$$

Hence, by Corollary 2.2.20, s_t = x_y.

Therefore, for all $[(a\_b, c\_d)]_E \in K$, there exists a unique $x\_y \in N_Q$ such that, for any $m\_q \in N_Q$, $[(a\_b, c\_d)]_E = [\{(m\_q + x\_y), m\_q\}]_E$, as desired. □

Theorem 3.2.22. *$(K \cup \{0\_0\}, <, +, *)$ is ring isomorphic to $(N_Q, <, +, *)$.*

Proof. By Lemma 3.2.21, every $[(a\_b, c\_d)]_E \in K$ can be represented by $[\{(m\_q + x\_y), m\_q\}]_E$, where $m\_q \in N_Q$ is arbitrary and x_y is unique. Then there is an obvious isomorphism, $f:[K \cup \{0\_0\}] \to N_Q$, defined by $[f(0\_0) = 0\_0] \wedge \{[f([(a\_b, c\_d)]_E) = x\_y] \text{ iff } ([(a\_b, c\_d)]_E = [\{(m\_q + x\_y), m\_q\}]_E)\}$. Let $[(a\_b, c\_d)]_E$, $[(e\_f, g\_h)]_E \in K$ be arbitrary, then there exists unique $v\_w, x\_y \in N_Q$ such that, for any $k\_p, m\_q \in N_Q$, $[(a\_b, c\_d)]_E = [\{(k\_p + v\_w), k\_p\}]_E$ and $[(e\_f, g\_h)]_E = [\{(m\_q + x\_y), m\_q\}]_E$. Then:

Addition. By Definition 3.1.06 and Theorem 2.2.11 and 2.2.13:

$$f([\{(k\_p + v\_w), k\_p\}]_E + [\{(m\_q + x\_y), m\_q\}]_E) = f([\{[(k\_p + v\_w) + (m\_q + x\_y)], (k\_p + m\_q)\}]_E);$$

$$= f([\{[(k\_p + m\_q) + (v\_w + x\_y)], (k\_p + m\_q)\}]_E);$$

= (v_w + x_y);

= f([{(k_p + v_w), k_p}]$_E$) + f([{(m_q + x_y), m_q}]$_E$).

By Theorem 2.2.08 and 3.2.13, this result extends to K U {0_0}.

Multiplication. Letting A = f([{(k_p + v_w), k_p}]$_E$ * [{(m_q + x_y), m_q}]$_E$), by Definition 3.1.06 and Theorem 2.2.11, 2.2.13, 2.2.15, and 2.2.16 and Corollary 2.2.17:

A = f([{[{(k_p + v_w) * (m_q + x_y)} + (k_p * m_q)], [{(k_p + v_w) * m_q} + {k_p * (m_q + x_y)}]}]$_E$);

= f([{(k_p * m_q + [k_p * x_y + v_w * m_q] + [v_w * x_y + k_p * m_q]), (k_p * m_q + v_w * m_q + k_p * m_q + k_p * x_y)}]$_E$);

= f([{(k_p * m_q + v_w * m_q + [k_p * x_y + k_p * m_q] + v_w * x_y), (k_p * m_q + v_w * m_q + k_p * m_q + k_p * x_y)}]$_E$);

= f([{[(k_p * m_q + v_w * m_q + k_p * m_q + k_p * x_y) + v_w * x_y], (k_p * m_q + v_w * m_q + k_p * m_q + k_p * x_y)}]$_E$);

= (v_w * x_y);

= f([{(k_p + v_w), k_p}]$_E$) * f([{(m_q + x_y), m_q}]$_E$).

By Theorem 2.2.09 and 3.2.15, this result extends to K U {0_0}.

Therefore, (K U {0_0}, <, +, *) is ring isomorphic to (N_Q, <, +, *), as desired. □

Corollary 3.2.23. *Let L = {[(a_b, c_d)]$_E$ | ([(a_b, c_d)]$_E$ ϵ Z_Q^-) ^ (b ≤ d)}, then, for all [(a_b, c_d)]$_E$ ϵ L, there exists a unique x_y ϵ N_Q such that, for any m_q ϵ N_Q, [(a_b, c_d)]$_E$ = [{m_q, (m_q + x_y)}]$_E$.*

Proof. By Definition 3.1.10, for every [(a_b, c_d)]$_E$ ϵ Z_Q^-, [(c_d, a_b)]$_E$ ϵ Z_Q^+ and, by Lemma 3.2.21, [(c_d, a_b)]$_E$ = [{(m_q + x_y), m_q}]$_E$ for unique x_y ϵ N_Q and arbitrary m_q ϵ N_Q. Then, by substitution, [(a_b, c_d)]$_E$ = [{m_q, (m_q + x_y)}]$_E$ ϵ Z_Q^- and, by definition, Z_Q^- properly contains L, as desired. □

Lemma 3.2.24. *Let M = {[(a_b, c_d)]$_E$ | ([(a_b, c_d)]$_E$ ϵ Z_Q^+) ^ (b < d)}, then, for all [(a_b, c_d)]$_E$ ϵ K, there exists a unique x_y ϵ N_Q such that, for any m_q ϵ N_Q, [(a_b, c_d)]$_E$ = [{(m_q + x_- y), m_q}]$_E$.*

Proof. By Definition 2.1.12 and 3.1.08, (c < a) ^ (b < d) and, by Lemma 2.2.07, there exists x, y ϵ N such that (c + x = a) ^ (b + y = d) ^ (- y = b - d) (reference [CD], Chapters 4 and 5, pages 113, 158 – 165, and 178), hence, by Definition 2.1.13, a_b = c_d + x_- y. Then, by Lemma 2.2.07, Theorem 2.2.11 and 2.2.13, Corollary 2.2.20, and reference [CD], Chapter 5, pages 162 and 164:

a_b + m_q = (c_d + x_- y) + m_q;

= c_d + (x_- y + m_q);

= c_d + (m_q + x_- y).

Therefore, by Definition 3.1.02, (a_b, c_d) E [(m_q + x_- y), m_q].

Let s_t ϵ N_Q be such that (a_b, c_d) E [(m_q + s_- t), m_q], then, by Theorem 3.2.02, [(m_q + s_- t), m_q] E [(m_q + x_- y), m_q] and, by Theorem 2.2.11 and 2.2.13:

(m_q + s_- t) + m_q = m_q + (m_q + s_- t);

= (m_q + m_q) + s_- t;

= m_q + (m_q + x_- y);

= (m_q + m_q) + x_- y.

Hence, by Corollary 2.2.20, s_- t = x_- y.

Therefore, for all $[(a\_b, c\_d)]_E \in K$, there exists a unique $x\_y \in N_Q$ such that, for any $m\_q \in N_Q$, $[(a\_b, c\_d)]_E = [\{(m\_q + x\_- y), m\_q\}]_E$, as desired. □

Corollary 3.2.25. *Let* $O = \{[(a\_b, c\_d)]_E \mid ([(a\_b, c\_d)]_E \in Z_Q^-) \wedge (d < b)\}$, *then, for all* $[(a\_b, c\_d)]_E \in O$, *there exists a unique* $x\_y \in N_Q$ *such that, for any* $m\_q \in N_Q$, $[(a\_b, c\_d)]_E = [\{m\_q, (m\_q + x\_- y)\}]_E$.

Proof. By Definition 3.1.10, for every $[(a\_b, c\_d)]_E \in Z_Q^-$, $[(c\_d, a\_b)]_E \in Z_Q^+$ and, by Lemma 3.2.24, $[(c\_d, a\_b)]_E = [\{(m\_q + x\_- y), m\_q\}]_E$ for unique $x\_y \in N_Q$ and arbitrary $m\_q \in N_Q$. Then, by substitution, $[(a\_b, c\_d)]_E = [\{m\_q, (m\_q + x\_- y)\}]_E \in Z_Q^-$ and, by definition, Z_Q^- properly contains O, as desired. □

Theorem 3.2.26. $(x\_y \in Z_Q) \rightarrow [(x\_- y \in Z_Q) \wedge (- x\_y \in Z_Q) \wedge (- x\_- y \in Z_Q)]$.

Proof. This is an immediate consequence of Lemma 3.2.21 and 3.2.24 and Corollary 3.2.23 and 3.2.25, as desired. □

Corollary 3.2.27. *For all* $a\_b \in Z_Q$, $a,b \in Z$.

Proof. By Definition 3.1.01 and 3.1.05, Lemma 2.2.07, and reference [CD], Chapter 5, as desired. □

Lemma 3.2.28. *For all* $a\_b, c\_d \in Z_Q$, $a\_b * c\_d = (a * c)\_(b * c + a * d + b * d)$.

Proof. By Theorem 3.2.22, multiplication defined on Z_Q is consistent with multiplication defined on N_Q, hence, by Definition 2.1.14, Lemma 3.2.21 and 3.2.24, and Corollary 3.2.23 and 3.2.25, a_b * c_d = (a * c)_(b * c + a * d + b * d), as desired. □

Theorem 3.2.29. *Partition* Z_Q *in the following manner:*

1. $A = \{a\_b \mid (a\_b \in Z_Q) \wedge (a < 0) \wedge (b < 0)\}$;
2. $B = \{a\_b \mid (a\_b \in Z_Q) \wedge (a < 0) \wedge (0 \le b)\}$;
3. $C = \{a\_b \mid (a\_b \in Z_Q) \wedge (a = 0) \wedge (b < 0)\}$;
4. $D = \{a\_b \mid (a\_b \in Z_Q) \wedge (a = 0) \wedge (0 \le b)\}$;
5. $E = \{a\_b \mid (a\_b \in Z_Q) \wedge (0 < a) \wedge (b < 0)\}$;
6. $F = \{a\_b \mid (a\_b \in Z_Q) \wedge (0 < a) \wedge (0 \le b)\}$.

Then we demonstrate the following:

a. *For all* $a\_b, c\_d \in A$, $a\_b * c\_d \in F$.

   Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, $(0 < a * c) \wedge [0 < (b * c + a * d + b * d)]$ and a_b * c_d ∈ F, as desired. □

b. *For all* $a\_b \in A$, $c\_d \in B$, $a\_b * c\_d \in E \cup F$.

   Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, $(0 < a * c) \wedge \{[(b * c + a * d + b * d) < 0] \vee [0 < (b * c + a * d + b * d)]\}$ and a_b * c_d ∈ E U F, as desired. □

c. *For all* $a\_b \in A$, $c\_d \in C$, $a\_b * c\_d \in D$.

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 = a * c) ^ [0 < (b * c + a * d + b * d)] and a_b * c_d ϵ D, as desired. □

*d. For all a_b ϵ A, c_d ϵ D, a_b * c_d ϵ C.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 = a * c) ^ [(b * c + a * d + b * d) < 0] and a_b * c_d ϵ C, as desired. □

*e. For all a_b ϵ A, c_d ϵ E, a_b * c_d ϵ A U B.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (a * c < 0) ^ {[(b * c + a * d + b * d) < 0] ∨ [0 < (b * c + a * d + b * d)]} and a_b * c_d ϵ A U B, as desired. □

*f. For all a_b ϵ A, c_d ϵ F, a_b * c_d ϵ A.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (a * c < 0) ^ [(b * c + a * d + b * d) < 0] and a_b * c_d ϵ A, as desired. □

*g. For all a_b, c_d ϵ B, a_b * c_d ϵ E U F.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 < a * c) ^ {[(b * c + a * d + b * d) < 0] ∨ [0 < (b * c + a * d + b * d)]} and a_b * c_d ϵ E U F, as desired. □

*h. For all a_b ϵ B, c_d ϵ C, a_b * c_d ϵ C U D.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 = a * c) ^ {[(b * c + a * d + b * d) < 0] ∨ [0 < (b * c + a * d + b * d)]} and a_b * c_d ϵ C U D, as desired. □

i. *For all a_b ϵ B, c_d ϵ D, a_b * c_d ϵ C U D.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 = a * c) ^ {[(b * c + a * d + b * d) < 0] ∨ [0 < (b * c + a * d + b * d)]} and a_b * c_d ϵ C U D, as desired. □

j. *For all a_b ϵ B, c_d ϵ E, a_b * c_d ϵ A U B.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (a * c < 0) ^ {[(b * c + a * d + b * d) < 0] ∨ [0 < (b * c + a * d + b * d)]} and a_b * c_d ϵ A U B, as desired. □

k. *For all a_b ϵ B, c_d ϵ F, a_b * c_d ϵ A U B.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (a * c < 0) ^ {[(b * c + a * d + b * d) < 0] ∨ [0 < (b * c + a * d + b * d)]} and a_b * c_d ϵ A U B, as desired. □

*l. For all a_b, c_d ϵ C, a_b * c_d ϵ D.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 = a * c) ^ [0 < (b * c + a * d + b * d)] and a_b * c_d ϵ D, as desired. □

*m. For all a_b ϵ C, c_d ϵ D, a_b * c_d ϵ C.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 = a * c) ^ [(b * c + a * d + b * d) < 0] and a_b * c_d ϵ C, as desired. □

n. *For all a_b ϵ C, c_d ϵ E, a_b * c_d ϵ C U D.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 = a * c) ^ {[(b * c + a * d + b * d) < 0] ˅ [0 < (b * c + a * d + b * d)]} and a_b * c_d ϵ C U D, as desired. □

o. *For all a_b ϵ C, c_d ϵ F, a_b * c_d ϵ C.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 = a * c) ^ [(b * c + a * d + b * d) < 0] and a_b * c_d ϵ C, as desired. □

p. *For all a_b, c_d ϵ D, a_b * c_d ϵ D.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 = a * c) ^ [0 < (b * c + a * d + b * d)] and a_b * c_d ϵ D, as desired. □

q. *For all a_b ϵ D, c_d ϵ E, a_b * c_d ϵ C U D.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 = a * c) ^ {[(b * c + a * d + b * d) < 0] ˅ [0 < (b * c + a * d + b * d)]} and a_b * c_d ϵ C U D, as desired. □

r. *For all a_b ϵ D, c_d ϵ F, a_b * c_d ϵ D.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 = a * c) ^ [0 < (b * c + a * d + b * d)] and a_b * c_d ϵ D, as desired. □

s. *For all a_b, c_d ϵ E, a_b * c_d ϵ E U F.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 < a * c) ^ {[(b * c + a * d + b * d) < 0] ˅ [0 < (b * c + a * d + b * d)]} and a_b * c_d ϵ E U F, as desired. □

t. *For all a_b ϵ E, c_d ϵ F, a_b * c_d ϵ E U F.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 < a * c) ^ {[(b * c + a * d + b * d) < 0] ˅ [0 < (b * c + a * d + b * d)]} and a_b * c_d ϵ E U F, as desired. □

u. *For all a_b, c_d ϵ F, a_b * c_d ϵ F.*

Proof. By Lemma 3.2.28, a_b * c_d = (a * c)_(b * c + a * d + b * d), hence, by Theorem 3.2.06, Corollary 3.2.27, and reference [CD], Chapter 5, (0 < a * c) ^ [0 < (b * c + a * d + b * d)] and a_b * c_d ϵ F, as desired. □

Lemma 3.2.30. *For all a_b, c_d ϵ Z_Q, a_b - c_d = a_b + [- (c_d)].*

Proof. By Definition 3.1.11 and Theorem 3.2.18, as desired. □

Theorem 3.2.31. *Z_Q is closed under the operation "-" (subtraction) of Definition 3.1.11.*

Proof. By Theorem 3.2.06 and Lemma 3.2.30, as desired. □

Theorem 3.2.32. *Z_Q is not closed under the operation "÷" (division) of Definition 3.1.12.*

Proof. By Corollary 3.2.27, there exist a_b, c_d ϵ Z_Q such that "a" and "c" are both prime and a ≠ c. Suppose, for contradiction, that a_b ÷ c_d ϵ Z_Q, then, by Definition 3.1.12, there exists x_y ϵ Z_Q such that a_b = x_y * c_d and, by Lemma 3.2.28, a = x * c. But then [(x = a) ^ (c = 1)] ᵛ [(x = 1) ^ (c = a)], a contradiction in either case. Therefore, Z_Q is not closed under division, as desired. □

Corollary 3.2.33. *The relation "<" (strict order)on Z_Q, as defined by Definition 3.1.13, is well-defined relative to the E of Definition 3.1.02.*

Proof. By Theorem 3.2.20, as desired. □

Lemma 3.2.34. *For all a_b, c_d ϵ Z_Q, a_b + c_d = (a + c)_(b +d).*

Proof. By Theorem 3.2.22, addition defined on Z_Q is consistent with addition defined on N_Q, hence, by Definition 2.1.13, Lemma 3.2.21 and 3.2.24, and Corollary 3.2.23 and 3.2.25, a_b + c_d = (a + c)_(b +d), as desired. □

Theorem 3.2.35. *(Z_Q, <) is a linearly ordered set.*

Proof. The proof is in three parts:

1) *Transitivity.* Let a_b, c_d, e_f ϵ Z_Q be arbitrary but such that a_b < c_d ^ c_d < e_f. Then, by Definition 3.1.13, there exist k_p, m_q ϵ Z_Q^+ such that a_b + k_p = c_d ^ c_d + m_q = e_f, hence, by Theorem 2.2.13, a_b + (k_p + m_q) = e_f and a_b < e_f.
2) *Assymetry.* Let k_p, m_q ϵ Z_Q be arbitrary and suppose, for contradiction, that k_p < m_q ^ m_q < k_p, then, by transitivity, k_p < k_p, contradicting Definition 3.1.13.
3) *Linearity.* Let a_b, c_d ϵ Z_Q be arbitrary, then, by Corollary 3.2.27, [(a < c) ᵛ (a = c) ᵛ (c < a)] ^ [(b < d) ᵛ (b = d) ᵛ (d < b)] (reference [CD], Chapter 5, page 167) and nine cases arise, three of which are redundant, leaving six cases to consider:

   Case 1. Suppose (a < c) ^ (b < d), then, by Corollary 3.2.27, there exist x,y ϵ Z^+ such that (a + x = c) ^ (b + y = d) (reference [CD], Chapter 5, page 167), hence, by Lemma 3.2.34, a_b + x_y = c_d and, by Definition 3.1.13, a_b < c_d. This result reverses in the case (c < a) ^ (d < b).

   Case 2. Suppose (a < c) ^ (b = d), then, by Corollary 3.2.27, there exist x ϵ Z^+ such that (a + x = c) ^ (b + 0 = d) (reference [CD], Chapter 5, pages 163 and 167), hence, by Lemma 3.2.34, a_b + x_0 = c_d and, by Definition 3.1.13, a_b < c_d. This result reverses in the case (c < a) ^ (d = b).

   Case 3. Suppose (a < c) ^ (d < b), then, by Corollary 3.2.27, there exist x,y ϵ Z^+ such that (a + x = c) ^ (d + y = b) ^ (d = b - y) (reference [CD], Chapter 5, pages 165 and 167), hence, by Lemma 3.2.34, a_b + x_- y = c_d. By Theorem 3.2.26, x_- y ϵ Z_Q and, by Definition 3.1.13, a_b < c_d. This result reverses in the case (c < a) ^ (b < d).

   Case 4. Suppose (a = c) ^ (b < d), then, by Corollary 3.2.27, there exist y ϵ Z^+ such that (a + 0 = c) ^ (b + y = d) (reference [CD], Chapter 5, pages 163 and 167), hence, by Lemma 3.2.34, a_b + 0_y = c_d and, by Definition 3.1.13, a_b < c_d.

   Case 5. Suppose (a = c) ^ (b = d), then, a_b = c_d.

   Case 6. Suppose (a = c) ^ (d < b), then, by Corollary 3.2.27, there exist y ϵ Z^+ such that (a + 0 = c) ^ (d + y = b) (reference [CD], Chapter 5, pages 163 and 167), hence, by Lemma 3.2.34, c_d + 0_y = a_b and, by Definition 3.1.13, c_d < a_b.

In all nine cases, (a_b < c_d) $\vee$ (a_b = c_d) $\vee$ (c_d < a_b).

Therefore, (Z_Q, <) is a linearly ordered set, as desired. □

Lemma 3.2.36. *For all a_b, c_d ϵ Z_Q^+, a_b + c_d ϵ Z_Q^+.*

Proof. Suppose a_b, c_d ϵ Z_Q^+ and suppose, for contradiction, ⌐ (a_b + c_d ϵ Z_Q^+), then, by Definition 2.1.12 and 3.1.08 and Lemma 3.2.34, (a + c < 0) $\vee$ [(a + c = 0) ^ (b + d = 0)] $\vee$ [(a + c = 0) ^ (b + d < 0)] and three cases arise;

Case 1. Suppose a + c < 0, then, by Corollary 3.2.27, (a < 0) $\vee$ (c < 0) (reference [CD], Chapter 5, pages 163 – 165), a contradiction.

Case 2. Suppose (a + c = 0) ^ (b + d = 0), then, by Corollary 3.2.27, a = b =c = d =0 (reference [CD], Chapter 5, page 163), a contradiction.

Case 3. Suppose (a + c = 0) ^ (b + d < 0), then, by Corollary 3.2.27, (a = c = 0) ^ [(b < 0) $\vee$ (d < 0)] (reference [CD], Chapter 5, pages 163 – 165), a contradiction.

In all three cases, a contradiction.

Therefore, a_b + c_d ϵ Z_Q^+, as desired. □

Theorem 3.2.37. *For all a_b, c_d, e_f ϵ Z_Q, (a_b < c_d) iff [(a_b + e_f) < (c_d + e_f)].*

Proof. Suppose a_b < c_d, then, by Definition 3.1.13, there exists g_h ϵ Z_Q^+ such that a_b + g_h = c_d, hence, by Theorem 3.2.07 and 3.2.09:

c_d + e_f = (a_b + g_h) + e_f;

= a_b + (g_h + e_f);

= a_b + (e_f + g_h);

= (a_b + e_f) + g_h.

Therefore, (a_b + e_f) < (c_d + e_f), and (a_b < c_d) → [(a_b + e_f) < (c_d + e_f)].

Suppose (a_b + e_f) < (c_d + e_f), then, by Definition 3.1.13, there exists g_h ϵ Z_Q^+ such that (a_b + e_f) + g_h = (c_d + e_f), hence, by Theorem 3.2.07, 3.2.09, and 3.2.18 and Lemma 3.2.20:

(a_b + e_f) + g_h = (c_d + e_f);

a_b + (e_f + g_h) = (c_d + e_f);

a_b + (g_h + e_f) = (c_d + e_f);

a_b + (g_h + e_f) + [- (e_f)] = (c_d + e_f) + [- (e_f)];

(a_b + g_h) + (e_f + [- (e_f)]) = c_d + (e_f + [- (e_f)]);

a_b + g_h = c_d.

Therefore, by Definition 3.1.13, a_b < c_d and [(a_b + e_f) < (c_d + e_f)] → (a_b < c_d).

Therefore, (a_b < c_d) iff [(a_b + e_f) < (c_d + e_f)], as desired. □

Theorem 3.2.38. *For all a_b, c_d, e_f, g_h ϵ Z_Q, [(a_b < c_d) ^ (e_f < g_h)] → [(a_b + e_f) < (c_d + g_h)].*

Proof. Suppose (a_b < c_d) ^ (e_f < g_h), then, by Definition 3.1.13, there exist k_p, m_q ϵ Z_Q^+ such that (a_b + k_p = c_d) ^ (e_f + m_q = g_h) and, by Theorem 3.2.07 and 3.2.09:

c_d + g_h = (a_b + k_p) + (e_f + m_q);

= a_b + (k_p + e_f) + m_q;

= (a_b + e_f) + (k_p + m_q).

Therefore, by Definition 3.1.13 and Lemma 3.2.36, (a_b + e_f) < (c_d + g_h), as desired. □

Lemma 3.2.39. *For all a_b, c_d ϵ Z_Q, (a_b < c_d) iff {(a < c) ˅ [(a = c) ^ (b < d)]}.*

Proof. Suppose a_b < c_d, then, by Definition 3.1.13, there exists g_h ϵ Z_Q^+ such that a_b + g_h = c_d. By Lemma 3.2.34, (a + g = c) ^ (b + h = d) and, by Definition 3.1.08, Theorem 3.2.26, and Corollary 3.2.27, {(0 < g) ^ [(h < 0) ˅ (h = 0) ˅ (0 < h)]} ˅ [(g = 0) ^ (0 < h)] and four cases arise, two of which are redundant, leaving two cases to consider:

Case 1. Suppose (0 < g) ^ (h < 0), then a < c (reference [CD], Chapter 5, page 167). This result remains unchanged in the cases [(0 < g) ^ (h = 0)] and [(0 < g) ^ (0 < h)].

Case 2. Suppose (g = 0) ^ (0 < h), then (a = c) ^ (b < d) (reference [CD], Chapter 5, page 167).

In all four cases, (a < c) ˅ [(a = c) ^ (b < d)].

Therefore, (a_b < c_d) → {(a < c) ˅ [(a = c) ^ (b < d)]}.

Suppose (a < c) ˅ [(a = c) ^ (b < d)], then there are two cases to consider:

Case 1. Suppose a < c, then, by Corollary 3.2.27, (b < d) ˅ (b = d) ˅ (d < b) and three cases arise:

Case 1.a. Suppose (a < c) ^ (b < d), then, by Corollary 3.2.27, there exists g, h ϵ Z^+ such that (a + g = c) ^ (b + h = d) (reference [CD], Chapter 5, page 167). By Lemma 3.2.34, a_b + g_h = c_d and, by Definition 2.1.12 and 3.1.08, g_h ϵ Z_Q^+, hence, by Definition 3.1.13, a_b < c_d.

Case 1.b. Suppose (a < c) ^ (b = d), then, by Corollary 3.2.27, there exists g ϵ Z^+ such that (a + g = c) ^ (b + 0 = d) (reference [CD], Chapter 5, page 167). By Lemma 3.2.34, a_b + g_0 = c_d and, by Definition 2.1.12 and 3.1.08, g_0 ϵ Z_Q^+, hence, by Definition 3.1.13, a_b < c_d.

Case 1.c. Suppose (a < c) ^ (d < b), then, by Corollary 3.2.27, there exists g, h ϵ Z^+ such that (a + g = c) ^ (d + h = b) ^ (b + (- h) = d) (reference [CD], Chapter 5, page 167). By Lemma 3.2.34, a_b + g_- h = c_d and, by Definition 2.1.12 and 3.1.08 and Theorem 3.2.26, g_- h ϵ Z_Q^+, hence, by Definition 3.1.13, a_b < c_d.

In all three cases, a_b < c_d.

Therefore, (a < c) → (a_b < c_d).

Case 2. Suppose (a = c) ^ (b < d), then, by Corollary 3.2.27, there exists h ϵ Z^+ such that (a + 0 = c) ^ (b + h = d) (reference [CD], Chapter 5, page 167). By Lemma 3.2.34, a_b + 0_h = c_d and, by Definition 2.1.12 and 3.1.08, 0_h ϵ Z_Q^+, hence, by Definition 3.1.13, a_b < c_d.

Therefore, [(a = c) ^ (b < d)] → (a_b < c_d).

Therefore, {(a < c) ˅ [(a = c) ^ (b < d)]} → (a_b < c_d).

Therefore, (a_b < c_d) iff {(a < c) ˅ [(a = c) ^ (b < d)]}, as desired. □

Theorem 3.2.40. *Let S = {a_b | (a_b ϵ Z_Q^+) ^ [(a = 0) ∨ (b < 0)]}, then, for all a_b, c_d ϵ Z_Q, e_f ϵ Z_Q^+ – S, (a_b < c_d) iff [(a_b * e_f) < (c_d * e_f)].*

Proof. Suppose a_b < c_d, then, by Lemma 3.2.39, (a < c) ∨ [(a = c) ^ (b < d)] and, by Definition 2.1.12 and 3.1.08, Theorem 3.2.26, and Corollary 3.2.27, (0 < e) ^ (0 ≤ f) and two cases arise:

Case 1. Suppose (a < c) ^ (0 < e) ^ (0 ≤ f), then a * e < c * e (reference [CD], Chapter 5, page 167) and, by Lemma 3.2.28 and 3.2.29, (a_b * e_f) < (c_d * e_f).

Case 2. Suppose (a = c) ^ (b < d) ^ (0 < e) ^ (0 ≤ f), then (a * e = c * e) ^ [b * (e + f) < d * (e + f)] (reference [CD], Chapter 5, page 167) and, by Lemma 3.2.28 and 3.2.29, (a_b * e_f) < (c_d * e_f).

In both cases, (a_b * e_f) < (c_d * e_f).

Therefore, (a_b < c_d) → [(a_b * e_f) < (c_d * e_f)].

Suppose (a_b * e_f) < (c_d * e_f), then, by Lemma 3.2.28 and 3.2.29, (a * e < c * e) ∨ {(a * e = c * e) ^ [(b * e + a * f + b * f) < (d * e + c * f + d * f)]} and, by Definition 2.1.12 and 3.1.08, Theorem 3.2.26, and Corollary 3.2.27, (0 < e) ^ (0 ≤ f) and two cases arise:

Case 1. Suppose (a * e < c * e) ^ (0 < e) ^ (0 ≤ f), then, a < c (reference [CD], Chapter 5, page 168) and, by Lemma 3.2.39, a_b < c_d.

Case 2. Suppose (a * e = c * e) ^ [(b * e + a * f + b * f) < (d * e + c * f + d * f)] ^ (0 < e) ^ (0 ≤ f), then, (a = c) ^ (b < d) (reference [CD], Chapter 5, pages 162, 167, and 168) and, by Lemma 3.2.39, a_b < c_d.

In both cases, a_b < c_d.

Therefore, [(a_b * e_f) < (c_d * e_f)] → (a_b < c_d).

Therefore, (a_b < c_d) iff [(a_b * e_f) < (c_d * e_f)], as desired. □

Theorem 3.2.41. *Let T = {a_b | (a_b ϵ Z_Q^-) ^ [(a = 0) ∨ (0 < b)]}, then, for all a_b, c_d ϵ Z_Q, e_f ϵ Z_Q^- – T, (a_b < c_d) iff [(c_d * e_f) < (a_b * e_f)].*

Proof. Suppose a_b < c_d, then, by Lemma 3.2.39, (a < c) ∨ [(a = c) ^ (b < d)] and, by Definition 2.1.12 and 3.1.08, Theorem 3.2.26, and Corollary 3.2.27, (e < 0) ^ (f ≤ 0) and two cases arise:

Case 1. Suppose (a < c) ^ (e < 0) ^ (f ≤ 0), then c * e < a * e (reference [CD], Chapter 5, page 167) and, by Lemma 3.2.28 and 3.2.29, (c_d * e_f) < (a_b * e_f).

Case 2. Suppose (a = c) ^ (b < d) ^ (e < 0) ^ (f ≤ 0), then (c * e = a * e) ^ [d * (e + f) < b * (e + f)] (reference [CD], Chapter 5, page 167) and, by Lemma 3.2.28 and 3.2.29, (c_d * e_f) < (a_b * e_f).

In both cases, (c_d * e_f) < (a_b * e_f).

Therefore, (a_b < c_d) → [(c_d * e_f) < (a_b * e_f)].

Suppose (c_d * e_f) < (a_b * e_f), then, by Lemma 3.2.28 and 3.2.29, (c * e < a * e) ∨ {(c * e = a * e) ^ [(d * e + c * f + d * f) < (b * e + a * f + b * f)]} and, by Definition 2.1.12 and 3.1.08, Theorem 3.2.26, and Corollary 3.2.27, (e < 0) ^ (f ≤ 0) and two cases arise:

Case 1. Suppose (c * e < a * e) ^ (e < 0) ^ (f ≤ 0), then, a < c (reference [CD], Chapter 5, page 168) and, by Lemma 3.2.39, a_b < c_d.

Case 2. Suppose (c * e = a * e) ^ [(d * e + c * f + d * f) < (b * e + a * f + b * f)] ^ (e < 0) ^ (f ≤ 0), then, (a = c) ^ (b < d) (reference [CD], Chapter 5, pages 162, 167, and 168) and, by Lemma 3.2.39, a_b < c_d.

In both cases, a_b < c_d.

Therefore, [(c_d * e_f) < (a_b * e_f)] → a_b < c_d.

Therefore, (a_b < c_d) iff [(c_d * e_f) < (a_b * e_f)], as desired. □

Lemma 3.2.42. *For all a_b ϵ Z_Q, - 1_0 * a_b = - (a_b) represents the additive inverse of a_b.*

Proof. By Theorem 3.2.19, Corollary 3.2.23, and Lemma 3.2.28:

- 1_0 * a_b = (- 1 * a)_(0 * a +(- 1 * b) + 0 * b);

= - a_- b.

Therefore, by Definition 3.1.10, - 1_0 * a_b = - (a_b) represents the additive inverse of a_b, as desired. □

Corollary 3.2.43. *For all a_b, c_d ϵ Z_Q, - (a_b) * c_d = - (a_b * c_d).*

Proof. By Theorem 3.2.10 and Lemma 3.2.42:

- (a_b) * c_d = (- 1_0 * a_b) * c_d;

= - 1_0 * (a_b * c_d);

= - (a_b * c_d), as desired. □

Corollary 3.2.44. *For all a_b, c_d ϵ Z_Q, - (a_b) * - (c_d) = a_b * c_d.*

Proof. By Theorem 3.2.08, 3.2.10, and 3.2.16, Corollary 3.2.27, and Lemma 3.2.42:

- (a_b) * - (c_d) = (- 1_0 * a_b) * (- 1_0 * c_d);

= - 1_0 * (a_b * - 1_0) * c_d;

= - 1_0 * (- 1_0 * a_b) * c_d;

= (- 1_0 * - 1_0) * (a_b * c_d);

= 1_0 * (a_b * c_d);

= a_b * c_d, as desired. □

Theorem 3.2.45. *Partition Z_Q in the following manner:*

1. *A = {a_b | (a_b ϵ Z_Q) ^ (a < 0) ^ (b < 0)};*
2. *B = {a_b | (a_b ϵ Z_Q) ^ (a < 0) ^ (0 ≤ b)};*
3. *C = {a_b | (a_b ϵ Z_Q) ^ (a = 0) ^ (b < 0)};*
4. *D = {a_b | (a_b ϵ Z_Q) ^ (a = 0) ^ (0 ≤ b)};*
5. *E = {a_b | (a_b ϵ Z_Q) ^ (0 < a) ^ (b < 0)};*
6. *F = {a_b | (a_b ϵ Z_Q) ^ (0 < a) ^ (0 ≤ b)}.*

Thenwe demonstrate the following:

a. *For all a_b,c_d ϵ A, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.44:

|a_b| * |c_d| = - (a_b) * - (c_d);

= a_b * c_d;

= |a_b * c_d|, as desired. □

b. *For all a_b ϵ A, c_d ϵ B, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.44:

|a_b| * |c_d| = - (a_b) * - (c_d);

= a_b * c_d;

= |a_b * c_d|, as desired. □

c. *For all a_b ϵ A, c_d ϵ C, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.44:

|a_b| * |c_d| = - (a_b) * - (c_d);

= a_b * c_d;

= |a_b * c_d|, as desired. □

d. *For all a_b ϵ A, c_d ϵ D, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.43:

|a_b| * |c_d| = - (a_b) * c_d;

= - (a_b * c_d);

= |a_b * c_d|, as desired. □

e. *For all a_b ϵ A, c_d ϵ E, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.43:

|a_b| * |c_d| = - (a_b) * c_d;

= - (a_b * c_d);

= |a_b * c_d|, as desired. □

f. *For all a_b ϵ A, c_d ϵ F, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.43:

|a_b| * |c_d| = - (a_b) * c_d;

= - (a_b * c_d);

= |a_b * c_d|, as desired. □

g. *For all a_b, c_d ϵ B, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.44:

|a_b| * |c_d| = - (a_b) * - (c_d);

= a_b * c_d;

= |a_b * c_d|, as desired. □

h. *For all a_b ϵ B, c_d ϵ C, (|a_b| * |c_d| = |a_b * c_d|) iff (b ≤ |a|).*

Proof. Suppose |a| < b, then, by Corollary 3.2.27, (b * d < 0) ^ (0 < a * d) (reference [CD], Chapter 5, pages 165 and 167) and, by Definition 3.1.14 and Lemma 3.2.42:

|a_b| * |c_d| = - (a_b) * - (0_d);

= - a_- b * 0_- d;

= 0_0 + a * d + b * d;

= 0_a * d + b * d, hence, |a_b| * |c_d| ϵ C, while:

|a_b * c_d| = |a_b * 0_d|;

= |0_0 + a * d + b * d |;

= |0_a * d + b * d | ϵ D.

Therefore, by the Law of Contraposition, (|a_b| * |c_d| = |a_b * c_d|) → (b ≤ |a|).

Suppose b ≤ |a|, then, by Corollary 3.2.27, 0 ≤ d * (a + b) (reference [CD], Chapter 5, pages 165 and 167) and, by Definition 3.1.14, Theorem 3.2.29, and Lemma 3.2.42:

|a_b| * |c_d| = - (a_b) * - (c_d);

= - a_- b * 0_- d;

= 0_a * d + b * d;

= 0_d * (a + b);

= |a_b * c_d|.

Therefore, (b ≤ |a|) → (|a_b| * |c_d| = |a_b * c_d|).

Therefore, (|a_b| * |c_d| = |a_b * c_d|) iff (b ≤ |a|), as desired. □

i. *For all a_b ϵ B, c_d ϵ D, (|a_b| * |c_d| = |a_b * c_d|) iff (b ≤ |a|).*

Proof. Suppose |a| < b, then, by Corollary 3.2.27, (a * d < 0) ^ (0 < b * d) (reference [CD], Chapter 5, page 167) and, by Definition 3.1.14 and Lemma 3.2.42:

|a_b| * |c_d| = - (a_b) * c_d;

= - a_- b * 0_d;

= 0_0 - a * d - b * d;

= 0_- a * d - b * d, hence, |a_b| * |c_d| ϵ C (reference [CD], Chapter 5, pages 167 – 171), while:

|a_b * c_d| = |a_b * 0_d|;

= |0_0 + a * d + b * d |;

= |0_a * d + b * d | ϵ D.

Therefore, by the Law of Contraposition, (|a_b| * |c_d| = |a_b * c_d|) → (b ≤ |a|).

Suppose b ≤ |a|, then, by Corollary 3.2.27, (a * d < 0) ^ (0 < b * d) (reference [CD], Chapter 5, page 167) and, by Definition 3.1.14, Theorem 3.2.29, and Lemma 3.2.42:

|a_b| * |c_d| = - (a_b) * c_d;

= - a_- b * 0_d;

= 0_- a * d - b * d;

= 0_- d * (a + b);

` = - [0_d * (a + b)];

= |a_b * c_d|.

Therefore, (b ≤ |a|) → (|a_b| * |c_d| = |a_b * c_d|).

Therefore, (|a_b| * |c_d| = |a_b * c_d|) iff (b ≤ |a|), as desired. □

j. *For all a_b ϵ B, c_d ϵ E, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.43:

|a_b| * |c_d| = - (a_b) * c_d;

= - (a_b * c_d);

= |a_b * c_d|, as desired. □

k. *For all a_b ϵ B, c_d ϵ F, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.43:

|a_b| * |c_d| = - (a_b) * c_d;

= - (a_b * c_d);

= |a_b * c_d|, as desired. □

l. *For all a_b, c_d ϵ C, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.44:

|a_b| * |c_d| = - (a_b) * - (c_d);

= - (a_b * c_d);

= |a_b * c_d|, as desired. □

m. *For all a_b ϵ C, c_d ϵ D, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.43:

|a_b| * |c_d| = - (a_b) * c_d;

= - (a_b * c_d);

= |a_b * c_d|, as desired. □

n. *For all a_b ϵ C, c_d ϵ E, (|a_b| * |c_d| = |a_b * c_d|) iff (|d| ≤ c).*

Proof. Suppose c < |d|, then, by Corollary 3.2.27, (b * c < 0) ^ (0 < b * d) (reference [CD], Chapter 5, page 167) and, by Definition 3.1.14 and Lemma 3.2.42:

|a_b| * |c_d| = - (a_b) * c_d;

= 0_- b * c_d;

= 0_- b * c + 0 - b * d;

= 0_- b * c - b * d, hence, |a_b| * |c_d| ϵ C (reference [CD], Chapter 5, pages 167 – 171),

while:

|a_b * c_d| = |0_b * c_d|;

= |0_b * c + 0 + b * d |;

= |0_b * c + b * d | ϵ D.

Therefore, by the Law of Contraposition, (|a_b| * |c_d| = |a_b * c_d|) → (|d| ≤ c).

Suppose |d| ≤ c, then, by Corollary 3.2.27, (b * c < 0) ^ (0 < b * d) (reference [CD], Chapter 5, page 167) and, by Definition 3.1.14, Theorem 3.2.29, and Lemma 3.2.42:

|a_b| * |c_d| = - (a_b) * c_d;

= 0_- b * c_d;

= 0_- b * c + 0 - b * d;

= 0_- b * (c+ d);

= - [0_b * (c+ d)];

= |a_b * c_d|.

Therefore, (|d| ≤ c) → (|a_b| * |c_d| = |a_b * c_d|).

Therefore, (|a_b| * |c_d| = |a_b * c_d|) iff (|d| ≤ c), as desired. □

o. *For all a_b ϵ C, c_d ϵ F, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, Theorem 3.2.29, and Corollary 3.2.43:

|a_b| * |c_d| = - (a_b) * c_d;

= - (a_b * c_d);

= |a_b * c_d|, as desired. □

p. *For all a_b, c_d ϵ D, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, the definition of D, and Theorem 3.2.29, as desired. □

q. *For all a_b ϵ D, c_d ϵ E, (|a_b| * |c_d| = |a_b * c_d|) iff (|d| ≤ c).*

Proof. Suppose c < |d|, then, by Corollary 3.2.27, (b * d < 0) ^ (0 < b * c) (reference [CD], Chapter 5, page 167) and, by Definition 3.1.14:

|a_b| * |c_d| = a_b * c_d;

= 0_b * c_d;

= 0_b * c + 0 + b * d;

= 0_b * c + b * d, hence, |a_b| * |c_d| ϵ C (reference [CD], Chapter 5, pages 167 – 171),

while:

|a_b * c_d| = |0_b * c_d|;

= |0_b * c + 0 + b * d |;

= |0_b * c + b * d | ϵ D.

Therefore, by the Law of Contraposition, (|a_b| * |c_d| = |a_b * c_d|) → (|d| ≤ c).

Suppose |d| ≤ c, then, by Corollary 3.2.27, (b * d < 0) ^ (0 < b * c) (reference [CD], Chapter 5, page 167) and, by Definition 3.1.14 and Theorem 3.2.29:

|a_b| * |c_d| = a_b * c_d;

= 0_b * c_d;

= 0_b * c + 0 + b * d;

= 0_b * (c + d);

= |a_b * c_d|.

Therefore, (|d| ≤ c) → (|a_b| * |c_d| = |a_b * c_d|).

Therefore, (|a_b| * |c_d| = |a_b * c_d|) iff (|d| ≤ c), as desired. □

r. *For all a_b ϵ D, c_d ϵ F, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, the definition of D, the definition of F, and Theorem 3.2.29, as desired. □

s. *For all a_b, c_d ϵ E, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, the definition of E, and Theorem 3.2.29, as desired. □

t. *For all a_b ϵ E, c_d ϵ F, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, the definition of E, the definition of F, and Theorem 3.2.29, as desired. □

u. *For all a_b, c_d ϵ F, |a_b| * |c_d| = |a_b * c_d|.*

Proof. By Definition 3.1.14, the definition of F, and Theorem 3.2.29, as desired. □

Lemma 3.2.46. *For all x_y ϵ* Z_Q*, |x_y| = |- (x_y)|.*

Proof. This is an immediate consequence of Definition 3.1.10 and 3.1.14, as desired. □

Theorem 3.2.47. *For all a_b, c_d ϵ* Z_Q*, |a_b + c_d| ≤ |a_b| + |c_d|.*

Proof. By Definition 3.1.08, Corollary 3.2.23 and 3.2.25, and Theorem 3.2.35, [(a_b ϵ Z_Q^-) ˅ (a_b = 0_0) ˅ (a_b ϵ Z_Q^+)] ^ [(c_d ϵ Z_Q^-) ˅ (c_d = 0_0) ˅ (c_d ϵ Z_Q^+)] and nine cases arise, three of which are redundant, leaving six cases to consider:

Case 1. Suppose (a_b ϵ Z_Q^-) ^ (c_d ϵ Z_Q^-), then, by Definition 3.1.14 and Theorem 3.2.24:

|a_b + c_d| = - (a_b + c_d);

= - [(a + c)_(b +d)];

= - (a + c)_- (b +d);

= (- a - c)_(- b - d);

= - a_- b + - c_- d;

= - (a_b) + [- (c_d)];

= |a_b| + |c_d|.

Case 2. Suppose (a_b = 0_0) ^ (c_d = 0_0), then, by Definition 3.1.14 and Lemma 3.2.34 and 3.2.42:

|a_b + c_d| = |0_0 + 0_0|;

= |0_0|;

= 0_0;

= 0_0 + 0_0;

= |a_b| + |c_d|.

Case 3. Suppose (a_b ϵ Z_Q^+) ^ (c_d ϵ Z_Q^+), then, by Definition 3.1.14:

|a_b + c_d| = a_b + c_d;

= |a_b| + |c_d|.

Case 4. Suppose (a_b ϵ Z_Q^+) ^ (c_d = 0_0), then, by Definition 3.1.14 and Theorem 3.2.13:

|a_b + c_d| = |a_b + 0_0|;

= |a_b|;

= a_b;

= a_b + 0_0;

= |a_b| + |c_d|.

This result remains unchanged in the case (a_b = 0_0) ^ (c_d ϵ Z_Q^+).

Case 5. Suppose (a_b ϵ Z_Q^-) ^ (c_d = 0_0), then, by Definition 3.1.14, Theorem 3.2.13, and Lemma 3.2.42:

|a_b + c_d| = |a_b + 0_0|;

= |a_b|;

= - (a_b);

$= - (a\_b) + 0\_0$;

$= |a\_b| + |c\_d|$.

This result remains unchanged in the case $(a\_b = 0\_0) \wedge (c\_d \in Z_Q^-)$.

Case 6. Suppose $(a\_b \in Z_Q^-) \wedge (c\_d \in Z_Q^+)$, then, by Definition 3.1.14 and Theorem 3.2.30, $(|a\_b| = - a\_b < c\_d) \vee (c\_d < |a\_b| = - a\_b)$ and two cases arise:

Case 6.a. Suppose $|a\_b| = - (a\_b) < c\_d$, then, by Definition 3.1.14 and Lemma 3.2.34 and 3.2.42:

$|a\_b + c\_d| = a\_b + c\_d$;

$< - (a\_b) + c\_d$;

$= |a\_b| + |c\_d|$.

Case 6.b. Suppose $c\_d < |a\_b| = - (a\_b)$, then, by Definition 3.1.14 and Lemma 3.2.34 and 3.2.42:

$|a\_b + c\_d| = - (a\_b + c\_d)$;

$= - [(a + c)\_(b + d)]$;

$= - (a + c)\_- (b + d)$;

$< (c - a)\_(d - b)$;

$= - a\_- b + c\_d$;

$= - (a\_b) + c\_d$;

$= |a\_b| + |c\_d|$.

In both cases, $|a\_b + c\_d| < |a\_b| + |c\_d|$.

This result remains unchanged in the case $(a\_b \in Z_Q^+) \wedge (c\_d \in Z_Q^-)$.

In all six cases, $(|a\_b + c\_d| = |a\_b| + |c\_d|) \vee (|a\_b + c\_d| < |a\_b| + |c\_d|)$.

Therefore, $|a\_b + c\_d| \leq |a\_b| + |c\_d|$, as desired. □

Theorem 3.2.48. *For all $a\_b, c\_d \in Z_Q$, $|a\_b| - |c\_d| \leq |a\_b - c\_d|$.*

Proof. By Definition 3.1.11 and Theorem 3.2.21 and 3.2.47:

$|(a\_b - c\_d) + c\_d| \leq |a\_b - c\_d| + |c\_d|$;

$|a\_b| \leq |a\_b - c\_d| + |c\_d|$;

$|a\_b| - |c\_d| \leq |a\_b - c\_d|$, as desired. □

Theorem 3.2.49. *For all $a\_b, c\_d \in Z_Q$, $(0\_0 < a\_b) \rightarrow [(|c\_d| < a\_b)$ iff $(- (a\_b) < c\_d < a\_b)$.*

Proof. Suppose $(0\_0 < a\_b) \wedge (|c\_d| < a\_b)$, then, by Theorem 3.2.35, $(c\_d < 0\_0) \vee (0\_0 < c\_d)$ and two cases arise:

Case 1. Suppose $c\_d < 0\_0$, then, by Definition 3.1.14, $|c\_d| = - (c\_d) < a\_b$, hence, by Theorem 3.2.41, and Lemma 3.2.42, $- (a\_b) < c\_d$.

Case 2. Suppose $0\_0 < c\_d$, then, by Definition 3.1.14, $c\_d = |c\_d| < a\_b$.

In both cases, - (a_b) < c_d < a_b.

Therefore, (|c_d| < a_b) → [- (a_b) < c_d < a_b].

Suppose (0_0 < a_b) ^ [- (a_b) < c_d < a_b], then, by Theorem 3.2.35, (c_d < 0_0) ᵛ (0_0 < c_d) and two cases arise:

Case 1. Suppose c_d < 0_0, then, by Definition 3.1.14, Theorem 3.2.41, and Lemma 3.2.42, - (c_d) = |c_d| < - [- (a_b)] = a_b.

Case 2. Suppose 0_0 < c_d, then, by Definition 3.1.14, |c_d| = c_d < a_b.

In both cases, |c_d| < a_b.

Therefore, [- (a_b) < c_d < a_b] → (|c_d| < a_b).

Therefore, (0_0 < a_b) → {(|c_d| < a_b) iff [-(a_b) < c_d < a_b]}, as desired. □

Theorem 3.2.50. *Let S = {a_b | a_b ϵ Z_Q} ≠ ϕ be arbitrary but bounded below, then S has a least element.*

Proof. Suppose S = {a_b | a_b ϵ Z_Q} ≠ ϕ is bounded below, then there exists some x_y ϵ Z_Q such that, for all a_b ϵ S, x_y < a_b and, by Definition 3.1.13, there exists k_p ϵ Z_Q^+ such that a_b = x_y + k_p. Let B = {k_p | (k_p ϵ Z_Q^+) ^ (x_y + k_p = a_b, for some a_b ϵ S)}, then, by Theorem 2.2.25 and 3.2.22, B has a least element. Let m_q be that least element, then, by Theorem 3.2.37, x_y + m_q is the least element of S, as desired. □

Theorem 3.2.51. *For all a_b, c_d ϵ Z_Q, c_d ≠ 0_0, there exist unique m_q, r_s ϵ Z_Q such that (0_0 ≤ r_s < |c_d|) ^ (a_b = m_q * c_d + r_s).*

Proof. The proof is in two parts:

1) *Existence.* By Theorem 3.2.06, m_q * c_d + r_s = a_b ϵ Z_Q and, by Definition 3.1.13 and Theorem 3.2.14, (r_s = 0_0) ᵛ (0_0 < r_s). If r_s = 0_0, then, by Lemma 3.2.44, r_s < |c_d|. Otherwise, let S = {a_b - (k_p * c_d)|(a_b, c_d, k_p ϵ Z_Q) ^ [0_0 ≤ a_b - (k_p * c_d)]}, then, by Theorem 3.2.06 and 3.2.26, k_p = - (c_d) * (a_b)^2 ϵ Z_Q. Then, by Theorem 3.2.16 and Lemma 3.2.42, a_b - (k_p * c_d) = a_b - ([- (c_d) * (a_b)^2] * c_d) = a_b + (c_d)^2 * (a_b)^2 and, by Definition 3.1.08 and 3.1.13 and Theorem 3.2.29, 0_0 ≤ a_b + (a_b)^2 ≤ a_b + (c_d)^2 * (a_b)^2, hence, S ≠ ϕ. By Theorem 3.2.50, S has a least element, r_s. Let a_b - (k_p * c_d) = r_s ϵ S and suppose, for contradiction, that |c_d| ≤ r_s, then, by Theorem 3.2.35, (c_d < 0_0) ᵛ (0_0 < c_d) and two cases arise:

   Case 1. Suppose c_d < 0_0 and let k_p = m_q - 1_0, then, by Definition 3.1.14 and Theorem 3.2.09, 3.2.12, 3.2.16, and Lemma 3.2.30 and 3.2.42, a_b - (k_p * c_d) = a_b - [(m_q - 1_0) * c_d] = (a_b - m_q * c_d) - |c_d| = r_s - |c_d| ϵ S. But then, by Defintion 3.1.13, r_s - |c_d| < r_s, a contradiction.

   Case 2. Suppose 0_0 < c_d and let k_p = m_q + 1_0, then, by Definition 3.1.14 and Theorem 3.2.09, 3.2.12, 3.2.16, and Lemma 3.2.30, a_b - (k_p * c_d) = a_b - [(m_q + 1_0) * c_d] = (a_b - m_q * c_d) - |c_d| = r_s - |c_d| ϵ S. But then, by Defintion 3.1.13, r_s - |c_d| < r_s, a contradiction.

   In both cases, a contradiction, hence, r_s < |c_d|.

   Therefore, (0_0 ≤ r_s < |c_d|) ^ (a_b = m_q * c_d + r_s).

2) *Uniqueness.* Suppose there exists k_p, m_q, r_t, s_u ϵ Z_Q such that k_p * c_d + r_t = m_q * c_d + s_u = a_b, where (0_0 ≤ r_t < |c_d|) ^ (0_0 ≤ s_u < |c_d|). Then, by Definition 3.1.11 and 3.1.14, - |c_d| < s_u - r_t < |c_d| and, by Definition 3.1.11 again, Theorem 3.2.12, and Lemma 3.2.30, (k_p - m_q) * c_d = s_u - r_t. But then, by Defintion 3.1.12 and Theorem 3.2.31, c_d | (s_u - r_t), hence, since - |c_d| < s_u - r_t < |c_d|, s_u - r_t = k_p - m_q = 0_0. Therefore, m_q and r_s are unique.

Therefore, there exist unique m_q, r_s ϵ Z_Q such that (0_0 ≤ r_s < |c_d|) ^ (a_b = m_q * c_d + r_s), as desired. □

Theorem 3.2.52. *For all n ϵ N, a_b ϵ N_Q, $(a\_b)^n$ is defined.*

Proof. We proceed by induction on n. Let P(x) be the property, "$(a\_b)^x$ is defined," then:

P(0). By Definition 3.1.15, $(a\_b)^0$ = 1_0.

Suppose P(n) is true, then $(a\_b)^n$ is defined and:

P(n + 1). By Definition 3.1.06 and 3.1.15, $(a\_b)^{(n + 1)} = (a\_b)^n$ * a_b and, by Theorem 3.2.06, $(a\_b)^{(n + 1)}$ is defined.

Therefore, P(n) → P(n + 1) and, by the Principle of Induction (reference [HJ], Chapter 3, page 42), for all n ϵ N, a_b ϵ N_Q, $(a\_b)^n$ is defined, as desired. □

Lemma 3.2.53. *For all a_b ϵ Z_Q^-, $(a\_b)^n$ is negative if n is odd and positive if n is even.*

Proof. This is an immediate consequence of Definition 3.1.08 and 3.1.15 and Corollary 3.2.43 and 3.2.44, as desired. □

Theorem 3.2.54. *For all n ϵ N, a_b, c_d ϵ Z_Q, $(a\_b * c\_d)^n = (a\_b)^n * (c\_d)^n$.*

Proof. We proceed by induction on n. Let P(x) be the property, "$(a\_b * c\_d)^x = (a\_b)^x * (c\_d)^x$," then:

P(0). By Definition 3.1.15 and Theorem 3.2.16, $(a\_b * c\_d)^0 = 1\_0 = 1\_0 * 1\_0 = (a\_b)^0 * (c\_d)^0$.

Suppose P(n) is true, then $(a\_b * c\_d)^n = (a\_b)^n * (c\_d)^n$ and:

P(n + 1). By Definition 3.1.15 and Theorem 3.2.08 and 3.2.10:

$$
\begin{aligned}
(a\_b * c\_d)^{(n+1)} &= (a\_b * c\_d)^n * (a\_b * c\_d);\\
&= (a\_b)^n * (c\_d)^n * (a\_b * c\_d);\\
&= (a\_b)^n * [(c\_d)^n * a\_b] * c\_d;\\
&= (a\_b)^n * [a\_b * (c\_d)^n] * c\_d;\\
&= [(a\_b)^n * a\_b] * [(c\_d)^n * c\_d];\\
&= (a\_b)^{(n+1)} * (c\_d)^{(n+1)}.
\end{aligned}
$$

Therefore, P(n) → P(n + 1) and, by the Principle of Induction (reference [HJ], Chapter 3, page 42), for all n ϵ N, a_b, c_d ϵ N_Q, $(a\_b * c\_d)^n = (a\_b)^n * (c\_d)^n$, as desired. □

Theorem 3.2.55. *For all m, n ϵ N, a_b ϵ Z_Q, $(a\_b)^m * (a\_b)^n = (a\_b)^{(m+n)}$.*

Proof. We proceed by induction on n. Let P(x) be the property, "$(a\_b)^m * (a\_b)^x = (a\_b)^{(m+x)}$," then:

P(0). By Definition 3.1.15 and Theorem 3.2.16, $(a\_b)^m * (a\_b)^0 = (a\_b)^m * 1\_0 = (a\_b)^m = (a\_b)^{(m+0)}$.

Suppose P(n) is true, then $(a\_b)^m * (a\_b)^n = (a\_b)^{(m+n)}$ and:

P(n + 1). By Definition 3.1.15 and Theorem 3.2.10:

$$
\begin{aligned}
(a\_b)^m * (a\_b)^{(n+1)} &= (a\_b)^m * [(a\_b)^n * a\_b];\\
&= [(a\_b)^m * (a\_b)^n] * a\_b;\\
&= (a\_b)^{(m+n)} * a\_b;
\end{aligned}
$$

$= (a\_b)^{[(m + n) + 1]}$;

$= (a\_b)^{[m + (n + 1)]}$.

Therefore, P(n) → P(n + 1) and, by the Principle of Induction (reference [HJ], Chapter 3, page 42), for all m, n ϵ N, a_b ϵ N_Q, $(a\_b)^m * (a\_b)^n = (a\_b)^{(m + n)}$, as desired. □

Theorem 3.2.56. *For all m, n ϵ N, a_b ϵ Z_Q, $[(a\_b)^m]^n = (a\_b)^{(m * n)}$.*

Proof. We proceed by induction on n. Let P(x) be the property, "$[(a\_b)^m]^x = (a\_b)^{(m * x)}$," then:

P(0). By Definition 3.1.15, $[(a\_b)^m]^0 = 1\_0 = (a\_b)^0 = (a\_b)^{(m * 0)}$.

Suppose P(n) is true, then $[(a\_b)^m]^n = (a\_b)^{(m * n)}$ and:

P(n + 1). By Definition 3.1.15 and Theorem 3.2.55:

$[(a\_b)^m]^{(n + 1)} = [(a\_b)^m]^n * (a\_b)^m$;

$= (a\_b)^{(m * n)} * (a\_b)^m$;

$= (a\_b)^{[(m * n) + m]}$;

$= (a\_b)^{[m * (n + 1)]}$.

Therefore, P(n) → P(n + 1) and, by the Principle of Induction (reference [HJ], Chapter 3, page 42), for all m, n ϵ N, a_b ϵ N_Q, $[(a\_b)^m]^n = (a\_b)^{(m * n)}$, as desired. □

Theorem 3.2.57. *Let S be contained in Z_Q and such that:*

1. *0_0 ϵ S;*
2. *(a_b ϵ S) → {[(a + 1)_b ϵ S] ^ [a_(b + 1) ϵ S]};*
3. *(a_b ϵ S) → [- (a_b) ϵ S].*

Then S = Z_Q.

Proof. By Definition 2.1.09 and 2.1.10, "1" and "2" above define a q-inductive set I_Q. By Definition 2.1.11, that set, I_Q, contains N_Q and, by Definition 3.1.05 and 3.1.08 and Theorem 3.2.26, "3" above extends N_Q to Z_Q, as desired. □

Theorem 3.2.58. *Z_Q is countable.*

Proof. By Theorem 2.2.54, N_Q x N_Q is countable (reference [HJ], Chapter 4, page 75). By Definition 3.1.01 and 3.1.05, N_Q x N_Q contains $Z_Q{}'/E = Z_Q$, hence, Z_Q is at most countable (reference [HJ], Chapter 4, page 77). Finally, by Theorem 3.2.22, Z_Q is countable, as desired. □

Theorem 3.2.59. *Z_Q has neither a greatest nor a least element.*

Proof. By Theorem 2.2.55 and 3.2.22, Z_Q has no greatest element and, by Theorem 3.2.26, Z_Q has no least element, as desired. □

4. Q-Rationals. We develop the q-rationals as equivalence classes of ordered pairs of q-integers, where, for all (a_b, c_d) ϵ Z_Q x Z_Q, c_d ≠ 0_0, (a_b, c_d) is to be considered equivalent to a_b/c_d. Here "/" is equivalent to the "÷" of Definition 3.1.12, however, we extend "÷" in the sense that, if ¬ (c_d |a_b), per Definition 3.1.12, then a_b/c_d defines a new entity.

4.1. Definitions. We define our mathematical entities using standard terminology.

Definition 4.1.01. Let $Q_Q{}'$ = {(a_b, c_d) | (a_b, c_d) ϵ Z_Q, c_d ≠ 0_0}, a proper subset of Z_Q x Z_Q.

Definition 4.1.02. The relation E on $Q_Q{}'$ is defined by:

(k_p, m_q) E (n_r, o_s) iff k_p * o_s = m_q * n_r.

Definition 4.1.03. The operation "+" (addition) on $Q_Q{}'$ is defined by:

+ [(k_p, m_q), (n_r, o_s)] = [(k_p * o_s + m_q * n_r), m_q * o_s].

Definition 4.1.04. The operation "*" (multiplication) on $Q_Q{}''$ is defined by:

* [(k_p, m_q), (n_r, o_s)] = (k_p * n_r, m_q * o_s).

Definition 4.1.05. Let (a_b, c_d) ϵ $Q_Q{}'$ be arbitrary, then the equivalence class of (a_b, c_d) modulo E, $[(a\_b, c\_d)]_E$, which is subject to Definition 4.1.03 and 4.1.04, will be called q-rationals and the set of all such q-rationals will be designated Q_Q.

Definition 4.1.06. Let $[(k\_p, m\_q)]_E$, $[(n\_r, o\_s)]_E$ ϵ Q_Q be arbitrary, then:

+ {$[(k\_p, m\_q)]_E$, $[(n\_r, o\_s)]_E$} = $[\{(k\_p * o\_s + m\_q * n\_r), m\_q * o\_s\}]_E$;

* {$[(k\_p, m\_q)]_E$, $[(n\_r, o\_s)]_E$} = $[(k\_p * n\_r,\ m\_q * o\_s)]_E$.

In those special cases where m_q * o_s = 0_0, both addition and multiplication are undefined.

Definition 4.1.07. The "zero" q-rational, 0_0, is the equivalence class $[(0\_0, m\_q)]_E$ ϵ Q_Q.

Definition 4.1.08. Denote the q-rational number, $[(x\_y, 1\_0)]_E$, by x_y and the q-rational number, $[(a\_b, c\_d)]_E$, by a_b/c_d.

Definition 4.1.09. The operation "-" (subtraction) on Q_Q is defined by:

for all $[(k\_p, m\_q)]_E$, $[(n\_r, o\_s)]_E$ ϵ Q_Q, - ($[(k\_p, m\_q)]_E$, $[(n\_r, o\_s)]_E$) = $[(a\_b, c\_d)]_E$ iff $[(k\_p, m\_q)]_E$ = $[(a\_b, c\_d)]_E$ + $[(n\_r, o\_s)]_E$.

Definition 4.1.10. The operation "÷ " (division) on Q_Q is defined by:

for all $[(k\_p, m\_q)]_E$, $[(n\_r, o\_s)]_E$ ϵ Q_Q, ÷ ($[(k\_p, m\_q)]_E$, $[(n\_r, o\_s)]_E$) = $[(a\_b, c\_d)]_E$ iff $[(k\_p, m\_q)]_E$ = $[(a\_b, c\_d)]_E$ * $[(n\_r, o\_s)]_E$, where $[(n\_r, o\_s)]_E$ ≠ 0_0.

Definition 4.1.11. Denote the additive inverse of $[(a\_b, c\_d)]_E$ by - $[(a\_b, c\_d)]_E$ = - (a_b)/c_d.

Definition 4.1.12. Denote the multiplicative inverse, $[(c\_d, a\_b)]_E$ = c_d/a_b, of $[(a\_b, c\_d)]_E$, by $[(a\_b, c\_d)]_E{}^{-1}$.

Definition 4.1.13. The relation "<" (strict order) on Q_Q is defined by:

for all $[(k\_p, m\_q)]_E$, $[(n\_r, o\_s)]_E$ ϵ Q_Q, < ($[(k\_p, m\_q)]_E$, $[(n\_r, o\_s)]_E$) iff k_p * m_q * $(o\_s)^2$ < n_r * o_s * $(m\_q)^2$.

Definition 4.1.14. Let $[(k\_p, m\_q)]_E$ ϵ Q_Q be arbitrary, then $[(k\_p, m\_q)]_E$ is said to be in lowest terms iff (0_0 < m_q) ^ (gcd (k_p, m_q) = 1_0).

Definition 4.1.5 Let $[(k\_p, m\_q)]_E \in Q_Q$ be arbitrary, then $[(k\_p, m\_q)]_E$ is said to be positive if $[(0\_0, m\_q)]_E < [(k\_p, m\_q)]_E$ and negative if $[(k\_p, m\_q)]_E < [(0\_0, m\_q)]_E$. The set of all positive q-rationals will be designated by Q_Q^+ and the set of all negative q-rationals by Q_Q^-.

Definition 4.1.16. The operation "| |" (absolute value) on Q_Q is defined by:

$$\text{for all } [(k\_p, m\_q)]_E \in Q_Q,\ |[(k\_p, m\_q)]_E| = [(k\_p, m\_q)]_E, \text{ if } [(0\_0, m\_q)]_E \leq [(k\_p, m\_q)]_E;$$

$$= -[(k\_p, m\_q)]_E, \text{ if } [(k\_p, m\_q)]_E < [(0\_0, m\_q)]_E.$$

Definition 4.1.17. For all $a\_b/c\_d \in Q_Q - \{0\_0\}$, and all $k \in Z$, $[(a\_b/c\_d)^0 = 1\_0] \wedge [(a\_b/c\_d)^1 = a\_b/c\_d] \wedge [(a\_b/c\_d)^{k+1} = (a\_b/c\_d)^k * a\_b/c\_d] \wedge [(a\_b/c\_d)^{-k} = (c\_d/a\_b)^k] \wedge \{[(a\_b/c\_d)^{1/k} = e\_f/g\_h] \text{ iff } [(e\_f/g\_h)^k = a\_b/c\_d]\}$.

Definition 4.1.18. Let (P,<) be an arbitrary linearly ordered set, then a gap in (P, <) is an ordered pair, (A, B) of subsets of P such that:

1. $(A, B \neq \phi) \wedge (A \cap B = \phi) \wedge (A \cup B = P)$;
2. for all $a \in A$, $b \in B$, $a < b$;
3. A has no greatest element and B has no least element.

4.2. Arguments. We demonstrate our arguments using the standard methods and terminology of mathematical logic and ZFC/AFA or generalizations thereof. Specific to the current work, we generalize the Principle of Induction to the Principle of Q-Induction and we utilize results from reference [HJ] and [CD].

Theorem 4.2.01. *The set Q_Q' of Definition 4.1.01 exists.*

Proof. By Theorem 3.2.01, Z_Q exists, hence, by the Axiom of Power Set, the definition of ordered pair, and the definition of Cartesian product, Q_Q', which is contained in $Z_Q \times Z_Q$, exists, as desired. □

Theorem 4.2.02. *The relation E on Q_Q', from Definition 4.1.02, is an equivalence relation.*

Proof. The proof is in three parts:

1) *Reflexivity.* Let $(a\_b, c\_d) \in Q_Q'$ be arbitrary, then, by Theorem 3.2.08, $a\_b * c\_d = c\_d * a\_b$, hence, by Definition 4.1.02, $(a\_b, c\_d)\ E\ (a\_b, c\_d)$.
2) *Symmetry.* Let $(k\_p, m\_q), (n\_r, o\_s) \in Q_Q'$ be arbitrary but such that $(k\_p, m\_q)\ E\ (n\_r, o\_s)$, then, by Definition 4.1.02, $k\_p * o\_s = m\_q * n\_r$ and, by Theorem 3.2.08, $n\_r * m\_q = o\_s * k\_p$, hence, by Definition 4.1.02 again, $(n\_r, o\_s)\ E\ (k\_p, m\_q)$.
3) *Transitivity.* Let $(k\_p, m\_q), (n\_r, o\_s), (v\_t, w\_u) \in Q_Q'$ be arbitrary but such that $[(k\_p, m\_q)\ E\ (n\_r, o\_s)] \wedge [(n\_r, o\_s)\ E\ (v\_t, w\_u)]$, then, by Definition 4.1.02, $(k\_p * o\_s = m\_q * n\_r) \wedge (n\_r * w\_u = o\_s * v\_t)$. By Definition 3.1.12 and Theorem 3.2.08 and 3.2.10, $(k\_p * o\_s * w\_u = m\_q * n\_r * w\_u) \wedge (m\_q * n\_r * w\_u = m\_q * o\_s * v\_t)$, hence, $k\_p * o\_s * w\_u = m\_q * o\_s * v\_t$. By Definition 4.1.01, $o\_s \neq 0\_0$ and, by Theorem 3.2.08 and 3.2.10, $(k\_p * w\_u) * o\_s = (m\_q * v\_t) * o\_s$ and, by Definition 3.1.12 again, $k\_p * w\_u = m\_q * v\_t$, hence, by Definition 4.1.02, $(k\_p, m\_q)\ E\ (v\_t, w\_u)$.

Therefore, E is an equivalence relation on Q_Q', as desired. □

Theorem 4.2.03. *For all $(a\_b, c\_d) \in Q_Q'$ and $x\_y \in Z_Q$, $(a\_b, c\_d)\ E\ (a\_b * x\_y, c\_d * x\_y)$.*

Proof. By Theorem 3.2.08 and 3.2.10, $a\_b * c\_d * x\_y = c\_d * a\_b * x\_y$ and, by Definition 4.1.02, $(a\_b, c\_d)\ E\ (a\_b * x\_y, c\_d * x\_y)$, as desired. □

Theorem 4.2.04. *Addition on Q_Q', as defined by Definition 4.1.03, is well-defined relative to the E of Definition 4.1.02.*

Proof. Let (a_b, c_d), (e_f, g_h), (k_p, m_q), (n_r, o_s) ϵ Q_Q' be arbitrary but such that [(a_b, c_d) E (e_f, g_h)] ^ [k_p, m_q) E (n_r, o_s)], then, by Definition 4.1.02, (a_b * g_h = c_d * e_f) ^ (k_p * o_s = m_q * n_r). By Definition 4.1.03, (a_b, c_d) + (k_p, m_q) = [(a_b *m_q + c_d * k_p), c_d * m_q] and (e_f, g_h) + (n_r, o_s) = [(e_f * o_s + g_h * n_r), g_h * o_s] and, by Theorem 3.2.08, 3.2.10, 3.2.11, and 3.2.12:

(a_b *m_q + c_d * k_p) * g_h * o_s = (a_b *m_q) * g_h * o_s + (c_d * k_p) * g_h * o_s;

= a_b *(m_q * g_h) * o_s + c_d * (k_p * g_h) * o_s;

= a_b *(g_h * m_q) * o_s + c_d * (g_h * k_p) * o_s;

= (a_b *g_h) * m_q * o_s + c_d * g_h * (k_p * o_s);

= (c_d *e_f) * m_q * o_s + c_d * g_h * (m_q * n_r);

= c_d *(e_f * m_q) * o_s + c_d * (g_h * m_q) * n_r;

= (c_d * m_q) * e_f * o_s + (c_d * m_q) * g_h * n_r;

= (c_d * m_q) * (e_f * o_s + g_h * n_r).

Therefore, by Definition 4.1.02, [(a_b *m_q + c_d * k_p), c_d * m_q] E [(e_f * o_s + g_h * n_r), g_h * o_s], as desired. □

Theorem 4.2.05. *Multiplication on Q_Q', as defined by Definition 4.1.04, is well-defined relative to the E of Definition 4.1.02.*

Proof. Let (a_b, c_d), (e_f, g_h), (k_p, m_q), (n_r, o_s) ϵ Q_Q' be arbitrary but such that [(a_b, c_d) E (e_f, g_h)] ^ [(k_p, m_q) E (n_r, o_s)], then, by Definition 4.1.02, (a_b * g_h = c_d * e_f) ^ (k_p * o_s = m_q * n_r). By Definition 4.1.04, (a_b, c_d) * (k_p, m_q) = (a_b * k_p, c_d * m_q) and (e_f, g_h) * (n_r, o_s) = (e_f * n_r, g_h * o_s) and, by Theorem 3.2.08 and 3.2.10:

(a_b * k_p) * (g_h * o_s) = [(a_b * k_p) * g_h] * o_s;

= [a_b * (k_p * g_h)] * o_s;

= [a_b * (g_h * k_p)] * o_s;

= (a_b * g_h) * (k_p * o_s);

= (c_d * e_f) * (m_q * n_r);

= [(c_d * e_f) * m_q] * n_r;

= [c_d * (e_f * m_q)] * n_r;

= [c_d * (m_q * e_f)] * n_r;

= (c_d * m_q) * (e_f * n_r).

Therefore, by Definition 4.1.02, (a_b * k_p, c_d * m_q) E (e_f * n_r, g_h * o_s), as desired. □

Theorem 4.2.06. *The set of all q-rationals, Q_Q, is closed under the arithmetical operations "+" (addition) and "*" (multiplication).*

Proof. This is an immediate consequence of Definition 4.1.01, 4.1.05, and 4.1.06 and Theorem 3.2.06, as desired. □

Theorem 4.2.07. *For all* $[(a\_b, c\_d)]_E, [(e\_f, g\_h)]_E \in Q_Q$, $[(a\_b, c\_d)]_E + [(e\_f, g\_h)]_E = [(e\_f, g\_h)]_E + [(a\_b, c\_d)]_E$.

Proof. By Definition 4.1.06 and Theorem 3.2.07 and 3.2.08:

$$\begin{aligned}[(a\_b, c\_d)]_E + [(e\_f, g\_h)]_E &= [\{(a\_b * g\_h + c\_d * e\_f), c\_d * g\_h\}]_E;\\ &= [\{(c\_d * e\_f + a\_b * g\_h), c\_d * g\_h\}]_E;\\ &= [\{(e\_f * c\_d + g\_h * a\_b), g\_h * c\_d\}]_E;\\ &= [(e\_f, g\_h)]_E + [(a\_b, c\_d)]_E.\end{aligned}$$

Therefore, addition on Q_Q is commutative, as desired. □

Theorem 4.2.08. *For all* $[(a\_b, c\_d)]_E, [(e\_f, g\_h)]_E \in Q_Q$, $[(a\_b, c\_d)]_E * [(e\_f, g\_h)]_E = [(e\_f, g\_h)]_E * [(a\_b, c\_d)]_E$.

Proof. By Definition 4.1.06 and Theorem 3.2.08:

$$\begin{aligned}[(a\_b, c\_d)]_E * [(e\_f, g\_h)]_E &= [(a\_b * e\_f, c\_d * g\_h)]_E;\\ &= [(e\_f * a\_b, g\_h * c\_d)]_E;\\ &= [(e\_f, g\_h)]_E * [(a\_b, c\_d)]_E.\end{aligned}$$

Therefore, multiplication on Q_Q is commutative, as desired. □

Theorem 4.2.09. *For all* $[(a\_b, c\_d)]_E, [(e\_f, g\_h)]_E, [(k\_l, m\_n)]_E \in Q_Q$, $([(a\_b, c\_d)]_E + [(e\_f, g\_h)]_E) + [(k\_l, m\_n)]_E = [(a\_b, c\_d)]_E + ([(e\_f, g\_h)]_E + [(k\_l, m\_n)]_E)$.

Proof. Letting $A = ([(a\_b, c\_d)]_E + [(e\_f, g\_h)]_E) + [(k\_l, m\_n)]_E$, by Definition 4.1.06 and Theorem 3.2.10, 3.2.11, and 3.2.12:

$$\begin{aligned}A &= [\{(a\_b * g\_h + c\_d * e\_f), c\_d * g\_h\}]_E + [(k\_l, m\_n)]_E;\\ &= [\{([(a\_b * g\_h + c\_d * e\_f) * m\_n] + (c\_d * g\_h) * k\_l), (c\_d * g\_h) * m\_n\}]_E;\\ &= [\{(a\_b * g\_h * m\_n + c\_d * e\_f * m\_n + c\_d * g\_h * k\_l), c\_d * g\_h * m\_n\}]_E;\\ &= [\{(a\_b * [g\_h * m\_n] + c\_d * [e\_f * m\_n + g\_h * k\_l]), c\_d * [g\_h * m\_n]\}]_E;\\ &= [(a\_b, c\_d)]_E + [\{(e\_f * m\_n + g\_h * k\_l]), g\_h * m\_n\}]_E;\\ &= [(a\_b, c\_d)]_E + ([(e\_f, g\_h)]_E + [(k\_l, m\_n)]_E).\end{aligned}$$

Therefore, addition on Q_Q is associative, as desired. □

Theorem 4.2.10. *For all* $[(a\_b, c\_d)]_E, [(e\_f, g\_h)]_E, [(k\_l, m\_n)]_E \in Q_Q$, $([(a\_b, c\_d)]_E * [(e\_f, g\_h)]_E) * [(k\_l, m\_n)]_E = [(a\_b, c\_d)]_E * ([(e\_f, g\_h)]_E * [(k\_l, m\_n)]_E)$.

Proof. By Definition 4.1.06 and Theorem 3.2.10:

$$\begin{aligned}([(a\_b, c\_d)]_E * [(e\_f, g\_h)]_E) * [(k\_l, m\_n)]_E &= [(a\_b * e\_f, c\_d * g\_h)]_E * [(k\_l, m\_n)]_E;\\ &= [\{(a\_b * e\_f) * k\_l, (c\_d * g\_h) * m\_n\}]_E;\\ &= [\{a\_b * (e\_f * k\_l), c\_d * (g\_h * m\_n)\}]_E;\end{aligned}$$

$$= [(a\_b, c\_d)]_E * [(e\_f * k\_l, g\_h * m\_n)]_E;$$

$$= [(a\_b, c\_d)]_E * ([(e\_f, g\_h)]_E * [(k\_l, m\_n)]_E).$$

Therefore, multiplication on Q_Q is associative, as desired. □

Theorem 4.2.11. *For all $[(a\_b, c\_d)]_E$, $[(e\_f, g\_h)]_E$, $[(k\_l, m\_n)]_E \in Q_Q$, $[(a\_b, c\_d)]_E * ([(e\_f, g\_h)]_E + [(k\_l, m\_n)]_E) = [(a\_b, c\_d)]_E * [(e\_f, g\_h)]_E + [(a\_b, c\_d)]_E * [(k\_l, m\_n)]_E$.*

Proof. Letting $A = [(a\_b, c\_d)]_E * ([(e\_f, g\_h)]_E + [(k\_l, m\_n)]_E)$, by Definition 4.1.06 and Theorem 3.2.08, 3.2.10, 3.2.11, and 4.2.03:

$$A = [(a\_b, c\_d)]_E * [\{(e\_f * m\_n + g\_h * k\_l), g\_h * m\_n\}]_E;$$

$$= [\{[a\_b * (e\_f * m\_n + g\_h * k\_l)], c\_d * g\_h * m\_n\}]_E;$$

$$= [\{(a\_b * e\_f * m\_n + a\_b * g\_h * k\_l), c\_d * g\_h * m\_n\}]_E;$$

$$= [\{[c\_d * (a\_b * e\_f * m\_n + a\_b * g\_h * k\_l)], [c\_d * (c\_d * g\_h * m\_n)]\}]_E;$$

$$= [\{[c\_d * (a\_b * e\_f) * m\_n + c\_d * (a\_b * g\_h) * k\_l], [c\_d * (c\_d * g\_h) * m\_n]\}]_E;$$

$$= [\{[(a\_b * e\_f) * (c\_d * m\_n) + (a\_b * g\_h) * (c\_d * k\_l)], [(c\_d * g\_h) * (c\_d * m\_n)]\}]_E;$$

$$= [(a\_b * e\_f, c\_d * g\_h)]_E + [(a\_b * k\_l, c\_d * m\_n)]_E;$$

$$= [(a\_b, c\_d)]_E * [(e\_f, g\_h)]_E + [(a\_b, c\_d)]_E * [(k\_l, m\_n)]_E.$$

Therefore, multiplication is left distributive over addition on Q_Q, as desired. □

Theorem 4.2.12. *For all $[(a\_b, c\_d)]_E$, $[(e\_f, g\_h)]_E$, $[(k\_l, m\_n)]_E \in Q_Q$, $([(e\_f, g\_h)]_E + [(k\_l, m\_n)]_E) * [(a\_b, c\_d)]_E = [(e\_f, g\_h)]_E * [(a\_b, c\_d)]_E + [(k\_l, m\_n)]_E * [(a\_b, c\_d)]_E$.*

Proof. Letting $A = ([(e\_f, g\_h)]_E + [(k\_l, m\_n)]_E) * [(a\_b, c\_d)]_E$, by Definition 4.1.06 and Theorem 3.2.08, 3.2.10, 3.2.11, and 4.2.03:

$$A = [\{(e\_f * m\_n + g\_h * k\_l), g\_h * m\_n\}]_E * [(a\_b, c\_d)]_E;$$

$$= [\{[(e\_f * m\_n + g\_h * k\_l) * a\_b], (g\_h * m\_n) * c\_d\}]_E;$$

$$= [\{(e\_f * m\_n * a\_b + g\_h * k\_l * a\_b), g\_h * m\_n * c\_d\}]_E;$$

$$= [\{[c\_d * (e\_f * m\_n * a\_b + g\_h * k\_l * a\_b)], [c\_d * (g\_h * m\_n * c\_d)]\}]_E;$$

$$= [\{[c\_d * e\_f * (m\_n * a\_b) + (c\_d * g\_h) * k\_l * a\_b], [(c\_d * g\_h) * m\_n * c\_d]\}]_E;$$

$$= [\{[c\_d * (e\_f * a\_b) * m\_n + (g\_h * c\_d) * (k\_l * a\_b)], [(g\_h * c\_d) * (m\_n * c\_d)]\}]_E;$$

$$= [\{[(e\_f * a\_b) * (c\_d * m\_n) + (g\_h * c\_d) * (k\_l * a\_b)], [(g\_h * c\_d) * (m\_n * c\_d)]\}]_E;$$

$$= [(e\_f * a\_b, g\_h * c\_d)]_E + [(k\_l * a\_b, m\_n * c\_d)]_E;$$

$$= [(e\_f, g\_h)]_E * [(a\_b, c\_d)]_E + [(k\_l, m\_n)]_E * [(a\_b, c\_d)]_E.$$

Therefore, multiplication is right distributive over addition on Q_Q, as desired. □

Theorem 4.2.13. *For all $[(a\_b, c\_d)]_E \in Q_Q$, $[(a\_b, c\_d)]_E + [(0\_0, m\_q)]_E = [(a\_b, c\_d)]_E$.*

Proof. By Definition 4.1.06 and Theorem 3.2.13, 3.2.15, and 4.2.03:

[(a_b, c_d)]$_E$ + [(0_0, m_q)]$_E$ = [{(a_b * m_q + c_d * 0_0, c_d * m_q)]$_E$;

= [{(a_b * m_q + 0_0, c_d * m_q)]$_E$;

= [{(a_b * m_q, c_d * m_q)]$_E$;

= [(a_b, c_d)]$_E$.

Therefore, the zero q-rational, 0_0, is an additive identity for Q$_Q$, as desired. □

Corollary 4.2.14. *For all [(a_b, c_d)]$_E$, [(e_f, g_h)]$_E$ ϵ Q$_Q$, ([(a_b, c_d)]$_E$ + [(e_f, g_h)]$_E$ = [(a_b, c_d)]$_E$) → ([(e_f, g_h)]$_E$ = [(0_0, m_q)]$_E$).*

Proof. Suppose [(a_b, c_d)]$_E$ + [(e_f, g_h)]$_E$ = [(a_b, c_d)]$_E$, then, by Definition 4.1.06, [{(a_b * g_h + c_d * e_f), c_d * g_h)]$_E$ = [(a_b, c_d)]$_E$ and, by Definition 4.1.05 and 4.1.02, (a_b * g_h + c_d * e_f) * c_d = (c_d * g_h) * a_b. By Theorem 3.2.08 and 3.2.12, (a_b * g_h) * c_d + (c_d * e_f) * c_d = (a_b * g_h) * c_d, hence, by Theorem 3.2.13, (c_d * e_f) * c_d = 0_0. By Definition 4.1.01, c_d ≠ 0_0, hence, by Theorem 3.2.15, e_f = 0_0 and [(e_f, g_h)]$_E$ = [(0_0, m_q)]$_E$. Therefore, the additive identity for Q$_Q$ is unique, as desired. □

Theorem 4.2.15. *For all [(a_b, c_d)]$_E$ ϵ Q$_Q$, [(a_b, c_d)]$_E$ * [(m_q, m_q)]$_E$ = [(a_b, c_d)]$_E$.*

Proof. This is an immediate consequence of Definition 4.1.06 and Theorem 4.2.03; therefore, the unity q-rational, 1_0, is a multiplicative identity for Q$_Q$, as desired. □

Corollary 4.2.16. *For all [(a_b, c_d)]$_E$, [(e_f, g_h)]$_E$ ϵ Q$_Q$, ([(a_b, c_d)]$_E$ * [(e_f, g_h)]$_E$ = [(a_b, c_d)]$_E$) → ([(e_f, g_h)]$_E$ = [(m_q, m_q)]$_E$).*

Proof. Suppose [(a_b, c_d)]$_E$ * [(e_f, g_h)]$_E$ = [(a_b, c_d)]$_E$, then, by Definition 4.1.06, [(a_b * e_f, c_d * g_h)]$_E$ = [(a_b, c_d)]$_E$ and, by Definition 4.1.02 and 4.1.05, (a_b * e_f) * c_d = (c_d * g_h) * a_b. By Theorem 3.2.08 and 3.2.10:

(a_b * c_d) * e_f = a_b * (c_d * e_f);

= a_b * (e_f * c_d);

= (a_b * e_f) * c_d;

= (c_d * g_h) * a_b;

= c_d * (g_h * a_b);

= c_d * (a_b * g_h);

= (c_d * a_b) * g_h;

= (a_b * c_d) * g_h.

By Definition 3.1.12 and Theorem 3.2.16, e_f = g_h, hence, by Definition 4.1.02 and 4.1.05, [(e_f, g_h)]$_E$ = [(m_q, m_q)]$_E$.

Therefore, the multiplicative identity for Q$_Q$, 1_0, is unique, as desired. □

Lemma 4.2.17. *For all [(a_b, c_d)]$_E$, [(e_f, c_d)]$_E$ ϵ Q$_Q$, [(a_b, c_d)]$_E$ + [(e_f, c_d)]$_E$ = [{(a_b + e_f), c_d}]$_E$.*

Proof. By Definition 4.1.06 and Theorem 4.2.08 and 4.2.11:

[(a_b, c_d)]$_E$ + [(e_f, c_d)]$_E$ = [{(a_b * c_d + c_d * e_f), (c_d)^2}]$_E$;

= [{(c_d * a_b + c_d * e_f), (c_d)^2}]$_E$;

= [{c_d * (a_b + e_f), (c_d)^2}]$_E$.

Therefore, (c_d)^2 * (a_b + e_f) = (c_d)^2 * (a_b + e_f) and, by Definition 4.1.02, [(a_b, c_d)]$_E$ + [(e_f, c_d)]$_E$ = [{(a_b + e_f), c_d}]$_E$, as desired. □

Lemma 4.2.18. *For all [(a_b, 1_0)]$_E$, [(c_d, 1_0)]$_E$ ϵ Q_Q, [(a_b, 1_0)]$_E$ * [(c_d, 1_0)]$_E$ = [(a_b * c_d, 1_0)]$_E$.*

Proof. This is an immediate consequence of Definition 4.1.06 and Theorem 3.2.16, as desired. □

Theorem 4.2.19. *For all a_b/c_d ϵ Q_Q, a_b/c_d + - (a_b)/c_d) = 0_0.*

Proof. By Definition 3.1.11 and 4.1.07, Theorem 3.2.24, and Lemma 4.2.17:

a_b/c_d + [- (a_b)]/c_d) = {a_b + [- (a_b)]}/c_d;

= 0_0/c_d;

= 0_0, as desired. □

Theorem 4.2.20. *Q_Q is closed under subtraction.*

Proof. This is an immediate consequence of Definition 4.1.09 and Theorem 3.2.06 and 3.2.31, as desired. □

Theorem 4.2.21. *For all a_b/c_d, e_f/g_h ϵ Q_Q, (a_b/c_d) - (e_f/g_h) = (a_b/c_d) + [- (e_f)/g_h] = [(a_b * g_h) - (c_d * e_f)]/(c_d * g_h) .*

Proof. By Definition 4.1.01, 4.1.05, 4.1.06, and 4.1.08, Lemma 3.2.30, and Corollary 3.2.43:

(a_b/c_d) + [- (e_f)/g_h] = [(a_b * g_h) + ([- (e_f)] * c_d)]/(c_d * g_h);

= {(a_b * g_h) + [- (e_ f * c_d)]}/(c_d * g_h);

= {(a_b * g_h) + [- (c_d * e_f)]}/(c_d * g_h);

= [(a_b * g_h) - (c_d * e_f)]/(c_d * g_h).

By Definition 4.1.01, 4.1.05, 4.1.06, and 4.1.08, Theorem 3.2.12, 3.2.18, and 4.2.15, and Lemma 3.2.30:

a_b/c_d = [(a_b * g_h) - (c_d * e_f)]/(c_d * g_h) + e_f/g_h;

= {[(a_b * g_h) - (c_d * e_f)] * g_h + (c_d * g_h * e_f)} /[c_d * (g_h)^2];

= {[a_b * (g_h)^2] - (c_d * e_f * g_h) + (c_d * g_h * e_f)} /[c_d * (g_h)^2];

= [a_b * (g_h)^2] /[c_d * (g_h)^2];

= a_b/c_d.

Therefore, by Definition 4.1.09, (a_b/c_d) - (e_f/g_h) = (a_b/c_d) + [- (e_f)/g_h] = [(a_b * g_h) - (c_d * e_f)]/(c_d * g_h), as desired. □

Theorem 4.2.22. *For all a_b/c_d ϵ Q_Q – {0_0}, a_b/c_d * c_d/a_b = 1_0.*

Proof. By Definition 3.1.12 and 4.1.06 and Theorem 3.2.08:

a_b/c_d * c_d/a_b = (a_b * c_d)/(c_d * a_b);

= (a_b * c_d)/(a_b * c_d);

= 1_0.

Therefore, c_d/a_b is the multiplicative inverse for a_b/c_d, as desired. □

Lemma 4.2.23. *For all a_b/c_d, e_f/g_h ϵ Q_Q, - (a_b)/c_d * e_f/g_h = - (a_b * e_f)/(c_d * g_h).*

Proof. This is an immediate consequence of Definition 4.1.01, 4.1.05, and 4.1.06 and Corollary 3.2.43, as desired. □

Lemma 4.2.24. *For all a_b/c_d, e_f/g_h ϵ Q_Q, - (a_b)/c_d * -(e_f)/g_h = (a_b * e_f)/(c_d * g_h).*

Proof. This is an immediate consequence of Definition 4.1.01, 4.1.05, and 4.1.06 and Corollary 3.2.44, as desired. □

Theorem 4.2.25. *For all a_b/c_d ϵ Q_Q, a_b/c_d * 0_0 = 0_0.*

Proof. This is an immediate consequence of Definition 4.1.06 and 4.1.07 and Theorem 3.2.15, as desired. □

Corollary 4.2.26. *0_0 does not have a multiplicative inverse.*

Proof. By Theorem 4.2.25, as desired. □

Theorem 4.2.27. *For all a_b/c_d, e_f/g_h ϵ Q_Q, e_f/g_h ≠ 0_0, (a_b/c_d) ÷ (e_f/g_h) = (a_b/c_d) *(e_f/g_h)$^{-1}$ = (a_b * g_h)/(c_d * e_f).*

Proof. By Definition 4.1.06 and Theorem 4.2.22:

(a_b/c_d) *(e_f/g_h)$^{-1}$ = (a_b/c_d) *(g_h/e_f);

= (a_b * g_h)/(c_d * e_f).

Now, suppose, for contradiction, that (a_b/c_d) ÷ (e_f/g_h) ≠ (a_b * g_h)/(c_d * e_f), then, by Definition 4.1.10 and Theorem 4.2.10, 4.2.14, and 4.2.22:

a_b/c_d ≠ [(a_b * g_h)/(c_d * e_f)] * (e_f/g_h);

≠ [(a_b/c_d) * (g_h/e_f)] * (e_f/g_h);

≠ (a_b/c_d) * [(g_h/e_f) * (e_f/g_h)];

≠ (a_b/c_d) * 1_0;

≠ a_b/c_d, a contradiction.

Therefore, (a_b/c_d) ÷ (e_f/g_h) = (a_b/c_d) *(e_f/g_h)$^{-1}$ = (a_b * g_h)/(c_d * e_f), as desired. □

Theorem 4.2.28. *The relation "<" (strict order) on Q_Q, as defined by Definition 4.1.13, is well-defined relative to the E of Definition 4.1.02.*

Proof. Let (a_b, c_d), (e_f, g_h), (k_p, m_q), (n_r, o_s) ϵ Q_Q be arbitrary but such that [(a_b, c_d) E (e_f, g_h)] ^ [(k_p, m_q) E (n_r, o_s)], then, by Definition 4.1.02, (a_b * g_h = c_d * e_f) ^ (k_p * o_s = m_q * n_r) and, by Definition 3.1.12, [a_b = (c_d * e_f)/g_h] ^ [k_p = (m_q * n_r)/o_s]. Suppose (a_b, c_d) < (k_p, m_q), then, by Definition 4.1.13, (a_b * c_d) * (m_q)^2 < (k_p *m_q) * (c_d)^2 and, by Definition 3.1.13, there exists t_u ϵ Z_Q^+ such that (a_b * c_d) * (m_q)^2 + t_u = (k_p *m_q) * (c_d)^2. But then, by substitution, Definition 3.1.11, 3.1.12, 3.1.15, 4.1.06, and 4.1.10, and Theorem 3.2.08, 3.2.10, and 3.2.11:

a_b * c_d * (m_q)^2 + t_u = k_p *m_q * (c_d)^2;

[(c_d * e_f)/g_h] * c_d * (m_q)^2 + t_u = [(m_q * n_r)/o_s] *m_q * (c_d)^2;

$$[(e\_f * c\_d)/g\_h] * c\_d * (m\_q)^2 + t\_u = [(n\_r * m\_q)/o\_s] * m\_q * (c\_d)^2;$$

$$[e\_f * (c\_d)^2 * (m\_q)^2]/g\_h + t\_u = [n\_r * (m\_q)^2 * (c\_d)^2]/o\_s;$$

$$(g\_h)^2 * (o\_s)^2 * ([e\_f * (c\_d)^2 * (m\_q)^2]/g\_h + t\_u) = ([n\_r * (m\_q)^2 * (c\_d)^2]/o\_s) * (g\_h)^2 * (o\_s)^2;$$

$$[(g\_h)^2 * (o\_s)^2 * e\_f * (c\_d)^2 * (m\_q)^2]/g\_h + (g\_h)^2 * (o\_s)^2 * t\_u = [n\_r * (m\_q)^2 * (c\_d)^2 * (g\_h)^2 * (o\_s)^2]/o\_s;$$

$$\{[g\_h * (o\_s)^2] * e\_f\} * (c\_d)^2 * (m\_q)^2 + (g\_h)^2 * (o\_s)^2 * t\_u = n\_r * \{[(m\_q)^2 * (c\_d)^2] * [(g\_h)^2 * o\_s]\};$$

$$\{e\_f * [g\_h * (o\_s)^2]\} * (c\_d)^2 * (m\_q)^2 + (g\_h)^2 * (o\_s)^2 * t\_u = n\_r * \{[o\_s * (g\_h)^2] * [(m\_q)^2 * (c\_d)^2]\};$$

$$e\_f * g\_h * (o\_s)^2 * (c\_d)^2 * (m\_q)^2 + (g\_h)^2 * (o\_s)^2 * t\_u = n\_r * o\_s * (g\_h)^2 * (m\_q)^2 * (c\_d)^2.$$

Hence, by Definition 3.1.12 and Theorem 3.2.29, $e\_f * g\_h * (o\_s)^2 * (c\_d)^2 * (m\_q)^2 < n\_r * o\_s * (g\_h)^2 * (m\_q)^2 * (c\_d)^2$. Suppose, for contradiction, that $n\_r * o\_s * (g\_h)^2 < e\_f * g\_h * (o\_s)^2$, then, by Theorem 3.2.41, $(c\_d)^2 * (m\_q)^2 \in Z_Q^-$, contradicting Theorem 3.2.29. Therefore, by Definition 4.1.13, (e_f, g_h) < (n_r, o_s) and the relation "<" on Q_Q is well-defined, as desired. □

Theorem 4.2.29. *Let $S = \{[(a\_b, 1\_0)]_E \mid \{[(a\_b, 1\_0)]_E \in Q_Q\}$, then (S, <, +, *) is ring isomorphic to (Z_Q, <, +, *).*

Proof. There is an obvious isomorphism, $f:S \to Z_Q$, defined by $f([(a\_b, 1\_0)]_E) = a\_b$. That f is a ring isomorphism follows immediately from Lemma 4.2.17 and 4.2.18, as desired. □

Theorem 4.2.30. *(Q_Q, <) is a linearly ordered set.*

Proof. The proof is in three parts:

1) *Transitivity.* Let a_b/c_d, e_f/g_h, k_p/m_q ϵ Q_Q be arbitrary but such that (a_b/c_d < e_f/g_h) ^ (e_f/g_h < k_p/m_q), then, by Definition 4.1.13, $[a\_b * c\_d * (g\_h)^2 < e\_f * g\_h * (c\_d)^2] \wedge [e\_f * g\_h * (m\_q)^2 < k\_p * m\_q * (g\_h)^2]$. By Definition 4.1.01 and 4.1.05 and Theorem 3.2.06 and 3.2.35, $a\_b * c\_d * (g\_h)^2 < k\_p * m\_q * (g\_h)^2$, hence, by Definition 4.1.13 again, a_b/c_d < k_p/m_q.
2) *Asymmetry.* Let a_b/c_d, e_f/g_h ϵ Q_Q be arbitrary and suppose, for contradiction, that (a_b/c_d < e_f/g_h) ^ (e_f/g_h < a_b/c_d), then, by transitivity, a_b/c_d < a_b/c_d, a contradiction.
3) *Linearity.* Let a_b/c_d, e_f/g_h, ϵ Q_Q be arbitrary, then, by Definition 4.1.01 and 4.1.05 and Theorem 3.2.06, $a\_b * c\_d * (g\_h)^2, e\_f * g\_h * (c\_d)^2 \in Z_Q$ and, by Theorem 3.2.35, $[a\_b * c\_d * (g\_h)^2 < e\_f * g\_h * (c\_d)^2] \vee [a\_b * c\_d * (g\_h)^2 = e\_f * g\_h * (c\_d)^2] \vee [e\_f * g\_h * (c\_d)^2 < a\_b * c\_d * (g\_h)^2]$, hence, by Definition 4.1.13, (a_b/c_d < e_f/g_h) ˅ (a_b/c_d = e_f/g_h) ˅ (e_f/g_h < a_b/c_d).

Therefore, (Q_Q, <) is a linearly ordered set, as desired. □

Theorem 4.2.31. *For all a_b/c_d, e_f/g_h, k_p/m_q ϵ Q_Q, (a_b/c_d < e_f/g_h) iff [(a_b/c_d + k_p/m_q) < (e_f/g_h + k_p/m_q)].*

Proof. Suppose a_b/c_d < e_f/g_h, then, by Definition 4.1.13, $a\_b * c\_d * (g\_h)^2 < e\_f * g\_h * (c\_d)^2$. Now, suppose, for contradiction, that (e_f/g_h + k_p/m_q) < (a_b/c_d + k_p/m_q), then, by Definition 4.1.06, e_f/g_h + k_p/m_q = [(e_f * m_q + g_h * k_p)/g_h * m_q] < [(a_b * m_q + c_d * k_p)/c_d * m_q] = a_b/c_d + k_p/m_q. Letting $A = (e\_f * m\_q + g\_h * k\_p) * (g\_h * m\_q) * (c\_d * m\_q)^2$, by Definition 3.1.15 and 4.1.13 and Theorem 3.2.08, 3.2.10, and 3.2.12:

$$A = \{(e\_f * m\_q) * [g\_h * m\_q * (c\_d * m\_q)^2]\} + \{(g\_h * k\_p) * [g\_h * m\_q * (c\_d * m\_q)^2]\};$$

$$= \{(e\_f * m\_q) * [g\_h * m\_q * (c\_d)^2 * (m\_q)^2]\} + \{(k\_p * g\_h) * [g\_h * m\_q * (c\_d)^2 * (m\_q)^2]\};$$

$$= \{e\_f * m\_q * [g\_h * (c\_d)^2] * m\_q * (m\_q)^2\} + \{k\_p * (g\_h * g\_h) * (c\_d)^2 * m\_q * (m\_q)^2\};$$

= {e_f * [g_h * (c_d)^{2}] * m_q * (m_q)^{3}} + {k_p * (g_h)^{2} * (c_d)^{2} * (m_q)^{3}};

= {e_f * g_h * (c_d)^{2}* (m_q)^{4}} + {k_p * (g_h)^{2} * (c_d)^{2} * (m_q)^{3}};

< (a_b * m_q + c_d * k_p) * [c_d * m_q * (g_h * m_q)^{2}];

= {(a_b * m_q) * [c_d * m_q * (g_h)^{2} * (m_q)^{2}]} + {(c_d * k_p) * [c_d * m_q * (g_h)^{2} * (m_q)^{2}]};

= {a_b * (m_q * c_d) * [m_q * (g_h)^{2}] * (m_q)^{2}} + {(k_p * c_d) * c_d * [m_q * (g_h)^{2}] * (m_q)^{2}};

= {a_b * (c_d * m_q) * [(g_h)^{2} * m_q] * (m_q)^{2}} + {k_p * (c_d)^{2} * [(g_h)^{2} * m_q] * (m_q)^{2}};

= {a_b * c_d * [m_q * (g_h)^{2}] * (m_q)^{3}} + {k_p * [(c_d)^{2} * (g_h)^{2}] * (m_q)^{3}};

= {a_b * c_d * [(g_h)^{2} * m_q] * (m_q)^{3}} + {k_p * [(g_h)^{2} * (c_d)^{2}] * (m_q)^{3}};

= {a_b * c_d * (g_h)^{2} * (m_q)^{4}} + {k_p * (g_h)^{2} * (c_d)^{2} * (m_q)^{3}}.

But then, by Theorem 3.2.37, e_f * g_h * (c_d)^{2} * (m_q)^{4}< a_b * c_d * (g_h)^{2} * (m_q)^{4}. Suppose, for contradiction, that a_b * c_d * (g_h)^{2} < e_f * g_h * (c_d)^{2}, then, by Theorem 3.2.41, (m_q)^{4} ϵ Z_Q^-, contradicting Theorem 3.2.29 and Lemma 3.2.53, hence, under the current supposition, e_f * g_h * (c_d)^{2} < a_b * c_d * (g_h)^{2}, a contradiction.

Therefore, (a_b/c_d + k_p/m_q) < (e_f/g_h + k_p/m_q).

Therefore, (a_b/c_d < e_f/g_h) → [(a_b/c_d + k_p/m_q) < (e_f/g_h + k_p/m_q)].

Suppose (a_b/c_d + k_p/m_q) < (e_f/g_h + k_p/m_q), then, letting A = (a_b * m_q + k_p * c_d) * [(c_d * m_q) * (g_h * m_q)^{2}], by Definition 3.1.15, 4.1.06, and 4.1.13 and Theorem 3.2.08, 3.2.10, and 3.2.12:

A = {(a_b * m_q) * [(c_d * m_q) * (g_h * m_q)^{2}]} + {(k_p * c_d) * [(c_d * m_q) * (g_h * m_q)^{2}]};

= {a_b * (m_q * c_d) * [m_q * (g_h)^{2}] * (m_q)^{2}} + {k_p * (c_d * c_d) * m_q * (g_h)^{2} * (m_q)^{2}};

= {a_b * (c_d * m_q) * [(g_h)^{2} * m_q] * (m_q)^{2}} + {k_p * (c_d)^{2} * [m_q * (g_h)^{2}] * (m_q)^{2}};

= {a_b * c_d * [m_q * (g_h)^{2}] * (m_q)^{3}} + {k_p * (c_d)^{2} * [(g_h)^{2} * m_q] * (m_q)^{2}};

= {a_b * c_d * [(g_h)^{2} * m_q] * (m_q)^{3}} + {k_p * (c_d)^{2} * (g_h)^{2} * (m_q)^{3}};

= {a_b * c_d * (g_h)^{2} * (m_q)^{4}} + {k_p * (c_d)^{2} * (g_h)^{2} * (m_q)^{3}};

< {e_f * g_h * (c_d)^{2} * (m_q)^{4}} + {k_p * (g_h)^{2} * (c_d)^{2} * (m_q)^{3}};

= {e_f * g_h * [(c_d)^{2} * m_q] * (m_q)^{3}} + {k_p * (g_h)^{2} * (c_d)^{2} * (m_q)^{3}};

= {e_f * g_h * [m_q * (c_d)^{2}] * (m_q)^{3}} + {k_p * (g_h)^{2} * [(c_d)^{2} * m_q] * (m_q)^{2}};

= {e_f * (g_h * m_q) * [(c_d)^{2} * m_q] * (m_q)^{2}} + {k_p * (g_h)^{2} * [m_q * (c_d)^{2}] * (m_q)^{2}};

= {e_f * (m_q * g_h) * [m_q * (c_d)^{2}] * (m_q)^{2}} + {k_p * (g_h * g_h) * m_q * (c_d)^{2} * (m_q)^{2}};

= {(e_f * m_q) * [(g_h * m_q) * (c_d * m_q)^{2}]} + {(k_p * g_h) * [(g_h * m_q) * (c_d * m_q)^{2}]};

= (e_f * m_q + k_p * g_h) * [(g_h * m_q) * (c_d * m_q)^{2}].

Therefore, by Definition 4.1.13 and Theorem 3.2.37, a_b/c_d < e_f/g_h.

Therefore, [(a_b/c_d + k_p/m_q) < (e_f/g_h + k_p/m_q)] → (a_b/c_d < e_f/g_h).

Therefore, (a_b/c_d < e_f/g_h) iff [(a_b/c_d + k_p/m_q) < (e_f/g_h + k_p/m_q)], as desired. □

Theorem 4.2.32. *For all a_b/c_d, e_f/g_h, k_p/m_q, n_r/o_s ϵ Q_Q, [(a_b/c_d < k_p/m_q) ^ (e_f/g_h < n_r/o_s)] → [(a_b/c_d + e_f/g_h) < (k_p/m_q + n_r/o_s)].*

Proof. Suppose (a_b/c_d < k_p/m_q) ^ (e_f/g_h < n_r/o_s), then, by Theorem 4.2.07, 4.2.19, 4.2.21, 4.2.30, and 4.2.31:

a_b/c_d + (e_f/g_h - k_p/m_q) < k_p/m_q + (e_f/g_h - k_p/m_q);
< k_p/m_q + (e_f/g_h + [- (k_p)]/m_q);
< k_p/m_q + ([- (k_p)]/m_q + e_f/g_h);
< (k_p/m_q + [- (k_p)]/m_q) + e_f/g_h;
< e_f/g_h;
< n_r/o_s.

But then:

a_b/c_d + (e_f/g_h - k_p/m_q) + k_p/m_q < n_r/o_s + k_p/m_q;
a_b/c_d + e_f/g_h + ([- (k_p)]/m_q + k_p/m_q) < n_r/o_s + k_p/m_q;
a_b/c_d + e_f/g_h + (k_p/m_q + [- (k_p)]/m_q) < n_r/o_s + k_p/m_q;
a_b/c_d + e_f/g_h < n_r/o_s + k_p/m_q.

Therefore, [(a_b/c_d < k_p/m_q) ^ (e_f/g_h < n_r/o_s)] → [(a_b/c_d + e_f/g_h) < (k_p/m_q + n_r/o_s)], as desired. □

Theorem 4.2.33. *Let S = {a_b/c_d |(a_b/c_d ϵ Q_Q^+) ^ [(a≠ 0) ^ {(0 < b) ˅ [(b < 0) ^ (|b| < a)]}] ^ [(c≠ 0) ^ {(0 < d) ˅ [(d < 0) ^ (|d| < c)]}]}, then, for all a_b/c_d, e_f/g_h ϵ Q_Q, k_p/m_q ϵ S, (a_b/c_d < e_f/g_h) iff [(a_b/c_d * k_p/m_q) < (e_f/g_h * k_p/m_q)].*

Proof. Suppose (a_b/c_d < e_f/g_h), then, by Definition 3.1.15, 4.1.06, and 4.1.13 and Theorem 3.2.08, 3.2.10, 3.2.29, and 3.2.40:

(a_b * k_p) * (c_d * m_q) * (g_h * m_q)^2 = a_b * (k_p * c_d) * [m_q * (g_h)^2] * (m_q)^2;
= a_b * (c_d * k_p) * [(g_h)^2 * m_q] * (m_q)^2;
= a_b * c_d * [k_p * (g_h)^2] * (m_q)^3;
= a_b * c_d * [(g_h)^2 * k_p] * (m_q)^3;
= [a_b * c_d * (g_h)^2] * [k_p * (m_q)^3];
< [e_f * g_h * (c_d)^2] * [k_p * (m_q)^3];
= e_f * g_h * [(c_d)^2 * k_p] * (m_q)^3;
= e_f * g_h * [k_p * (c_d)^2] * (m_q)^3;
= e_f * (g_h * k_p) * [(c_d)^2 * m_q] * (m_q)^2;
= e_f * (k_p * g_h) * [m_q * (c_d)^2] * (m_q)^2;
= (e_f * k_p) * (g_h * m_q) * (c_d * m_q)^2.

Therefore, by Definition 4.1.13, (a_b/c_d * k_p/m_q) < (e_f/g_h * k_p/m_q).

Therefore, (a_b/c_d < e_f/g_h) → [(a_b/c_d * k_p/m_q) < (e_f/g_h * k_p/m_q)].

Suppose (a_b/c_d * k_p/m_q) < (e_f/g_h * k_p/m_q), then, by Definition 3.1.15, 4.1.06, and 4.1.13 and Theorem 3.2.08 and 3.2.10:

(a_b * k_p) * (c_d * m_q) * (g_h * m_q)^{2} = a_b * (k_p * c_d) * [m_q * (g_h)^{2}] * (m_q)^{2};

= a_b * (c_d * k_p) * [(g_h)^{2} * m_q] * (m_q)^{2};

= a_b * c_d * [k_p * (g_h)^{2}] * (m_q)^{3};

= a_b * c_d * [(g_h)^{2} * k_p] * (m_q)^{3};

= [a_b * c_d * (g_h)^{2}] * [k_p * (m_q)^{3}];

< [e_f * g_h * (c_d)^{2}] * [k_p * (m_q)^{3}];

= e_f * g_h * [(c_d)^{2} * k_p] * (m_q)^{3};

= e_f * g_h * [k_p * (c_d)^{2}] * (m_q)^{3};

= e_f * (g_h * k_p) * [(c_d)^{2} * m_q] * (m_q)^{2};

= e_f * (k_p * g_h) * [m_q * (c_d)^{2}] * (m_q)^{2};

= (e_f * k_p) * (g_h * m_q) * (c_d * m_q)^{2}.

Therefore, by Theorem 3.2.29, a_b/c_d < e_f/g_h.

Therefore, [(a_b/c_d * k_p/m_q) < (e_f/g_h * k_p/m_q)] → (a_b/c_d < e_f/g_h).

Therefore, (a_b/c_d < e_f/g_h) iff [(a_b/c_d * k_p/m_q) < (e_f/g_h * k_p/m_q)], as desired. □

Theorem 4.2.34. *Let T = {a_b/c_d | (a_b/c_d ϵ Q_{Q}^{-}) ^ [(a≠ 0) ^ {(b < 0) ˅ [(0 < b) ^ (b < |a|)]}] ^ [(c≠ 0) ^ {(d < 0) ˅ [(0 < d) ^ (d < |c|)]}]}, then, for all a_b/c_d, e_f/g_h ϵ Q_{Q}, k_p/m_q ϵ T, (a_b/c_d < e_f/g_h) iff [(e_f/g_h * k_p/m_q) < (a_b/c_d * k_p/m_q)].*

Proof. Suppose (a_b/c_d < e_f/g_h), then, by Definition 3.1.15, 4.1.06, and 4.1.13, and Theorem 3.2.08, 3.2.10, 3.2.29, and 3.2.41:

(e_f * k_p) * (g_h * m_q) * (c_d * m_q)^{2} = e_f * (k_p * g_h) * [m_q * (c_d)^{2}] * (m_q)^{2};

= e_f * (g_h * k_p) * [(c_d)^{2} * m_q] * (m_q)^{2};

= e_f * g_h * [k_p * (c_d)^{2}] * (m_q)^{3};

= e_f * g_h * [(c_d)^{2} * k_p] * (m_q)^{3};

= [e_f * g_h * (c_d)^{2}] * [k_p * (m_q)^{3}];

< [a_b * c_d * (g_h)^{2}] * [k_p * (m_q)^{3}];

= a_b * c_d * [(g_h)^{2} * k_p] * (m_q)^{3};

= a_b * c_d * [k_p * (g_h)^{2}] * (m_q)^{3};

= a_b * (c_d * k_p) * [(g_h)^{2} * m_q] * (m_q)^{2};

= a_b * (k_p * c_d) * [m_q * (g_h)^{2}] * (m_q)^{2};

$$= (a\_b * k\_p) * (c\_d * m\_q) * (g\_h * m\_q)^2.$$

Therefore, by Definition 4.1.13, $(e\_f/g\_h * k\_p/m\_q) < (a\_b/c\_d * k\_p/m\_q)$.

Therefore, $(a\_b/c\_d < e\_f/g\_h) \rightarrow [(e\_f/g\_h * k\_p/m\_q) < (a\_b/c\_d * k\_p/m\_q)]$.

Suppose $(e\_f/g\_h * k\_p/m\_q) < (a\_b/c\_d * k\_p/m\_q)$, then, by Definition 3.1.15, 4.1.06, and 4.1.13 and Theorem 3.2.08 and 3.2.10:

$$(e\_f * k\_p) * (g\_h * m\_q) * (c\_d * m\_q)^2 = e\_f * (k\_p * g\_h) * [m\_q * (c\_d)^2] * (m\_q)^2;$$

$$= e\_f * (g\_h * k\_p) * [(c\_d)^2 * m\_q] * (m\_q)^2;$$

$$= e\_f * g\_h * [k\_p * (c\_d)^2] * (m\_q)^3;$$

$$= e\_f * g\_h * [(c\_d)^2 * k\_p] * (m\_q)^3;$$

$$= [e\_f * g\_h * (c\_d)^2] * [k\_p * (m\_q)^3];$$

$$< [a\_b * c\_d * (g\_h)^2] * [k\_p * (m\_q)^3];$$

$$= a\_b * c\_d * [(g\_h)^2 * k\_p] * (m\_q)^3;$$

$$= a\_b * c\_d * [k\_p * (g\_h)^2] * (m\_q)^3;$$

$$= a\_b * (c\_d * k\_p) * [(g\_h)^2 * m\_q] * (m\_q)^2;$$

$$= a\_b * (k\_p * c\_d) * [m\_q * (g\_h)^2] * (m\_q)^2;$$

$$= (a\_b * k\_p) * (c\_d * m\_q) * (g\_h * m\_q)^2.$$

Therefore, by Theorem 3.2.41, $a\_b/c\_d < e\_f/g\_h$.

Therefore, $[(e\_f/g\_h * k\_p/m\_q) < (a\_b/c\_d * k\_p/m\_q)] \rightarrow (a\_b/c\_d < e\_f/g\_h)$.

Therefore, $(a\_b/c\_d < e\_f/g\_h)$ iff $[(e\_f/g\_h * k\_p/m\_q) < (a\_b/c\_d * k\_p/m\_q)]$, as desired. □

Lemma 4.2.35. *For all $a\_b/c\_d, e\_f/g\_h \in Q_Q^+$, $(a\_b/c\_d < e\_f/g\_h)$ iff $(g\_h/e\_f < c\_d/a\_b)$.*

Proof. Suppose $a\_b/c\_d < e\_f/g\_h$, then, by Definition 3.1.15, and 4.1.13 and Theorem 4.2.08, 4.2.10, 4.2.15, 4.2.22, 4.2.29, and 4.2.33:

$$a\_b * c\_d * (g\_h)^2 < e\_f * g\_h * (c\_d)^2;$$

$$a\_b * c\_d * (g\_h * g\_h) * (1\_0/c\_d * 1\_0/g\_h) < e\_f * g\_h * (c\_d * c\_d) * (1\_0/g\_h * 1\_0/c\_d);$$

$$a\_b * c\_d * (1\_0/c\_d * 1\_0/g\_h) * (g\_h * g\_h) < e\_f * g\_h * (1\_0/g\_h * 1\_0/c\_d) * (c\_d * c\_d);$$

$$a\_b * (c\_d * 1\_0/c\_d) * (1\_0/g\_h * g\_h) * g\_h < e\_f * (g\_h * 1\_0/g\_h) * (1\_0/c\_d * c\_d) * c\_d;$$

$$a\_b * 1\_0 * 1\_0 * g\_h < e\_f * 1\_0 * 1\_0 * c\_d;$$

$$a\_b * g\_h < e\_f * c\_d.$$

Now suppose, for contradiction, that $c\_d/a\_b \leq g\_h/e\_f$, then, by Definition 3.1.15, and 4.1.13 and Theorem 4.2.08, 4.2.10, 4.2.15, 4.2.22, 4.2.29, and 4.2.33:

$$c\_d * a\_b * (e\_f)^2 \leq g\_h * e\_f * (a\_b)^2;$$

$$c\_d * a\_b * (e\_f * e\_f) * (1\_0/a\_b * 1\_0/e\_f) \leq g\_h * e\_f * (a\_b * a\_b) * (1\_0/ e\_f * 1\_0/ a\_b);$$

c_d * a_b * (1_0/a_b * 1_0/e_f) * (e_f * e_f) ≤ g_h * e_f * (1_0/ e_f * 1_0/ a_b) * (a_b * a_b);

c_d * (a_b * 1_0/a_b) * (1_0/e_f * e_f) * e_f ≤ g_h * (e_f * 1_0/ e_f) * (1_0/ a_b * a_b) * a_b;

c_d * 1_0 * 1_0 * e_f ≤ g_h * 1_0 * 1_0 * a_b) * a_b;

c_d * e_f ≤ g_h * a_b;

e_f * c_d ≤ a_b * g_h, a contradiction.

Therefore, (a_b/c_d < e_f/g_h) → (g_h/e_f < c_d/a_b).

Suppose g_h/e_f < c_d/a_b, then by Definition 3.1.15, and 4.1.13 and Theorem 4.2.08, 4.2.10, 4.2.15, 4.2.22, 4.2.29, and 4.2.33:

g_h * e_f * (a_b)^2 < c_d * a_b * (e_f)^2;

g_h * e_f * (a_b * a_b) * (1_0/ e_f * 1_0/ a_b) < c_d * a_b * (e_f * e_f) * (1_0/a_b * 1_0/e_f);

g_h * e_f * (1_0/ e_f * 1_0/ a_b) * (a_b * a_b) < c_d * a_b * (1_0/a_b * 1_0/e_f) * (e_f * e_f);

g_h * (e_f * 1_0/ e_f) * (1_0/ a_b * a_b) * a_b < c_d * (a_b * 1_0/a_b) * (1_0/e_f * e_f) * e_f;

g_h * 1_0 * 1_0 * a_b) * a_b < c_d * 1_0 * 1_0 * e_f;

g_h * a_b < c_d * e_f;

a_b * g_h < e_f * c_d.

Now suppose, for contradiction, that e_f/g_h ≤ a_b/c_d, then, by Definition 3.1.15, and 4.1.13 and Theorem 4.2.08, 4.2.10, 4.2.15, 4.2.22, 4.2.29, and 4.2.33:

e_f * g_h * (c_d)^2 ≤ a_b * c_d * (g_h)^2;

e_f * g_h * (c_d * c_d) * (1_0/g_h * 1_0/c_d) ≤ a_b * c_d * (g_h * g_h) * (1_0/c_d * 1_0/g_h);

e_f * g_h * (1_0/g_h * 1_0/c_d) * (c_d * c_d) ≤ a_b * c_d * (1_0/c_d * 1_0/g_h) * (g_h * g_h);

e_f * (g_h * 1_0/g_h) * (1_0/c_d * c_d) * c_d ≤ a_b * (c_d * 1_0/c_d) * (1_0/g_h * g_h) * g_h;

e_f * 1_0 * 1_0 * c_d ≤ a_b * 1_0 * 1_0 * g_h;

e_f * c_d < a_b * g_h, a contradiction.

Therefore, (g_h/e_f < c_d/a_b) → (a_b/c_d < e_f/g_h).

Therefore, (a_b/c_d < e_f/g_h) iff (g_h/e_f < c_d/a_b), as desired. □

Lemma 4.2.36 *For all a_b/c_d ϵ Q_Q, a_b/c_d can be represented uniquely in lowest terms.*

Proof. Suppose a_b/c_d is not represented in lowest terms, then, by Definition 4.1.13, (c_d < 0_0) ˅ (gcd(a_b, c_d) ≠ 1_0) and two cases arise:

Case 1. Suppose c_d < 0_0, then, by Definition 3.1.06 and 4.1.06 and Lemma 3.2.42:

a_b/c_d = (- 1_0/- 1_0) * (a_b/c_d);

= (- 1_0 * a_b/- 1_0 * c_d);

= - (a_b)/- (c_d); and, by Definition 3.1.10 and 3.1.13 and Theorem 3.2.18, 0_0 < c_d.

Case 2. Suppose gcd(a_b, c_d) ≠ 1_0, then gcd(a_b, c_d) = x_y ϵ Z_Q^+ and, by Definition 3.1.12, there exists k_p, m_q ϵ Z_Q^+ such that a_b = k_p + x_y and c_d = m_q + x_y and, by Theorem 4.2.03, (k_p/m_q) E (a_b/c_d), where gcd(k_p,m_q) = 1_0.

Suppose, for contradiction, that [(e_f/g_h) E (a_b/c_d)] ^ [gcd(e_f, g_h) = 1_0] ^ [(k_p/m_q) E (a_b/c_d)] ^ [gcd(k_p, m_q) = 1_0] ^ (e_f/g_h ≠ k_p/m_q), then, by Theorem 4.2.03, there exists x_y ϵ Z_Q, x_y ≠ 1_0, such that [(e_f * x_y)/(g_h * x_y) = k_p/m_q] ˅ [(k_p * x_y)/(m_q * x_y) = e_f/g_h]. But then, [(x_y | k_p) ^ (x_y | m_q)] ˅ [(x_y | e_f) ^ (x_y | g_h)], a contradiction.

In both cases, a_b/c_d has a unique representation in lowest terms.

Therefore, a_b/c_d can be represented uniquely in lowest terms, as desired. □

Lemma 4.2.37. *For all a_b/c_d ϵ Q_Q, a_b/c_d in lowest terms, |a_b/c_d| = |a_b|/c_d.*

Proof. By Definition 3.1.14, 4.1.01, 4.1.05, 4.1.11, 4.1.14, and 4.1.16, as desired. □

Theorem 4.2.38. *Let G = {a_b|(a_b ϵ Z_Q^+) ^ (a = 0) ^ (b ≠ 0)}, let H = { a_b|(a_b ϵ Z_Q^+) ^ (0 < a) ^ (b < 0) ^ (a < |b|)}, and let:*

1. *A = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (a < 0) ^ (b < 0)};*
2. *B = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (a < 0) ^ (0 ≤ b)};*
3. *C = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (a = 0) ^ (b < 0)};*
4. *D = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (a = 0) ^ (0 ≤ b)};*
5. *E = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (0 < a) ^ (b < 0)};*
6. *F = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (0 < a) ^ (0 ≤ b)}, where the elements of each set are in lowest terms.*

Then we demonstrate the following:

a. *For all a_b/c_d, e_f/g_h ϵ A, ¬ {[(c_d ϵ G) ^ (g_h ϵ H)] ˅ [(g_h ϵ G) ^ (c_d ϵ H)]}, |a_b/c_d| * |e_f/g_h| = |a_b/c_d * e_f/g_h|.*

   Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

b. *For all a_b/c_d ϵ A, e_f/g_h ϵ B, ¬ {[(c_d ϵ G) ^ (g_h ϵ H)] ˅ [(g_h ϵ G) ^ (c_d ϵ H)]}, |a_b/c_d| * |e_f/g_h| = |a_b/c_d * e_f/g_h|.*

   Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

c. *For all a_b/c_d ϵ A, e_f/g_h ϵ C, ¬ {[(c_d ϵ G) ^ (g_h ϵ H)] ˅ [(g_h ϵ G) ^ (c_d ϵ H)]}, |a_b/c_d| * |e_f/g_h| = |a_b/c_d * e_f/g_h|.*

   Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

d. *For all a_b/c_d ϵ A, e_f/g_h ϵ D, ¬ {[(c_d ϵ G) ^ (g_h ϵ H)] ˅ [(g_h ϵ G) ^ (c_d ϵ H)]}, |a_b/c_d| * |e_f/g_h| = |a_b/c_d * e_f/g_h|.*

   Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

e. *For all a_b/c_d ϵ A, e_f/g_h ϵ E, ¬ {[(c_d ϵ G) ^ (g_h ϵ H)] ˅ [(g_h ϵ G) ^ (c_d ϵ H)]}, |a_b/c_d| * |e_f/g_h| = |a_b/c_d * e_f/g_h|.*

   Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

f. *For all a_b/c_d ϵ A, e_f/g_h ϵ F, ¬ {[(c_d ϵ G) ^ (g_h ϵ H)] ˅ [(g_h ϵ G) ^ (c_d ϵ H)]}, |a_b/c_d| * |e_f/g_h| = |a_b/c_d * e_f/g_h|.*

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

g. *For all* $a\_b/c\_d, e\_f/g\_h \in B, \neg \{[(c\_d \in G) \wedge (g\_h \in H)] \vee [(g\_h \in G) \wedge (c\_d \in H)]\}, |a\_b/c\_d| * |e\_f/g\_h| = |a\_b/c\_d * e\_f/g\_h|$.

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

h. *For all* $a\_b/c\_d \in B, e\_f/g\_h \in C, \neg \{[(c\_d \in G) \wedge (g\_h \in H)] \vee [(g\_h \in G) \wedge (c\_d \in H)]\}, (|a\_b/c\_d| * |e\_f/g\_h| = |a\_b/c\_d * e\_f/g\_h|)$ *iff* $(b \leq |a|)$.

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

i. *For all* $a\_b/c\_d \in B, e\_f/g\_h \in D, \neg \{[(c\_d \in G) \wedge (g\_h \in H)] \vee [(g\_h \in G) \wedge (c\_d \in H)]\}, (|a\_b/c\_d| * |e\_f/g\_h| = |a\_b/c\_d * e\_f/g\_h|)$ *iff* $(b \leq |a|)$.

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

j. *For all* $a\_b/c\_d \in B, e\_f/g\_h \in E, \neg \{[(c\_d \in G) \wedge (g\_h \in H)] \vee [(g\_h \in G) \wedge (c\_d \in H)]\}, |a\_b/c\_d| * |e\_f/g\_h| = |a\_b/c\_d * e\_f/g\_h|$.

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

k. *For all* $a\_b/c\_d \in B, e\_f/g\_h \in F, \neg \{[(c\_d \in G) \wedge (g\_h \in H)] \vee [(g\_h \in G) \wedge (c\_d \in H)]\}, |a\_b/c\_d| * |e\_f/g\_h| = |a\_b/c\_d * e\_f/g\_h|$.

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

l. *For all* $a\_b/c\_d, e\_f/g\_h \in C, \neg \{[(c\_d \in G) \wedge (g\_h \in H)] \vee [(g\_h \in G) \wedge (c\_d \in H)]\}, |a\_b/c\_d| * |e\_f/g\_h| = |a\_b/c\_d * e\_f/g\_h|$.

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

m. *For all* $a\_b/c\_d \in C, e\_f/g\_h \in D, \neg \{[(c\_d \in G) \wedge (g\_h \in H)] \vee [(g\_h \in G) \wedge (c\_d \in H)]\}, |a\_b/c\_d| * |e\_f/g\_h| = |a\_b/c\_d * e\_f/g\_h|$.

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

n. *For all* $a\_b/c\_d \in C, e\_f/g\_h \in E, \neg \{[(c\_d \in G) \wedge (g\_h \in H)] \vee [(g\_h \in G) \wedge (c\_d \in H)]\}, (|a\_b/c\_d| * |e\_f/g\_h| = |a\_b/c\_d * e\_f/g\_h|)$ *iff* $(|f| \leq e)$.

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

o. *For all* $a\_b/c\_d \in C, e\_f/g\_h \in F, \neg \{[(c\_d \in G) \wedge (g\_h \in H)] \vee [(g\_h \in G) \wedge (c\_d \in H)]\}, |a\_b/c\_d| * |e\_f/g\_h| = |a\_b/c\_d * e\_f/g\_h|$.

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

p. *For all* $a\_b/c\_d, e\_f/g\_h \in D, \neg \{[(c\_d \in G) \wedge (g\_h \in H)] \vee [(g\_h \in G) \wedge (c\_d \in H)]\}, |a\_b/c\_d| * |e\_f/g\_h| = |a\_b/c\_d * e\_f/g\_h|$.

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

q. *For all* $a\_b/c\_d \in D, e\_f/g\_h \in E, \neg \{[(c\_d \in G) \wedge (g\_h \in H)] \vee [(g\_h \in G) \wedge (c\_d \in H)]\}, (|a\_b/c\_d| * |e\_f/g\_h| = |a\_b/c\_d * e\_f/g\_h|)$ *iff* $(|f| \leq e)$.

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

r. *For all a_b/c_d ϵ D, e_f/g_h ϵ F, ¬ {[(c_d ϵ G) ^ (g_h ϵ H)] ˅ [(g_h ϵ G) ^ (c_d ϵ H)]}, |a_b/c_d| * |e_f/g_h| = |a_b/c_d * e_f/g_h|.*

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

s. *For all a_b/c_d, e_f/g_h ϵ E, ¬ {[(c_d ϵ G) ^ (g_h ϵ H)] ˅ [(g_h ϵ G) ^ (c_d ϵ H)]}, |a_b/c_d| * |e_f/g_h| = |a_b/c_d * e_f/g_h|.*

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

t. *For all a_b/c_d ϵ E, e_f/g_h ϵ F, ¬ {[(c_d ϵ G) ^ (g_h ϵ H)] ˅ [(g_h ϵ G) ^ (c_d ϵ H)]}, |a_b/c_d| * |e_f/g_h| = |a_b/c_d * e_f/g_h|.*

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

u. *For all a_b/c_d, e_f/g_h ϵ F, ¬ {[(c_d ϵ G) ^ (g_h ϵ H)] ˅ [(g_h ϵ G) ^ (c_d ϵ H)]}, |a_b/c_d| * |e_f/g_h| = |a_b/c_d * e_f/g_h|.*

Proof. By Definition 4.1.06, Theorem 3.2.45, and Lemma 4.2.37, as desired. □

Theorem 4.2.39. *For all a_b/c_d ϵ Q_Q, a_b/c_d in lowest terms, |a_b/c_d| = |- (a_b)/c_d|.*

Proof. By Lemma 3.2.46 and 4.2.37, as desired. □

Theorem 4.2.40. *For all a_b/c_d, e_f/g_h ϵ Q_Q, a_b/c_d in lowest terms, |a_b/c_d + e_f/g_h| ≤ |a_b/c_d| + |e_f/g_h|.*

Proof. By Definition 4.1.15 and Theorem 4.2.30, [(a_b/c_d ϵ Q_Q^-) ˅ (a_b/c_d = 0_0) ˅ (a_b/c_d ϵ Q_Q^+)] ^ [(e_f/g_h ϵ Q_Q^-) ˅ (e_f/g_h = 0_0) ˅ (e_f/g_h ϵ Q_Q^+)] and nine cases arise, three of which are redundant, leaving six cases to consider:

Case 1. Suppose (a_b/c_d ϵ Q_Q^-) ^ (e_f/g_h ϵ Q_Q^-), then, by Definition 4.1.06 and 4.1.16, Theorem 3.2.08 and 3.2.10, Corollary 3.2.43, and Lemma 4.2.37:

|a_b/c_d + e_f/g_h| = - (a_b/c_d + e_f/g_h);

= - [(a_b * g_h + c_d * e_f)/(c_d * g_h)];

= - (a_b * g_h + c_d * e_f)/(c_d * g_h);

= {- (a_b) * g_h + c_d * [-(e_f)]}/(c_d * g_h)];

= - (a_b)/c_d + - (e_f)/g_h;

= |a_b/c_d| + |e_f/g_h|.

Case 2. Suppose (a_b/c_d = 0_0) ^ (e_f/g_h = 0_0), then, by Definition 4.1.06 and 4.1.16 and Theorem 4.2.13:

|a_b/c_d + e_f/g_h| = |0_0 + 0_0|;

= |0_0|;

= 0_0;

= 0_0 + 0_0;

= |a_b/c_d| + |e_f/g_h|.

Case 3. Suppose (a_b/c_d ϵ Q_Q^+) ^ (e_f/g_h ϵ Q_Q^+), then, by Definition 4.1.16:

|a_b/c_d + e_f/g_h| = a_b/c_d + e_f/g_h;

= |a_b/c_d| + |e_f/g_h|.

Case 4. Suppose (a_b/c_d ϵ Q_Q^+) ^ (e_f/g_h = 0_0), then, by Definition 4.1.16 and Theorem 4.2.13:

|a_b/c_d + e_f/g_h| = |a_b/c_d + 0_0|;

= |a_b/c_d|;

= a_b/c_d;

= a_b/c_d + 0_0;

= |a_b/c_d| + |e_f/g_h|.

This result remains unchanged in the case (a_b/c_d = 0_0) ^ (e_f/g_h ϵ Q_Q^+).

Case 5. Suppose (a_b/c_d ϵ Q_Q^-) ^ (e_f/g_h = 0_0), then, by Definition 4.1.16 and Theorem 4.2.13:

|a_b/c_d + e_f/g_h| = |a_b/c_d + 0_0|;

= |a_b/c_d|;

= - (a_b)/c_d;

= - (a_b)/c_d + 0_0;

= |a_b/c_d| + |e_f/g_h|.

This result remains unchanged in the case (a_b/c_d = 0_0) ^ (e_f/g_h ϵ Q_Q^-).

Case 6. Suppose (a_b/c_d ϵ Q_Q^-) ^ (e_f/g_h ϵ Q_Q^+), then, by Definition 4.1.16 and Theorem 4.2.30, (|a_b/c_d| = - (a_b)/c_d < e_f/g_h) ˅ (e_f/g_h < |a_b/c_d| = - (a_b)/c_d) and two cases arise:

Case 6.a. Suppose |a_b/c_d| = - (a_b)/c_d < e_f/g_h, then, by Definition 4.1.09, 4.1.13, and 4.1.16 and Theorem 4.2.07:

|a_b/c_d + e_f/g_h| = e_f/g_h - |a_b/c_d|;

= e_f/g_h - [- (a_b)/c_d];

< e_f/g_h + [- (a_b)/c_d];

= [- (a_b)/c_d] + e_f/g_h;

= |a_b/c_d| + |e_f/g_h|.

Case 6.b. Suppose e_f/g_h < |a_b/c_d| = - (a_b)/c_d, then, by Definition 4.1.09, 4.1.13, and 4.1.16:

|a_b/c_d + e_f/g_h| = |a_b/c_d| - (e_f/g_h);

= [- (a_b)/c_d] - (e_f/g_h);

< [- (a_b)/c_d] + e_f/g_h;

= |a_b/c_d| + |e_f/g_h|.

In both cases, |a_b/c_d + e_f/g_h| < |a_b/c_d| + |e_f/g_h|.

This result remains unchanged in the case (a_b/c_d ϵ Q_Q^+) ^ (e_f/g_h ϵ Q_Q^-).

In all six cases, (|a_b/c_d + e_f/g_h| = |a_b/c_d| + |e_f/g_h|) ᵛ (|a_b/c_d + e_f/g_h| < |a_b/c_d| + |e_f/g_h|).

Therefore, |a_b/c_d + e_f/g_h| ≤ |a_b/c_d| + |e_f/g_h|, as desired. □

Corollary 4.2.41. *For all a_b/c_d, e_f/g_h ϵ Q_Q, |a_b/c_d| - |e_f/g_h| ≤ |a_b/c_d - e_f/g_h|.*

Proof. By Definition 4.1.09 and Theorem 4.2.13 and 4.2.19:

|(a_b/c_d - e_f/g_h) + e_f/g_h| ≤ |a_b/c_d - e_f/g_h| + |e_f/g_h|;

|a_b/c_d| ≤ |a_b/c_d - e_f/g_h| + |e_f/g_h|;

|a_b/c_d| - |e_f/g_h| ≤ |a_b/c_d - e_f/g_h|, as desired. □

Theorem 4.2.42. *For all a_b/c_d, e_f/g_h ϵ Q_Q, (0_0 < a_b/c_d) → [(|e_f/g_h| < a_b/c_d) iff [- (a_b)/c_d < e_f/g_h < a_b/c_d].*

Proof. Suppose (0_0 < a_b/c_d) ^ (|e_f/g_h| < a_b/c_d), then, by Theorem 4.2.30, (e_/g_h < 0_0) ᵛ (0_0 ≤ e_f/g_h) and two cases arise:

Case 1. Suppose e_f/g_h < 0_0, then, by Definition 4.1.06 and 4.1.16 and Theorem 4.2.34, - (a_b)/c_d < e_f/g_h.

Case 2. Suppose 0_0 ≤ e_f/g_h, then, by Definition 4.1.16, 0_0 ≤ e_f/g_h < a_b/c_d.

In both cases, - (a_b)/c_d < e_f/g_h < a_b/c_d.

Therefore, (|e_f/g_h | < a_b/c_d) → [- (a_b)/c_d < e_f/g_h < a_b/c_d].

Suppose (0_0 < a_b/c_d) ^ [- (a_b)/c_d < e_f/g_h < a_b/c_d], then, by Theorem 4.2.30, (e_f/g_h < 0_0) ᵛ (0_0 ≤ e_f/g_h) and two cases arise:

Case 1. Suppose e_f/g_h < 0_0, then, by Definition 4.1.06 and 4.1.16 and Theorem 4.2.34, - (e_f)/g_h = |e_f/g_h | < - [- (a_b)/c_d] = a_b/c_d.

Case 2. Suppose 0_0 ≤ e_f/g_h, then, by Definition 4.1.16, | e_f/g_h |< a_b/c_d.

In both cases, | e_f/g_h |< a_b/c_d.

Therefore, [- (a_b)/c_d < e_f/g_h < a_b/c_d] → (|e_f/g_h |< a_b/c_d).

Therefore, (0_0 < a_b/c_d) → {(| e_f/g_h | < a_b/c_d) iff [- (a_b)/c_d < e_f/g_h < a_b/c_d]}, as desired. □

Theorem 4.2.43. *For all n ϵ Z, a_b/c_d ϵ Q_Q, $(a\_b/c\_d)^n$ is defined.*

Proof. We proceed by induction on n. Let P(x) be the property, "$(a\_b/c\_d)^x$ is defined," then:

P(0). By Definition 4.1.17, $(a\_b/c\_d)^0$ = 1_0.

Suppose P(n) is true, then $(a\_b/c\_d)^n$ is defined and:

P(n + 1). By Definition 4.1.17, $(a\_b/c\_d)^{(n+1)}$ = $(a\_b/c\_d)^n$ * a_b/c_d, hence, by Definition 4.1.06 and Theorem 4.2.06, $(a\_b/c\_d)^{(n+1)}$ is defined.

P(–n). By Definition 4.1.17, $(a\_b/c\_d)^{-n}$ = $(c\_d/a\_b)^n$, hence, since a_b/c_d ϵ Q_Q was arbitrary, $(a\_b/c\_d)^{-n}$ is defined.

Therefore, $P(n) \rightarrow [P(n + 1) \wedge P(- n)]$ and, by the Principle of Induction on Z (reference [CD], Chapter 5, page 173), for all $n \in Z$, $a\_b/c\_d \in Q_Q$, $(a\_b/c\_d)^n$ is defined, as desired. □

Lemma 4.2.44. *For all $a\_b/c\_d \in Q_Q^-$, $(a\_b/c\_d)^n$ is negative if n is odd and positive if n is even.*

Proof. This is an immediate consequence of Definition 4.1.06, 4.1.15, and 4.1.17 and Corollary 3.2.43 and 3.2.44, as desired. □

Theorem 3.2.45. *For all $n \in Z$, $a\_b/c\_d, e\_f/g\_h \in Q_Q$, $(a\_b/c\_d * e\_f/g\_h)^n = (a\_b/c\_d)^n * (e\_f/g\_h)^n$.*

Proof. We proceed by induction on n. Let P(x) be the property, "$(a\_b/c\_d * e\_f/g\_h)^x = (a\_b/c\_d)^x * (e\_f/g\_h)^x$," then:

P(0). By Definition 4.1.17 and Theorem 4.2.15, $(a\_b/c\_d * e\_f/g\_h)^0 = 1\_0 = 1\_0 * 1\_0 = (a\_b/c\_d)^0 * (e\_f/g\_h)^0$.

Suppose P(n) is true, then $(a\_b/c\_d * e\_f/g\_h)^n = (a\_b/c\_d)^n * (e\_f/g\_h)^n$ and:

P(n + 1). By Definition 4.1.17 and Theorem 4.2.08 and 4.2.10:

$$\begin{aligned}
(a_b/c_d * e_f/g_h)^{(n+1)} &= (a_b/c_d * e_f/g_h)^n * (a_b/c_d * e_f/g_h);\\
&= (a_b/c_d)^n * (e_f/g_h)^n * (a_b/c_d * e_f/g_h);\\
&= (a_b/c_d)^n * [(e_f/g_h)^n * (a_b/c_d * e_f/g_h)];\\
&= (a_b/c_d)^n * [(e_f/g_h)^n * a_b/c_d] * e_f/g_h;\\
&= (a_b/c_d)^n * [a_b/c_d * (e_f/g_h)^n] * e_f/g_h;\\
&= [(a_b/c_d)^n * a_b/c_d] * [(e_f/g_h)^n * e_f/g_h];\\
&= (a_b/c_d)^{(n+1)} * (e_f/g_h)^{(n+1)}.
\end{aligned}$$

P(– n). By Definition 4.1.06 and 4.1.17 and by the fact that $a\_b/c\_d, e\_f/g\_h \in Q_Q$ are arbitrary:

$$\begin{aligned}
(a_b/c_d * e_f/g_h)^{-n} &= [(a_b * e_f)/(c_d * g_h)]^{-n};\\
&= [(c_d * g_h)/(a_b * e_f)]^n;\\
&= (c_d/a_b)^n * (g_h / e_f)^n;\\
&= (a_b/c_d)^{-n} * (e_f/g_h)^{-n}.
\end{aligned}$$

Therefore, $P(n) \rightarrow [P(n + 1) \wedge P(- n)]$ and, by the Principle of Induction on Z (reference [CD], Chapter 5, page 173), for all $n \in Z$, $a\_b/c\_d, e\_f/g\_h \in Q_Q$, $(a\_b/c\_d * e\_f/g\_h)^n = (a\_b/c\_d)^n * (e\_f/g\_h)^n$, as desired. □

Theorem 4.2.46. *For all $m, n \in Z$, $a\_b/c\_d \in Q_Q$, $(a\_b/c\_d)^m * (a\_b/c\_d)^n = (a\_b/c\_d)^{(m+n)}$.*

Proof. We proceed by induction on n. Let P(x) be the property, "$(a\_b/c\_d)^m * (a\_b/c\_d)^x = (a\_b/c\_d)^{(m+x)}$," then:

P(0). By Definition 4.1.17 and Theorem 4.2.15, $(a\_b/c\_d)^m * (a\_b/c\_d)^0 = (a\_b/c\_d)^m * 1\_0 = (a\_b/c\_d)^m = (a\_b/c\_d)^{(m+0)}$.

Suppose P(n) is true, then $(a\_b/c\_d)^m * (a\_b/c\_d)^n = (a\_b/c\_d)^{(m+n)}$ and:

P(n + 1). By Definition 4.1.17 and Theorem 4.2.10:

$$\begin{aligned}
(a_b/c_d)^m * (a_b/c_d)^{(n+1)} &= (a_b/c_d)^m * [(a_b/c_d)^n * a_b/c_d];\\
&= [(a_b/c_d)^m * (a_b/c_d)^n] * a_b/c_d;
\end{aligned}$$

$= (a\_b/c\_d)^{(m + n)} * a\_b/c\_d;$

$= (a\_b/c\_d)^{[(m + n) + 1]};$

$= (a\_b/c\_d)^{[m + (n + 1)]}.$

P(– n). This is an immediate consequence of the above argument together with the fact that, for all m, n ϵ Z, m – n = m + (– n) (reference [CD], Chapter 5, page 165).

Therefore, P(n) → [P(n + 1) ^ P(– n)] and, by the Principle of Induction on Z (reference [CD], Chapter 5, page 173), for all m, n ϵ Z, *a_b/c_d* ϵ Q_Q, $(a\_b/c\_d)^m * (a\_b/c\_d)^n = (a\_b/c\_d)^{(m + n)}$, as desired. □

Theorem 4.2.47. *For all m, n ϵ Z, a_b/c_d ϵ Q_Q, $[(a\_b/c\_d)^m]^n = (a\_b/c\_d)^{(m * n)}$.*

Proof. We proceed by induction on n. Let P(x) be the property, "$[(a\_b/c\_d)^m]^x = (a\_b/c\_d)^{(m * x)}$," then:

P(0). By Definition 4.1.17, $[(a\_b/c\_d)^m]^0 = 1\_0 = (a\_b/c\_d)^0 = (a\_b/c\_d)^{(m * 0)}$.

Suppose P(n) is true, then $[(a\_b/c\_d)^m]^n = (a\_b/c\_d)^{(m * n)}$ and:

P(n + 1). By Definition 4.1.17 and Theorem 4.2.46:

$[(a\_b/c\_d)^m]^{(n + 1)} = [(a\_b/c\_d)^m]^n * (a\_b/c\_d)^m;$

$= (a\_b/c\_d)^{(m * n)} * (a\_b/c\_d)^m;$

$= (a\_b/c\_d)^{[(m * n) + m]};$

$= (a\_b/c\_d)^{[m * (n + 1)]}.$

P(– n). By Definition 4.1.17 and the fact that a_b/c_d is arbitrary:

$[(a\_b/c\_d)^m]^{- n} = [(c\_d/a\_b)^m]^{n};$

$= (c\_d/a\_b)^{(m * n)};$

$= (a\_b/c\_d)^{- (m * n)};$

$= (a\_b/c\_d)^{[m * (- n)]}.$

Therefore, P(n) → [P(n + 1) ^ P(– n)] and, by the Principle of Induction on Z (reference [CD], Chapter 5, page 173), for all m, n ϵ Z, *a_b/c_d* ϵ Q_Q, $[(a\_b/c\_d)^m]^n = (a\_b/c\_d)^{(m * n)}$, as desired. □

Theorem 4.2.48. *Q_Q is dense.*

Proof. Let a_b/c_d, e_f/g_h ϵ Q_Q be arbitrary but such that a_b/c_d < e_f/g_h, then, by Definition 4.1.13, $a\_b * c\_d * (g\_h)^2 < e\_f * g\_h * (c\_d)^2$. By Definition3.1.15, 4.1.06, and 4.1.10 and Theorem 3.2.11, 3.2.16, and 4.2.12:

$(a\_b/c\_d + e\_f/g\_h) * 1\_0/2\_0 = (a\_b/c\_d * 1\_0/2\_0) + (e\_f/g\_h * 1\_0/2\_0);$

$= [(a\_b * 1\_0)/(c\_d * 2\_0)] + [(e\_f * 1\_0)/(g\_h * 2\_0)];$

$= a\_b/(c\_d * 2\_0) + e\_f/(g\_h * 2\_0);$

$= (a\_b * g\_h * 2\_0 + e\_f * c\_d * 2\_0)/ [c\_d * g\_h * (2\_0)^2];$

$= 2\_0 * (a\_b * g\_h + e\_f * c\_d)/ (2\_0)^2 * (c\_d * g\_h);$

$= (a\_b * g\_h + e\_f * c\_d)/ (2\_0 * c\_d * g\_h).$

By Definition 3.1.06, 3.1.13, and 3.1.15 and Theorem 3.2.08, 3.2.10, 3.2.12, and 3.2.37:

$$a\_b * c\_d * (2\_0 * c\_d * g\_h)^2 = (a\_b * c\_d) * (2\_0)^2 * (c\_d)^2 * (g\_h)^2;$$
$$= [(a\_b * c\_d) * 2\_0] * 2\_0 * [(c\_d)^2 * (g\_h)^2];$$
$$= 2\_0 * a\_b * c\_d * [2\_0 * (g\_h)^2] * (c\_d)^2;$$
$$= [2\_0 * a\_b * c\_d * (g\_h)^2] * [2\_0 * (c\_d)^2];$$
$$= [a\_b * c\_d * (g\_h)^2 + a\_b * c\_d * (g\_h)^2] * [2\_0 * (c\_d)^2];$$
$$< [a\_b * c\_d * (g\_h)^2 + e\_f * g\_h * (c\_d)^2] * [2\_0 * (c\_d)^2];$$
$$= [a\_b * (c\_d * g\_h) * g\_h + e\_f * (g\_h * c\_d) * c\_d] * [2\_0 * (c\_d)^2];$$
$$= [a\_b * g\_h * (c\_d * g\_h) + e\_f * c\_d * (c\_d * g\_h)] * [2\_0 * (c\_d)^2];$$
$$= (a\_b * g\_h + e\_f * c\_d) * [(c\_d * g\_h) * 2\_0] * (c\_d)^2;$$
$$= (a\_b * g\_h + e\_f * c\_d) * (2\_0 * c\_d * g\_h) * (c\_d)^2.$$

Therefore, by Definition 4.1.13, a_b/c_d < [(a_b/c_d + e_f/g_h) * 1_0/2_0].

Letting A = (a_b * g_h + e_f * c_d) * (2_0 * c_d * g_h) * $(g\_h)^2$, by Definitiion 3.1.06, 3.1.13, and 3.1.15 and Theorem 3.2.08, 3.2.10, 3.2.12, and 3.2.37:

$$A = (a\_b * g\_h + e\_f * c\_d) * [2\_0 * (c\_d * g\_h)] * (g\_h)^2;$$
$$= (a\_b * g\_h + e\_f * c\_d) * [(c\_d * g\_h) * 2\_0] * (g\_h)^2;$$
$$= (a\_b * g\_h + e\_f * c\_d) * (c\_d * g\_h) * [2\_0 * (g\_h)^2];$$
$$= [(a\_b * g\_h) * (c\_d * g\_h) + (e\_f * c\_d) * (c\_d * g\_h)] * [2\_0 * (g\_h)^2];$$
$$= \{[a\_b * (g\_h * c\_d) * g\_h] + [e\_f * (c\_d * c\_d) * g\_h]\} * [2\_0 * (g\_h)^2];$$
$$= \{[a\_b * (c\_d * g\_h) * g\_h] + [e\_f * g\_h * (c\_d * c\_d)]\} * [2\_0 * (g\_h)^2];$$
$$= \{[a\_b * c\_d * (g\_h)^2] + [e\_f * g\_h * (c\_d)^2]\} * [2\_0 * (g\_h)^2];$$
$$< \{[e\_f * g\_h * (c\_d)^2] + [e\_f * g\_h * (c\_d)^2]\} * [2\_0 * (g\_h)^2];$$
$$= [2\_0 * (e\_f * g\_h) * (c\_d)^2] * [2\_0 * (g\_h)^2];$$
$$= (e\_f * g\_h) * \{2\_0 * [(c\_d)^2 * 2\_0] * (g\_h)^2\};$$
$$= e\_f * g\_h * [2\_0 * 2\_0 * (c\_d)^2 * (g\_h)^2];$$
$$= e\_f * g\_h * (2\_0)^2 * (g\_h)^2 * (c\_d)^2;$$
$$= e\_f * g\_h * (2\_0 * g\_h * c\_d)^2.$$

Therefore, by Definition 4.1.13, (a_b/c_d + e_f/g_h) * 1_0/2_0 < e_f/g_h.

Therefore, by Theorem 4.2.30, a_b/c_d < [(a_b/c_d + e_f/g_h) * 1_0/2_0] < e_f/g_h, and Q_Q is dense, as desired. □

Theorem 4.2.49. *For all a_b/c_d ϵ Q_Q, there exists x_y ϵ Z_Q such that a_b/c_d < x_y.*

Proof. Let a_b/c_d be in lowest terms, then, by Definition 3.1.12 and Theorem 3.2.16, for all x_y ϵ Z_Q, x_y = x_y/1_0 ϵ Z_Q. Let x_y = |a_b| + 1_0, then, by Definition 3.1.13 and 3.1.14, Theorem 3.2.16, and Corollary 3.2.44:

$$\begin{aligned} a\_b * c\_d * (1\_0)^2 &= a\_b * c\_d; \\ &\leq |a\_b| * c\_d; \\ &< |a\_b| * (c\_d)^2 + (c\_d)^2; \\ &= [(|a\_b| + 1\_0) * 1\_0 * (c\_d)^2. \end{aligned}$$

Therefore, by Definition 4.1.13, a_b/c_d < x_y, as desired. □

Theorem 4.2.50. *For all a_b/c_d ϵ Q_Q^+, there exists x_y ϵ Z_Q^+ such that 0_1/x_y < a_b/c_d.*

Proof. Let a_b/c_d ϵ Q_Q^+ be in lowest terms and let x_y = (c_d + 1_0), then, by Definition 3.1.15 and Theorem 3.2.12, 3.2.29, and 3.2.40:

$$\begin{aligned} 0\_1 * (c\_d + 1\_0) * (c\_d)^2 &= 0\_1 * (c\_d)^3 + (c\_d)^2; \\ &< a\_b * (c\_d)^3 + 2\_0 * (c\_d)^2 + c\_d; \\ &= a\_b * c\_d * [(c\_d)^2 + 2\_0 * c\_d + 1\_0]; \\ &= a\_b * c\_d * (c\_d + 1\_0)^2. \end{aligned}$$

Therefore, by Defintion 4.1.13, 0_1/x_y < a_b/c_d, as desired. □

Theorem 4.2.51. *(Q_Q, <, +, *) forms a field.*

Proof. That (Q_Q, <, +, *) forms a field is an immediate consequence of Theorem 4.2.07, 4.2.08, 4.2.09, 4.2.10, 4.2.11, 4.2.13, 4.2.14, 4.2.15, 4.2.16, 4.2.20, and 4.2.23, as desired. □

Corollary 4.2.52. *(Q_Q, <, +, *) is not an ordered field.*

Proof. Let a_b, c_d ϵ Z_Q^+ be such that (a = 0) ^ (d < 0) ^ (c < |d|), then, by Theorem 4.2.29, a_b, c_d ϵ Q_Q^+ and, by Lemma 3.2.28, a_b * c_d = 0_b * c + b * d, where (b * c + b * d) < 0, hence, a_b * c_d ϵ Q_Q^-. Therefore, (Q_Q, <, +, *) violates the closure axiom and is not an ordered field, as desired. □

Theorem 4.2.53. *Q_Q is countable.*

Proof. By Theorem 3.2.58, Z_Q x Z_Q is countable (reference [HJ], Chapter 4, page 75) and, by Definition 4.1.01 and 4.1.05, Q_Q = Q_Q'/E is a proper subset of Z_Q x Z_Q and at most countable (reference [HJ], Chapter 4, page 77). By Theorem 4.2.29, Q_Q properly contains Z_Q, hence, Q_Q is countable, as desired. □

Theorem 4.2.54. *Q_Q has neither a greatest nor a least element.*

Proof. This is an immediate consequence of Definition 4.1.01 and 4.1.05 and Theorem 3.2.59, as desired. □

Theorem 4.2.55. ¬ *[$(2\_2)^{1/2}$ ϵ Q_Q].*

Proof. Suppose, for contradiction, that $(2\_2)^{1/2}$ ϵ Q_Q, then, by Theorem 4.2.34, $(2\_2)^{1/2}$ can be represented uniquely as a q-rational fraction, a_b/c_d, in lowest terms. Let a_b/c_d = $(2\_2)^{1/2}$, then, by Definition 4.1.06 and 4.1.16, $(a\_b)^2/(c\_d)^2$ = 2_2/1_0, hence, $(a\_b)^2$ = 2_2 and c_d = 1_0. Then, by Definition 2.1.14, Lemma 2.2.07, and Theorem 3.2.20, (a^2 ϵ N) ^ (a^2 = 2). But $1^2 = 1 < 2^2 = 4$, hence, 1 < a < 2, a contradiction. Therefore, ¬ [$(2\_2)^{1/2}$ ϵ Q_Q], as desired. □

Theorem 4.2.56. *Q_Q has gaps.*

Proof. Let A = {a_b/c_d | (a_b/c_d ϵ Q_Q) ^ [(a_b/c_d ≤ 0_0) ˅ (0_0 < a_b/c_d ^ $(a\_b/c\_d)^{1/2}$ < e_f)]} and let B = Q_Q – A, then, by Theorem 4.2.30, A and B exist. Clearly A ∩ B = A ∩ (Q_Q – A) = ϕ and A U B = A U (Q_Q – A) = Q_Q. Suppose, for contradiction, that A has a greatest element, a_b/c_d, and B a least element, e_f/g_h, then a_b/c_d is the greatest lower bound of B and e_f/g_h is the least upper bound of A, hence, a_b/c_d = e_f/g_h (reference [HJ], Chapter 2, pages 35 and 36), contradicting the fact that A ∩ B = ϕ. Therefore, by Definition 4.1.18, Q_Q has gaps, as desired. □

Lemma 4.2.57. *Let W = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (a ≠ 0) ^ (c = 0)}, then, for all a_b/c_d ϵ Q_Q – W, a_b/c_d = x_y, where x,y ϵ R.*

Proof. Let a_b/c_d ϵ Q_Q – W be arbitrary, then, by Definition 3.1.12, 4.1.01, and 4.1.05 and Lemma 3.2.28:

a_b = x_y * c_d;

= (x * c)_(y * c + x * d + y * d);

= (x * c)_[x * d + y * (c + d)].

Hence, x = a/c, y = (b * c - a * d)/[c * (c + d)], and a_b/c_d = (a/c)_{(b * c - a * d)/[c * (c + d)}, as desired. □

5. Q-Reals. We develop the q-reals as suprema of Dedekind cuts on the q-rationals. This development adheres rather faithfully to the standard Dedekind cut method; however, due to the definition of multiplication on Q_Q, the fact that Z_Q^+ is not closed under multiplication, and that there exist elements of Z_Q^- which remain in Z_Q^- when multiplied, there is a slight deviation, in the form of two added cases under Dedekind cut multiplication. The modification necessary to accommodate these additional cases is rather straight-forward and the elegance of the Dedekind cut method is largely retained, allowing this development to fill the gaps in the q-rationals resulting in a unique, q-naturally lattice complete field. Due to the aforementioned violation of closure, the q-rational completion is not an ordered field.

Historically, the standard reals were invented after it was discovered that the standard rationals were insufficient for the general quantification of geometrical elements. Through analogous geometrical considerations, we discover that the q-components of the q-reals must be allowed to take complex values; in fact, there is a proper subset of the q-reals which, from a practical perspective, could be defined as elements of R x C, where, for all (a, b + ci) ϵ R x C, (a, b + ci) = a_b + ci. Given that, in the geometrical context, these q-reals quantify the magnitude of linear elements, it is difficult if not impossible to physically interpret the imaginary components as rotations in the plane.

5.1. Definitions. We define our mathematical entities using standard terminology.

Definition 5.1.01. Given a dense, linearly ordered set, (P, <), a Dedekind cut on P is an ordered pair of sets, (A, B), such that:

1. (A, B ≠ ϕ) ^ (A ∩ B = ϕ) ^ (A U B = P);
2. for all a ϵ A, b ϵ B, a < b;
3. A does not have a greatest element.

Since B = P – A, we can fully define a Dedekind cut simply by specifying the set A in the ordered pair (A, B).

Definition 5.1.02. Let R_D = {A|(A, Q_Q – A) is a Dedekind cut on (Q_Q, <)}.

Definition 5.1.03. The relation "<" (strict order) on R_D is defined by:

for all A_1, A_2 ϵ R_D, A_1 < A_2 iff A_2 properly contains A_1.

Definition 5.1.04. The operation "+" (addition) on R_D is defined by:

for all A_1, A_2 ϵ R_D, $A_1 + A_2$ = {a_b/c_d + e_f/g_h|(a_b/c_d ϵ A_1) ^ (e_f/g_h ϵ A_2)}.

Definition 5.1.05. Let Z = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (a_b/c_d < 0_0)} ϵ R_D, then we call Z the zero element of R_D.

Definition 5.1.06. Let U = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (a_b/c_d < 1_0)} ϵ R_D, then we call U the unity element of R_D.

Definition 5.1.07. Let A ϵ R_D be arbitrary, then the additive inverse of A is:

- A = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (there exists e_f/g_h ϵ Q_Q such that, for all k_p/m_q ϵ A, a_b/c_d < e_f/g_h < - (k_p)/m_q}.

Definition 5.1.08. Let A ϵ R_D be arbitrary, then:

1. if there exist a_b/c_d ϵ A such that 0_0 < a_b/c_d, then Z < A and A is called positive;
2. if there exist a_b/c_d ϵ Q_Q – A such that a_b/c_d < 0_0, then A < Z and A is called negative;
3. the set of all positive elements of R_D will be designated R_D^+ and the set of all negative elements R_D^-.

Definition 5.1.09. The relation "||" (absolute value) on Q_Q is defined by:

for all A ϵ R_D, |A| = A U (- A).

Definition 5.1.10. Let K represent the statement:

for all a_b/c_d ϵ A_1 – Z (|A_1| – Z), e-f/g_h ϵ A_2 – Z (|A_2| – Z), {[(a = 0) ^ (f < 0) ^ (e < |f|)] ˅ [(e = 0) ^ (b < 0) ^ (a < |b|)]}, where |A_i| – Z applies is A_i < Z.

Let L represent the statement:

for all a_b/c_d ϵ A_1 – Z (|A_1| – Z), e-f/g_h ϵ A_2 – Z (|A_2| – Z), {[(c = 0) ^ (h < 0) ^ (g < |h|)] ˅ [(g = 0) ^ (d < 0) ^ (c < |d|)]}, where |A_i| – Z applies is A_i < Z.

Let M = {a_b/c_d * e_f/g_h|[a_b/c_d ϵ A_1 – Z (|A_1| – Z)] ^ [e_f/g_h ϵ A_2 – Z (|A_2| – Z)]}, where |A_i| – Z applies is A_i < Z.

Let - M = {a_b/c_d|- (a_b)/c_d ϵ M}.

Let |M| = M U (- M).

Then the operation, "*" (multiplication) on R_D is defined by:

for all A_1, A_2 ϵ R_D, A_1 * A_2 = M U Z, if (Z < A_1, A_2) ^ ¬ (K ˅ L);

= - (|M| U Z) if (Z < A_1, A_2) ^ [(K ^ ¬ L) ˅ (L ^ ¬ K)];

= - (M U Z) if [(A_1 < Z) ^ (Z < A_2)] ˅ [(A_2 < Z) ^ (Z < A_1)];

= M U Z, if (A_1, A_2 < Z) ^ ¬ (K ˅ L);

= - (|M| U Z) if (A_1, A_2 < Z) ^ [(K ^ ¬ L) ˅ (L ^ ¬ K)];

= Z, if (A_1 = Z) ˅ (A_2 = Z).

Definition 5.1.11. Let A ϵ R_D – Z be arbitrary, then the multiplicative inverse of A is:

A^{-1} = { a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (there exists e_f/g_h ϵ Q_Q – A (Q_Q – |A|) such that e_f/g_h < c_d/a_b} U Z U {0_0}, if Z < A;

$A^{-1} = -(|A|^{-1})$, if A < Z.

Definition 5.1.12. The operation "-" (subtraction) on R_D is defined by:

for all $A_1, A_2 \in R_D$, $A_1 - A_2 = A_3$ iff $A_1 = A_3 * A_2$.

Definition 5.1.13. The operation "÷ " (division) on R_D is defined by:

for all $A_1, A_2 \in R_D$, $A_1 \div A_2 = A_1 * A_2^{-1}$.

Definition 5.1.14. Let $A \in R_D$ be arbitrary, then if $Q_Q - A$ has a least element in Q_Q, A is called a q-rational cut; otherwise, A is called a q-irrational cut. The set of all q-rational cuts will be designated by Q_D.

Definition 5.1.15. Let R_Q = {a_b|a_b = sup A, $A \in R_D$}, then the elements of R_Q will be called q-real numbers. If sup $A \in Q_Q$, then sup A can be represented by a_b/c_d or e_f, where a_b = e_f * c_d, provided e_f exists. If ¬ (sup $A \in Q_Q$), then sup A can be represented by k_p, where k, p ϵ R, the set of real numbers.

Definition 5.1.16. For all a_b ϵ R_Q – {0_0}, p/q ϵ Q, k ϵ Z, $[(a\_b)^0 = 1\_0] \wedge [(a\_b)^1 = a\_b] \wedge [(a\_b)^{k+1} = (a\_b)^k * a\_b] \wedge [(a\_b)^{-k} = 1\_0/(a\_b)^k] \wedge \{[(a\_b)^{1/k} = c\_d]$ iff $[(c\_d)^k = a\_b]\} \wedge \{(a\_b)^{p/q} = [(a\_b)^p]^{1/q}\}$.

Definition 5.1.17. Let (C, <, +, *) be an arbitrary field, then (C, <, +, *) is Dedekind complete iff every Dedekind cut on (C, <) has a supremum in (C, <).

Definition 5.1.18. Let (C, <, +, *) be an arbitrary field, then (C, <, +, *) has the infimum property iff every non-empty subset of (C, <) which is bounded below, has an infimum in (C, <).

Definition 5.1.19. Let (C, <, +, *) be an arbitrary field, then (C, <, +, *) is lattice complete iff it is Dedekind complete and has the infimum property.

Definition 5.1.20. A lattice complete field constructed set-theoretically on the foundation (N_Q, <, +, *) is said to be q-naturally lattice complete or, equivalently, first-order lattice complete.

5.2. Arguments. We demonstrate our arguments using the standard methods and terminology of mathematical logic and ZFC/AFA or generalizations thereof. Specific to the current work, we generalize the Principle of Induction to the Principle of Q-Induction and we utilize results from reference [HJ] and [CD].

Theorem 5.2.01. *The set R_D of Definition 5.1.02 exists.*

Proof. By Definition 5.1.01, R_D is properly containded in $P(Q_Q)$, hence, by Theorem 4.2.01 and the Axiom of Power Set, R_D exists, as desired. □

Theorem 5.2.02. *The set R_D is closed under the arithmetical operation addition.*

Proof. Consistent with Definition 5.1.01, the proof is in three parts:

1) Let $A_1, A_2 \in R_D$ be arbitrary, then, by Definition 5.1.01 and 5.1.02, $(A_1 \neq \phi) \wedge (A_2 \neq \phi)$, hence, by Definition 5.1.04, $A_1 + A_2 \neq \phi$. Let (a_b/c_d ϵ A_1) ^ (e_f/g_h ϵ A_2) ^ (k_p/m_q ϵ $Q_Q - A_1$) ^ (n_r/o_s ϵ $Q_Q - A_2$) be arbitrary, then, by Definition 5.1.01 and 5.1.02, (a_b/c_d < k_p/m_q) ^ (e_f/g_h < n_r/o_s), hence, by Theorem 4.2.32, (a_b/c_d + e_f/g_h) < (k_p/m_q + n_r/o_s) and $Q_Q - (A_1 + A_2) \neq \phi$.
2) Suppose, for contradiction, that there exists some a_b/c_d ϵ $Q_Q - (A_1 + A_2)$ such that for some e_f/g_h ϵ $(A_1 + A_2)$, a_b/c_d < e_f/g_h. Then, by Theorem 4.2.07, 4.2.19, and 4.2.21, a_b/c_d + [e_f/g_h - a_b/c_d] = e_f/g_h and, by Definition 5.1.01, 5.1.02, and 5.1.04 and Theorem 4.2.20, (a_b/c_d ϵ A_1) ∨ (a_b/c_d ϵ A_2), a

contradiction. Therefore, for all e_f/g_h ϵ ($A_1 + A_2$), a_b/c_d ϵ Q_Q – ($A_1 + A_2$), e_f/g_h < a_b/c_d. From this immediately follows ($A_1 + A_2$) U [Q_Q – ($A_1 + A_2$)] = Q_Q.

3) By Definition 5.1.01 and 5.1.02, neither A_1 nor A_2 have greatest elements, hence, by Definition 5.1.04 and Theorem 4.2.32, ($A_1 + A_2$) has no greatest element.

Therefore, ($A_1 + A_2$) ϵ R_D and R_D is closed under addition, as desired. □

Theorem 5.2.03. *(A ϵ R_D) → (- A ϵ R_D).*

Proof. Suppose A ϵ R_D, then, consistent with Definition 5.1.01, the proof is in three parts:

1) By Definition 5.1.01 and 5.1.02, A ≠ ϕ and, by Definition 5.1.07 and Theorem 4.2.51, - A ≠ ϕ and, for every k_p/m_q ϵ A, - (k_p)/m_q ϵ Q_Q – (- A), hence, [Q_Q – (- A)] ≠ ϕ.
2) Suppose, for contradiction, that there exists some a_b/c_d ϵ Q_Q – (- A) such that for some e_f/g_h ϵ (- A), a_b/c_d < e_f/g_h. Then, by Definition 5.1.07 and Theorem 4.2.30, a_b/c_d ϵ - A, a contradiction. Therefore, for all e_f/g_h ϵ (- A), a_b/c_d ϵ Q_Q – (- A), e_f/g_h < a_b/c_d. From this immediately follows (- A) U [Q_Q – (- A)] = Q_Q.
3) By Definition 5.1.01 and 5.1.02, A has no greatest element, hence, by Definition 5.1.07 and Theorem 4.2.51, - A has no greatest element.

Therefore, - A ϵ R_D, as desired. □

Lemma 5.2.04. *For all A ϵ R_D, A = - (- A).*

Proof. Let a_b/c_d ϵ A be arbitrary, then, by Definition 5.1.07, there exists e_f/g_h ϵ Q_Q, k_p/m_q ϵ - A such that k_p/m_q < e_f/g_h < - (a_b)/c_d. But then, by Theorem 4.2.15 and 4.2.34 and Lemma 4.2.23 and 4.2.24, a_b/c_d < - (e_f)/g_h < - (k_p)/m_q, hence, a_b/c_d ϵ - (- A) and - (- A) contains A.

Let a_b/c_d ϵ - (- A) be arbitrary, then, by Definition 5.1.07, for all k_p/m_q ϵ - A, there exists e_f/g_h ϵ Q_Q such that a_b/c_d < e_f/g_h < - (k_p)/m_q. But then, by Theorem 4.2.15 and 4.2.34 and Lemma 4.2.23 and 4.2.24, k_p/m_q < - (e_f)/g_h < - (a_b)/c_d, hence, a_b/c_d ϵ A and A contains - (- A).

Therefore, A = - (- A), as desired. □

Theorem 5.2.05. *For all A ϵ R_D, [(A < Z) iff (Z < - A)] ^ [(- A < Z) iff (Z < A)].*

Proof. Suppos A < Z, then, by Definition 5.1.03 and 5.1.05, for all a_b/c_d ϵ A, a_b/c_d < 0_0 and, by Theorem 4.2.15, 4.2.25, and 4.2.34 and Lemma 4.2.23 and 4.2.24, 0_0 < - (a_b)/c_d, hence, by Definition 5.1.07 and 5.1.08, Z < - A. Therefore, (A < Z) → (Z < - A).

Suppose Z < - A, then, by Definition 5.1.08, there exists a_b/c_d ϵ - A such that 0_0 < a_b/c_d and, by Definition 5.1.07 and Lemma 5.2.04, there exists k_p/m_q ϵ Q_Q – A such that k_p/m_q < 0_0, hence, A < Z. Therefore, (Z < - A) → (A < Z).

Therefore, (A < Z) iff (Z < - A).

Suppose - A < Z, then, by Definition 5.1.03 and 5.1.05, for all a_b/c_d ϵ - A, a_b/c_d < 0_0 and, by Theorem 4.2.15, 4.2.25, and 4.2.34 and Lemma 4.2.23 and 4.2.24, 0_0 < - (a_b)/c_d, hence, by Definition 5.1.07 and 5.1.08 and Lemma 5.2.04, Z < A. Therefore, (- A < Z) → (Z < A).

Suppose Z < A, then, by Definition 5.1.08, there exists a_b/c_d ϵ A such that 0_0 < a_b/c_d and, by Definition 5.1.07, there exists k_p/m_q ϵ Q_Q – (- A) such that k_p/m_q < 0_0, hence, - A < Z. Therefore, (Z < A) → (- A < Z).

Therefore, (- A < Z) iff (Z < A).

Therefore, [(A < Z) iff (Z < - A)] ^ [(- A < Z) iff (Z < A)], as desired. □

Corollary 5.2.06. *For all $A \in R_D - Z$, $|A| = |-A| \in R_D^+$.*

Proof. By Definition 5.1.09, Lemma 5.2.04, and Theorem 5.2.05, as desired. □

Theorem 5.2.07. *The set R_D is closed under the arithmetical operation multiplication.*

Proof. By Definition 5.1.10, there are seven cases, three of which are redundant, leaving four cases to consider:

Case 1. Suppose $(Z < A_1, A_2) \wedge \neg (K \vee L)$, then, consistent with Definition 5.1.01, the proof is in three parts:

1) By Definition 5.1.01 and 5.1.02, $(A_1 \neq \phi) \wedge (A_2 \neq \phi)$, hence, by Definition 5.1.10, $A_1 * A_2 \neq \phi$. Let $(a\_b/c\_d \in A_1 - Z) \wedge (e\_f/g\_h \in A_2 - Z) \wedge (k\_p/m\_q \in Q_Q - A_1) \wedge (n\_r/o\_s \in Q_Q - A_2)$ be arbitrary, then, by Definition 5.1.01, 5.1.02, and 5.1.05, $(0\_0 \leq a\_b/c\_d < k\_p/m\_q) \wedge (0\_0 \leq e\_f/g\_h < n\_r/o\_s)$ and, by Definition 4.1.13 and Theorem 4.2.40, $(a\_b/c\_d * e\_f/g\_h) < (k\_p/m\_q * n\_r/o\_s)$, hence, $Q_Q - (A_1 * A_2) \neq \phi$.
2) Suppose, for contradiction, that there exists some $a\_b/c\_d \in Q_Q - (A_1 * A_2)$ such that for some $e\_f/g\_h \in (A_1 * A_2)$, $a\_b/c\_d < e\_f/g\_h$. By Theorem 4.2.30, $(a\_b/c\_d < 0\_0) \vee (a\_b/c\_d = 0\_0) \vee (0\_0 < a\_b/c\_d)$. If $a\_b/c\_d < 0\_0$, by Definition 5.1.10, $a\_b/c\_d \in (A_1 * A_2)$, a contradiction. Otherwise, if $0\_0 \leq a\_b/c\_d$, then $0\_0 \leq a\_b/c\_d < k\_p/m\_q * n\_r/o\_s$ for some $k\_p/m\_q \in A_1 - Z$, $n\_r/o\_s \in A_2 - Z$. By Definition 4.1.12 and 4.1.13 and Theorem 4.2.22 and 4.2.33, $0\_0 \leq a\_b/c\_d * o\_s/n\_r < k\_p/m\_q$ and, by Definition 5.1.01 and 5.1.02 and Theorem 4.2.30, $a\_b/c\_d * o\_s/n\_r \in A_1 - Z$. But then, by Definition 5.1.10 and Theorem 4.2.22 again, $a\_b/c\_d \in (A_1 * A_2)$, a contradiction. Therefore, for all $e\_f/g\_h \in (A_1 * A_2)$, $a\_b/c\_d \in Q_Q - (A_1 * A_2)$, $e\_f/g\_h < a\_b/c\_d$. From this immediately follows $(A_1 * A_2) \cup [Q_Q - (A_1 * A_2)] = Q_Q$.
3) By Definition 5.1.01 and 5.1.02, neither $A_1 - Z$ nor $A_2 - Z$ have greatest elements, hence, by Definition 5.1.10 and Theorem 4.2.33, $(A_1 * A_2)$ has no greatest element.

Case 2. Suppose $(Z < A_1, A_2) \wedge [(K \wedge \neg L) \vee (L \wedge \neg K)]$, then, consistent with Definition 5.1.01, the proof is in three parts:

1) By Definition 5.1.01 and 5.1.02, $(A_1 \neq \phi) \wedge (A_2 \neq \phi)$, hence, by Definition 5.1.10, $A_1 * A_2 \neq \phi$. Let $(a\_b/c\_d \in A_1 - Z) \wedge (e\_f/g\_h \in A_2 - Z) \wedge (k\_p/m\_q \in Q_Q - A_1) \wedge (n\_r/o\_s \in Q_Q - A_2)$ be arbitrary, then, by Definition 5.1.10, $a\_b/c\_d * e\_f/g\_h < 0\_0 \leq k\_p/m\_q * n\_r/o\_s$, hence, by Theorem 4.2.30, $Q_Q - (A_1 * A_2) \neq \phi$.
2) Suppose, for contradiction, that there exists some $a\_b/c\_d \in Q_Q - (A_1 * A_2)$ such that for some $e\_f/g\_h \in (A_1 * A_2)$, $a\_b/c\_d < e\_f/g\_h$, then, by Definition 5.1.10, $(a\_bc\_d \in |M|) \vee (a\_bc\_d \in Z - M)$, a contradiction in either case. Therefore, for all $e\_f/g\_h \in (A_1 * A_2)$, $a\_b/c\_d \in Q_Q - (A_1 * A_2)$, $e\_f/g\_h < a\_b/c\_d$. From this immediately follows $(A_1 * A_2) \cup [Q_Q - (A_1 * A_2)] = Q_Q$.
3) By Definition 5.1.01 and 5.1.02, neither $A_1 - Z$ nor $A_2 - Z$ have greatest elements. By Definition 4.1.06, 4.1.13, and 5.1.01, as $a\_b/c\_d \in A_1 - Z$, $e\_f/g\_h \in A_2 - Z$ get larger, $a\_b/c\_d * e\_f/g\_h$ gets smaller, hence, M has no least element, - M has no greatest element, and $A_1 * A_2 = -(|M| \cup Z)$ has no greatest element.

Case 3. That $-(M \cup Z) \in R_D$ is an immediate consequence of Case 1 and Theorem 5.2.03.

Case 4. That $-(M \cup Z) \in R_D$ is an immediate consequence of Case 1 and Theorem 5.2.03.

Case 5. That $M \cup Z \in R_D$ is an immediate consequence of Case 1.

Case 6. That $-((|M| \cup Z) \in R_D$ is an immediate consequence of Case 2.

Case 7. $Z \in R_D$ by Definition 5.1.05.

In all seven cases, $(A_1 * A_2) \in R_D$.

Therefore, R_D is closed under multiplication, as desired. □

Theorem 5.2.08. *For all $A_1, A_2 \in R_D$, $A_1 + A_2 = A_2 + A_1$.*

Proof. By Definition 5.1.04 and Theorem 4.2.07, addition on R_D is commutative, as desired. □

Theorem 5.2.09. *For all $A_1, A_2 \in R_D$, $A_1 * A_2 = A_2 * A_1$.*

Proof. By Definition 5.1.10 and Theorem 4.2.08, multiplication on R_D is commutative, as desired. □

Theorem 5.2.10. *For all $A_1, A_2, A_3 \in R_D$, $(A_1 + A_2) + A_3 = A_1 + (A_2 + A_3)$.*

Proof. By Definition 5.1.04 and Theorem 4.2.09, addition on R_D is associative, as desired. □

Theorem 5.2.11. *For all $A_1, A_2, A_3 \in R_D$, $(A_1 * A_2) * A_3 = A_1 * (A_2 * A_3)$.*

Proof. By Definition 5.1.10 and Theorem 4.2.10, multiplication on R_D is associative, as desired. □

Theorem 5.2.12. *For all $A_1, A_2, A_3 \in R_D$, $A_1 * (A_2 + A_3) = A_1 * A_2 + A_1 * A_3$.*

Proof. By Definition 5.1.04 and 5.1.10 and Theorem 4.2.11, multiplication is left distributive over addition on R_D, as desired. □

Theorem 5.2.13. *For all $A_1, A_2, A_3 \in R_D$, $(A_1 + A_2) * A_3 = A_1 * A_3 + A_2 * A_3$.*

Proof. By Definition 5.1.04 and 5.1.10 and Theorem 4.2.12, multiplication is right distributive over addition on R_D, as desired. □

Theorem 5.3.14. *For all $A \in R_D$, $A + Z = A$.*

Proof. Let a_b/c_d ϵ A, e_f/g_h ϵ Z be arbitrary, then, by Definition 5.1.05, e_f/g_h < 0_0 and, by Theorem 4.2.31, e_f/g_h + a_b/c_d < 0_0 + a_b/c_d. Therefore, by Definition 5.1.01, A contains A + Z.

Conversely, let a_b/c_d, e_f/g_h ϵ A be arbitrary but such that a_b/c_d < e_f/g_h, then, by Theorem 4.2.19 and 4.2.31, a_b/c_d + [- (e_f)]/g_h < 0_0. But then, by Definition 5.1.05, a_b/c_d + [- (e_f)]/g_h ϵ Z and, by Theorem 4.2.19 again, a_b/c_d ϵ A + Z. Therefore, A + Z contains A.

Therefore, A + Z = A and Z is the additive identity for R_D, as desired. □

Corollary 5.2.15. *For all $A, X \in R_D$, $(A + X = A) \rightarrow (X = Z)$.*

Proof. Suppose A + X = A and let a_b/c_d ϵ A, e_f/g_h ϵ X be arbitrary, then, by Theorem 5.2.14, a_b/c_d + e_f/g_h ϵ A + Z and, since a_b/c_d ϵ A, e_f/g_h ϵ Z, hence, Z contains X.

Conversely, let a_b/c_d ϵ A, e_f/g_h ϵ Z be arbitrary, then, by supposition and Theorem 5.2.14, a_b/c_d + e_f/g_h ϵ A + X and, since a_b/c_d ϵ A, e_f/g_h ϵ X, hence, X contains Z.

Therefore, X = Z and the additive identity is unique, as desired. □

Theorem 5.2.16. *For all $A \in R_D$, $A + (- A) = Z$.*

Proof. Let a_b/c_d ϵ - A be arbitrary, then, by Definition 5.1.07, there exists e_f/g_h ϵ Q_Q such that, for all k_p/m_q ϵ A, a_b/c_d < e_f/g_h < - (k_p)/m_q. But then, by Theorem 4.2.07, 4.2.19, and 4.2.32, a_b/c_d + k_p/m_q < 0_0 and, by Definition 5.1.05, Z contains A + (- A).

By Defintion 5.1.04 and 5.1.07 and Theorem 4.2.19 and 4.2.48, for all a_b/c_d ϵ A, e_f/g_h ϵ - A, a_b/c_d + e_f/g_h < 0_0, yet there exists a_b/c_d ϵ A, e_f/g_h ϵ - A such that a_b/c_d + e_f/g_h approaches arbitrarily close to 0_0 without attaining 0_0. But, by Definition 5.1.05, this precisely describes the elements of Z, hence, A + (- A) contains Z.

Therefore, A + (- A) = Z and - A is the additive inverse of A, as desired. □

Corollary 5.2.17. *For all $A, X \in R_D$, $(A + X = Z) \rightarrow (X = - A)$.*

Proof. Suppose A + X = Z, then, by Theorem 5.2.08, 5.2.10, 5.2.14, and 5.2.16:

(A + X) + (- A) = Z + (- A);

(X + A) + (- A) = (- A);

X + [A + (- A)] = (- A);

X + Z = (- A);

X = (- A).

Therefore, the additive inverse is unique, as desired. □

Lemma 5.2.18. *For all A ϵ R_D, A * Z = Z.*

Proof. By Definition 5.1.10, as desired. □

Theorem 5.2.19. *(R_D, <) is a linearly ordered set.*

Proof. The proof is in three parts:

1) *Transitivity.* Let A_1, A_2, A_3 ϵ R_D be arbitrary but such that ($A_1 < A_2$) ^ ($A_2 < A_3$). Then, by Definition 5.1.03, A_2 properly contains A_1 and A_3 properly contains A_2, hence, A_3 properly contains A_1 and $A_1 < A_3$.
2) *Asymmetry.* Let A_1, A_2 ϵ R_D be arbitrary and suppose, for contradiction, that ($A_1 < A_2$) ^ ($A_2 < A_1$), then, by transitivity, $A_1 < A_1$, a contradiction.
3) *Linearity.* Let A_1, A_2 ϵ R_D be arbitrary, then, by Definition 5.1.01 and 5.1.02, there are three possible cases:

   Case 1. ($A_1 – A_2$ = ϕ) ^ ($A_2 – A_1$ ≠ ϕ), then, by Definition 5.1.03, $A_1 < A_2$.

   Case 2. ($A_1 – A_2$ = ϕ) ^ ($A_2 – A_1$ = ϕ), then, by the Axiom of Extensionality, $A_1 = A_2$.

   Case 3. ($A_1 – A_2$ ≠ ϕ) ^ ($A_2 – A_1$ = ϕ), then, by Definition 5.1.03, $A_2 < A_1$.

Therefore, (R_D, <) is a linearly ordered set, as desired. □

Theorem 5.2.20. *For all A ϵ R_D, A * U = A.*

Proof. By Theorem 5.2.19, (A < Z) ˅ (A = Z) ˅ (Z < A) and three cases arise:

Case 1. Suppose Z < A and let a_b/c_d ϵ A – Z, e_f/g_h ϵ U be arbitrary. Then, by Definition 5.1.06, e_f/g_h < 1_0 and, by Theorem 4.2.08, 4.2.15, and 4.2.33, a_b/c_d * e_f/g_h < a_b/c_d, hence, by Definition 5.1.10, A contains A * U.

Let a_b/c_d ϵ A be arbitrary, then, by Definition 5.1.01 and 5.1.02, there exists e_f/g_h ϵ A such that a_b/c_d < e_f/g_h. Let k_p.m_q ϵ U be arbitrary, then, by Definition 5.1.06, k_p/m_q approaches arbitrarily close to 1_0. Let k_p/m_q approach arbitrarily close to 1_0, then, by Theorem 4.2.15, e_f/g_h * k_p/m_q approaches arbitrarily close to e_f/g_h and a_b/c_d < e_f/g_h * k_p/m_q. Therefore, by Definition 5.1.10, A * U contains A.

Therefore, (Z < A) → (A * U = A).

Case 2. Suppose A = Z, then, by Lemma 5.2.18, A * U = A.

Case 3. Suppose A < Z, then, by Definition 5.1.10, Case 1 immediately above, and Lemma 5.2.04, A * U = A.

In all three cases, A * U = A.

Therefore, A * U = A and U is a multiplicative identity for R_D, as desired. □

Corollary 5.2.21. *For all $A, X \in R_D$, $(A * X = A) \rightarrow (X = U)$.*

Proof. Suppose $A * X = A$, then, by Theorem 5.2.20, $A * X = A * U$ and, by Definition 5.1.10, $X = U$. Therefore, the multiplicative identity is unique, as desired. □

Lemma 5.2.22. *For all $A_1, A_2 \in R_D$, $A_1 - A_2 = A_1 + -(A_2)$.*

Proof. Suppose, for contradiction, that $A_1 - A_2 \neq A_1 + -(A_2)$, then, by Definition 5.1.12 and Theorem 5.2.08, 5.2.10, 5.2.14, and 5.2.16:

$$A_1 - A_2 \neq A_1 + (-A_2);$$

$$A_1 \neq [A_1 + (-A_2)] + A_2;$$

$$\neq A_1 + [(-A_2) + A_2];$$

$$\neq A_1 + [A_2 + (-A_2)];$$

$$\neq A_1 + Z;$$

$$\neq A_1, \text{ a contradiction.}$$

Therefore, $A_1 - A_2 = A_1 + -(A_2)$, as desired. □

Theorem 5.2.23. *R_D is closed under subtraction.*

Proof. By Theorem 5.2.02 and Lemma 5.2.22, as desired. □

Theorem 5.2.24. *$(A \in R_D - Z) \rightarrow (A^{-1} \in R_D)$.*

Proof. Suppose $A \in R_D - Z$, then, by Theorem 5.2.19, $(A < Z) \vee (Z < A)$ and two cases arise:

Case 1. Suppose $Z < A$, then, consistent with Definition 5.1.01, the proof is in three parts:

1) By Definition 5.1.01 and 5.1.02, $(Q_Q - A \neq \phi)$, hence, by Definition 5.1.01, 5.1.02, and 5.1.11 and Lemma 4.2.35, $A^{-1} \neq \phi$.
2) Suppose, for contradiction, that there exists some a_b/c_d $\in Q_Q - A^{-1}$ such that, for some e_f/g_h $\in A^{-1}$, a_b/c_d < e_f/g_h. Then, by Definition 5.1.11, there exists k_p/m_q $\in Q_Q - A$ such that k_p/m_q < g_h/e_f and, by Lemma 4.2.35, g_h/e_f < c_d/a_b, hence, by Theorem 4.2.30, a_b/c_d $\in A^{-1}$, a contradiction. Therefore, for all e_f/g_h $\in A^{-1}$, a_b/c_d $\in Q_Q - A^{-1}$, e_f/g_h < a_b/c_d. From this immediately follows $A^{-1} \cup (Q_Q - A^{-1}) = Q_Q$.
3) Suppose, for contradiction, that A^{-1} has a greatest element a_b/c_d. Then, by Lemma 4.2.35, for all e_f/g_h $\in A^{-1}$, there exists k_p/m_q $\in Q_Q - A$ such that k_p/m_q < c_d/a_b ≤ g_h/e_f. But then, by Theorem 4.2.48, there exists o_s/n_r $\in Q_Q$ such that k_p/m_q < o_s/n_r < c_d/a_b and a_b/c_d < n_r/o_s $\in A^{-1}$, a contradiction.

Therefore, $[(A \in R_D - Z) \wedge (Z < A)] \rightarrow (A^{-1} \in R_D)$.

Case 2. Suppose $A < Z$, then, by Definition 5.1.11, Case 1 immediately above, and Lemma 5.2.04, $A^{-1} \in R_D$.

Therefore, $[(A \in R_D - Z) \wedge (A < Z)] \rightarrow (A^{-1} \in R_D)$.

In both cases, $A^{-1} \in R_D$.

Therefore, $[(A \in R_D - Z) \rightarrow (A^{-1} \in R_D)$, as desired. □

Theorem 5.2.25. *For all $A \in R_D - Z$, $A * A^{-1} = U$.*

Proof. By Theorem 5.2.19, (A < Z) ˅ (Z < A) and two cases arise:

Case 1. Suppose Z < A, then, by Definition 5.1.03 and 5.1.11, Z < A^{-1}. Let a_b/c_d ϵ A – Z, e_f/g_h ϵ A^{-1} – Z be arbitrary, then, by Definition 5.1.01, 5.1.02, and 5.1.11, there exist k_p/m_q ϵ Q_Q – A such that a_b/c_d < k_p/m_q < g_h/e_f and, by Theorem 4.2.30 and Lemma 4.2.35, e_f/g_h < c_d/a_b, hence, by Theorem 4.2.08, 4.2.22, and 4.2.33:

a_b/c_d * e_f/g_h = e_f/g_h * a_b/c_d;

< c_d/a_b * a_b/c_d;

= a_b/c_d * c_d/a_b;

= 1_0.

Therefore, by Definition 5.1.06, U contains A * A^{-1}.

Let glb(Q_Q – A) = x_y, then, by Definition 5.1.01 and 5.1.02 and Lemma 4.2.35, for any e_f/g_h ϵ Q_Q – A, g_h/e_f < $(x\_y)^{-1}$ and, by Definition 5.1.11 and Theorem 4.2.48, g_h/e_f ϵ A^{-1}. Furthermore, by Theorem 4.2.48, e_f/g_h approaches arbitrarily close to x_y, hence, g_h/e_f approaches arbitrarily close to $(x\_y)^{-1}$. Let a_b/c_d * g_h/e_f ϵ A * A^{-1} be arbitrary, then, as g_h/e_f approaches arbitrarily close to $(x\_y)^{-1}$, by Definition 5.1.01, 5.1.02, and 5.1.10 and Theorem 4.2.22, a_b/c_d * g_h/e_f approaches arbitrarily close to x_y * $(x\_y)^{-1}$ = 1_0, hence, by Definition 5.1.06, A * A^{-1} contains U.

Therefore, (Z < A) → (A * A^{-1} = U).

Case 2. Suppose A < Z, then, by Definition 5.1.10, Case 1 immediately above, and Lemma 5.2.04, A * A^{-1} = U.

Therefore, (A < Z) → (A * A^{-1} = U).

In both cases, A * A^{-1} = U.

Therefore, A * A^{-1} = U, as desired. □

Theorem 5.2.26. *For all A_1, A_2 ϵ R_D, A_2 ≠ Z, A_1 ÷ A_2 ϵ R_D.*

Proof. By Definition 5.1.13 and Theorem 5.2.07 and 5.2.24, as desired. □

Theorem 5.2.27. *For all A_1, A_2, A_3 ϵ R_D, (A_1 < A_2) iff [(A_1 + A_3) < (A_2 + A_3)].*

Proof. Suppose A_1 < A_2, then, by Definition 5.1.03, for all a_b/c_d ϵ A_1, a_b/c_d ϵ A_2, but there exist e_f/g_h ϵ A_2 such that ¬ (e_f/g_h ϵ A_1). But then, by Definition 5.1.04, for all k_p/m_q ϵ A_3, a_b/c_d ϵ A_1, [a_b/c_d + k_p/m_q ϵ (A_1 + A_3)] ^ [a_b/c_d + k_p/m_q ϵ (A_2 + A_3)], while, for those e_f/g_h ϵ A_2, ¬ (e_f/g_h ϵ A_1), {¬ [e_f/g_h + k_p/m_q ϵ (A_1 + A_3)]} ^ [e_f/g_h + k_p/m_q ϵ (A_2 + A_3)], hence, by Definition 5.1.03, (A_1 + A_3) < (A_2 + A_3). Therefore, (A_1 < A_2) → [(A_1 + A_3) < (A_2 + A_3)].

Suppose (A_1 + A_3) < (A_2 + A_3), then, by Definition 5.1.03, for all a_b/c_d ϵ A_1, k_p/m_q ϵ A_3, a_b/c_d + k_p/m_q ϵ (A_2 + A_3), but there exists e_f/g_h ϵ A_2 such that {¬ [e_f/g_h + k_p/m_q ϵ (A_1 + A_3)]} ^ [e_f/g_h + k_p/m_q ϵ (A_2 + A_3)]. But then, by Definition 5.1.04, for all a-b/c_d ϵ A_1, a_b/c_d ϵ A_2 while there exists e_f/g_h ϵ A_2 such that ¬ (e_f/g_h ϵ A_1), hence, by Definition 5.1.03, A_1 < A_2. Therefore, [(A_1 + A_3) < (A_2 + A_3)] → (A_1 < A_2).

Therefore, (A_1 < A_2) iff [(A_1 + A_3) < (A_2 + A_3)], as desired. □

Theorem 5.2.28. *For all A_1, A_2, A_3, A_4 ϵ R_D, [(A_1 < A_2) ^ (A_3 < A_4)] → [(A_1 + A_3) < (A_2 + A_4)].*

Proof. Suppose (A_1 < A_2) ^ (A_3 < A_4), then, by Theorem 5.2.27, (A_1 + A_3) < (A_2 + A_3) and, by Theorem 5.2.08 and 5.2.27, (A_2 + A_3) < (A_2 + A_4), hence, by Theorem 5.2.19, (A_1 + A_3) < (A_2 + A_4), as desired. □

Theorem 5.2.29. *Let $S = \{a\_b/c\_d \mid (a\_b/c\_d \in Q_Q^+) \wedge [(a \neq 0) \wedge \{(0 < b) \vee [(b < 0) \wedge (|b| < a)]\}] \wedge [(c \neq 0) \wedge \{(0 < d) \vee [(d < 0) \wedge (|d| < c)]\}]\}$ and let $V = \{A \mid (A \in R_D^+) \wedge$ [for any $a\_b/c\_d \in (A - Z) - S$, there exists $e\_f/g\_h \in (A - Z) \cap S$ such that $a\_b/c\_d < e\_f/g\_h]\}$. Then, for all $A_1, A_2 \in R_D$, $A_3 \in V$, $(A_1 < A_2)$ iff $[(A_1 * A_3) < (A_2 * A_3)]$.*

Proof. Let $A_1, A_2 \in R_D$, $A_3 \in V$, be arbitrary, then, by Theorem 5.2.19, $[(A_1 < Z) \vee (A_1 = Z) \vee (Z < A_1)] \wedge [(A_2 < Z) \vee (A_2 = Z) \vee (Z < A_2)]$ and five cases arise:

Case 1. Suppose $(Z < A_1) \wedge (Z < A_2) \wedge (A_1 < A_2)$, then, by Definition 5.1.01, 5.1.02, and 5.1.03, $A_2 - Z$ properly contains $A_1 - Z$, hence, by Definition 5.1.10 and Theorem 4.2.33, $A_2 * A_3$ properly contains $A_1 * A_3$ and $(A_1 * A_3) < (A_2 * A_3)$.

Suppose $(Z < A_1) \wedge (Z < A_2) \wedge [(A_1 * A_3) < (A_2 * A_3)]$, then, by Definition 5.1.01, 5.1.02, and 5.1.03 and Theorem 5.2.07, $(A_2 * A_3) - Z$ properly contains $(A_1 * A_3) - Z$, hence, by Definition 5.1.10 and Theorem 4.2.33, A_2 properly contains A_1 and $A_1 < A_2$.

Therefore, $[(Z < A_1) \wedge (Z < A_2)] \rightarrow \{(A_1 < A_2)$ iff $[(A_1 * A_3) < (A_2 * A_3)]\}$.

Case 2. Suppose $(A_1 < Z) \wedge (A_2 < Z) \wedge (A_1 < A_2)$, then, by Definition 5.1.01, 5.1.02, and 5.1.03, A_2 properly contains A_1, hence, by Definition 5.1.07 and 5.1.09 and Theorem 4.2.34, $|A_1|$ properly contains $|A_2|$ and $|A_2| < |A_1|$. By Case 1 immediately above, $(|A_2| * A_3) < (|A_1| * A_3)$ and, by Definition 5.1.10 and Theorem 4.2.34 again, $(A_1 * A_3) < (A_2 * A_3)$.

Suppose $(A_1 < Z) \wedge (A_2 < Z) \wedge [(A_1 * A_3) < (A_2 * A_3)]$, then, by Definition 5.1.01, 5.1.02, and 5.1.03, $(A_2 * A_3)$ properly contains $(A_1 * A_3)$, and, by Definition 5.1.10 and Theorem 4.2.34, $(|A_2| * A_3) < (|A_1| * A_3)$. By Case 1 immediately above, $|A_2| < |A_1|$ and, by Definition 5.1.07 and 5.1.09 and Theorem 4.2.34 again, $A_1 < A_2$.

Therefore, $[(A_1 < Z) \wedge (A_2 < Z)] \rightarrow \{(A_1 < A_2)$ iff $[(A_1 * A_3) < (A_2 * A_3)]\}$.

Case 3. Suppose $(A_1 < Z) \wedge (A_2 = Z)$, then, by Definition 5.1.10 and Lemma 5.2.18, $(A_1 < A_2)$ iff $[(A_1 * A_3) < (A_2 * A_3)]$.

Therefore, $[(A_1 < Z) \wedge (A_2 = Z)] \rightarrow \{(A_1 < A_2)$ iff $[(A_1 * A_3) < (A_2 * A_3)]\}$.

Case 4. Suppose $(A_1 < Z) \wedge (Z < A_2)$, then, by Definition 5.1.10, $(A_1 * A_3) < Z < (A_2 * A_3)$ and, by Theorem 5.2.19, $(A_1 < A_2)$ iff $[(A_1 * A_3) < (A_2 * A_3)]$.

Therefore, $[(A_1 < Z) \wedge (Z < A_2)] \rightarrow \{(A_1 < A_2)$ iff $[(A_1 * A_3) < (A_2 * A_3)]\}$.

Case 5. Suppose $(A_1 = Z) \wedge (Z < A_2)$, then, by Definition 5.1.10 and Lemma 5.2.18, $(A_1 < A_2)$ iff $[(A_1 * A_3) < (A_2 * A_3)]$.

Therefore, $[(A_1 = Z) \wedge (Z < A_2)] \rightarrow \{(A_1 < A_2)$ iff $[(A_1 * A_3) < (A_2 * A_3)]\}$.

In all five cases, $(A_1 < A_2)$ iff $[(A_1 * A_3) < (A_2 * A_3)]$.

Therefore, $(A_1 < A_2)$ iff $[(A_1 * A_3) < (A_2 * A_3)]$, as desired. □

Theorem 5.2.30. *Let $T = \{a\_b/c\_d \mid (a\_b/c\_d \in Q_Q^-) \wedge [(a \neq 0) \wedge \{(0 < b) \vee [(b < 0) \wedge (|b| < a)]\}] \wedge [(c \neq 0) \wedge \{(0 < d) \vee [(d < 0) \wedge (|d| < c)]\}]\}$ and let $W = \{A \mid (A \in R_D^-) \wedge$ [for any $a\_b/c\_d \in A - T$, there exists $e\_f/g\_h \in A \cap T$ such that $e\_f/g\_h < a\_b/c\_d]\}$. Then, for all $A_1, A_2 \in R_D$, $A_3 \in W$, $(A_1 < A_2)$ iff $[(A_2 * A_3) < (A_1 * A_3)]$.*

Proof. Let $A_1, A_2 \in R_D$, $A_3 \in W$, be arbitrary, then, by Theorem 5.2.19, $[(A_1 < Z) \vee (A_1 = Z) \vee (Z < A_1)] \wedge [(A_2 < Z) \vee (A_2 = Z) \vee (Z < A_2)]$ and five cases arise:

Case 1. Suppose $(Z < A_1) \wedge (Z < A_2) \wedge (A_1 < A_2)$, then, by Definition 5.1.07 and 5.1.09, $|A_3| \in V$, where V is defined in Theorem 5.2.29, and, by Theorem 5.2.28, $(A_1 * |A_3|) < (A_2 * |A_3|)$. But then, by Definition 5.1.10 and Theorem 4.2.34, $(A_2 * A_3) < (A_1 * A_3)$.

Suppose $(Z < A_1) \wedge (Z < A_2) \wedge [(A_2 * A_3) < (A_1 * A_3)]$, then, by Definition 5.1.07 and 5.1.09 and Theorem 4.2.34, $(A_1 * |A_3|) < (A_2 * |A_3|)$ and $|A_3| \in V$, where V is defined in Theorem 5.2.29. But then, by Theorem 5.2.29, $A_1 < A_2$.

Therefore, $[(Z < A_1) \wedge (Z < A_2)] \rightarrow \{(A_1 < A_2) \text{ iff } [(A_2 * A_3) < (A_1 * A_3)]\}$.

Case 2. Suppose $(A_1 < Z) \wedge (A_2 < Z) \wedge (A_1 < A_2)$, then, by Definition 5.1.07 and 5.1.09, $|A_3| \in V$, where V is defined in Theorem 5.2.29, and, by Theorem 4.2.34, $|A_2| < |A_1|$. But then, by Theorem 5.2.29, $(|A_2| * |A_3|) < (|A_1| * |A_3|)$ and, by Definition 5.1.10, $(A_2 * A_3) < (A_1 * A_3)$.

Suppose $(A_1 < Z) \wedge (A_2 < Z) \wedge [(A_1 * A_3) < (A_2 * A_3)]$, then, by Definition 5.1.10, $(|A_2| * |A_3|) < (|A_1| * |A_3|)$ and, by Theorem 5.2.29, $|A_2| < |A_1|$. But then, by Definition 5.1.07 and 5.1.09 and Theorem 4.2.34, $A_1 < A_2$.

Therefore, $[(A_1 < Z) \wedge (A_2 < Z)] \rightarrow \{(A_1 < A_2) \text{ iff } [(A_2 * A_3) < (A_1 * A_3)]\}$.

Case 3. Suppose $(A_1 < Z) \wedge (A_2 = Z)$, then, by Definition 5.1.10 and Lemma 5.2.18, $(A_1 < A_2)$ iff $[(A_2 * A_3) < (A_1 * A_3)]$.

Therefore, $[(A_1 < Z) \wedge (A_2 = Z)] \rightarrow \{(A_1 < A_2) \text{ iff } [(A_2 * A_3) < (A_1 * A_3)]\}$.

Case 4. Suppose $(A_1 < Z) \wedge (Z < A_2)$, then, by Definition 5.1.10, $(A_1 * A_3) < Z < (A_2 * A_3)$ and, by Theorem 5.2.19, $(A_1 < A_2)$ iff $[(A_2 * A_3) < (A_1 * A_3)]$.

Therefore, $[(A_1 < Z) \wedge (Z < A_2)] \rightarrow \{(A_1 < A_2) \text{ iff } [(A_2 * A_3) < (A_1 * A_3)]\}$.

Case 5. Suppose $(A_1 = Z) \wedge (Z < A_2)$, then, by Definition 5.1.10 and Lemma 5.2.18, $(A_1 < A_2)$ iff $[(A_2 * A_3) < (A_1 * A_3)]$.

Therefore, $[(A_1 = Z) \wedge (Z < A_2)] \rightarrow \{(A_1 < A_2) \text{ iff } [(A_2 * A_3) < (A_1 * A_3)]\}$.

In all five cases, $(A_1 < A_2)$ iff $[(A_2 * A_3) < (A_1 * A_3)]$.

Therefore, $(A_1 < A_2)$ iff $[(A_2 * A_3) < (A_1 * A_3)]$, as desired. □

Theorem 5.2.31. *For all $A_1, A_2 \in R_D$, $|A_1 + A_2| \leq |A_1| + |A_2|$.*

Proof. Let $A_1, A_2 \in R_D$ be arbitrary, then, by Theorem 5.2.19, $[(A_1 < Z) \vee (A_1 = Z) \vee (Z < A_1)] \wedge [(A_2 < Z) \vee (A_2 = Z) \vee (Z < A_2)]$ and nine cases arise, three of which are redundant, leaving six cases to consider:

Case 1. Suppose $(A_1 < Z) \wedge (A_2 < Z)$, then, by Definition 5.1.04, 5.1.07 and 5.1.09 and Lemma 5.2.22:

$$
\begin{aligned}
|A_1 + A_2| &= (A_1 + A_2) \cup [-(A_1 + A_2)]; \\
&= (A_1 + A_2) \cup [(-A_1) + (-A_2)]; \\
&= [A_1 \cup (-A_1)] + [A_2 \cup (-A_2)]; \\
&= |A_1| + |A_2|.
\end{aligned}
$$

Case 2. Suppose $(A_1 < Z) \wedge (A_2 = Z)$, then, by Definition 5.1.05, 5.1.07 and 5.1.09 and Theorem 5.2.14:

$$
\begin{aligned}
|A_1 + A_2| &= |A_1 + Z|; \\
&= |A_1|; \\
&= |A_1| + Z; \\
&= |A_1| + |Z|; \\
&= |A_1| + |A_2|.
\end{aligned}
$$

Case 3. Suppose $(A_1 < Z) \wedge (Z < A_2)$, then, by Theorem 5.2.19, $(|A_1| < A_2) \vee (|A_1| = A_2) \vee (A_2 < |A_1|)$ and three cases arise:

Case 3.a. Suppose $|A_1| < A_2$, then, by Definition 5.1.04 and 5.1.12, $Z < (A_1 + A_2)$ and, by Theorem 5.2.05 and 5.2.21, $A_1 < |A_1|$, hence, by Definition 5.1.09 and Theorem 5.2.27:

$$|A_1 + A_2| = A_1 + A_2;$$

$$< |A_1| + A_2;$$

$$= |A_1| + |A_2|.$$

Case 3.b. Suppose $|A_1| = A_2$, then, by Definition 5.1.05 and 5.1.12, $Z = (A_1 + A_2)$ and, by Definition 5.1.09 and Theorem 5.2.05, $A_1 < |A_1|$, hence, by Theorem 5.2.27:

$$|A_1 + A_2| = |Z|;$$

$$= Z;$$

$$= A_1 + A_2;$$

$$< |A_1| + A_2;$$

$$= |A_1| + |A_2|.$$

Case 3.c. Suppose $A_2 < |A_1|$, then, by Definition 5.1.04 and 5.1.12, $(A_1 + A_2) < Z$ and, by Theorem 5.2.05, $Z < -(A_1 + A_2)$, hence, by Definition 5.2.09, $|A_1 + A_2| = -(A_1 + A_2)$. By Theorem 5.2.05, $(-A_2) < A_2$ and, by Definition 5.1.09, $(-A_1) = |A_1|$, hence, by Lemma 5.2.22 and Theorem 5.2.27:

$$|A_1 + A_2| = -(A_1 + A_2);$$

$$= (-A_1) + (-A_2);$$

$$< |A_1| + A_2;$$

$$= |A_1| + |A_2|.$$

In all three cases, $|A_1 + A_2| < |A_1| + |A_2|$.

Therefore, $[(A_1 < Z) \wedge (Z < A_2)] \rightarrow (|A_1 + A_2| < |A_1| + |A_2|)$.

This result remains unchanged in the case $(Z < A_1) < (A_2 < Z)$.

Case 4. Suppose $(A_1 = Z) \wedge (A_2 = Z)$, then, by Definition 5.1.05, 5.1.07, and 5.1.09 and Theorem 5.2.14:

$$|A_1 + A_2| = |Z + Z|;$$

$$= |Z|;$$

$$= Z;$$

$$= Z + Z;$$

$$= |Z| + |Z|;$$

$$= |A_1| + |A_2|.$$

Case 5. Suppose $(A_1 = Z) \wedge (Z < A_2)$, then, by Definition 5.1.05, 5.1.07, and 5.1.09 and Theorem 5.2.14:

$$|A_1 + A_2| = |Z + A_2|;$$

$= |A_2|;$

$= Z + |A_2|;$

$=|Z| + |A_2|;$

$= |A_1| + |A_2|.$

This result remains unchanged in the case $(Z < A_1) \wedge (A_2 = Z)$.

Case 6. Suppose $(Z < A_1) \wedge (Z < A_2)$, then, by Definition 5.1.09:

$|A_1 + A_2| = A_1 + A_2;$

$= |A_1| + |A_2|.$

In all nine cases, $|A_1 + A_2| \leq |A_1| + |A_2|$.

Therefore, $|A_1 + A_2| \leq |A_1| + |A_2|$, as desired. □

Corollary 5.2.32. *For all $A_1, A_2 \in R_D$, $|A_1| - |A_2| \leq |A_1 - A_2|$.*

Proof. By Definition 5.1.12 and Theorem 5.2.16 and 5.2.31:

$|(A_1 - A_2) + A_2| \leq | A_1 - A_2| + |A_2|;$

$|A_1 + [(- A_2) + A_2]| \leq | A_1 - A_2| + |A_2|;$

$|A_1 + [A_2 + (- A_2)]| \leq | A_1 - A_2| + |A_2|;$

$|A_1| \leq | A_1 - A_2| + |A_2|;$

$|A_1| - |A_2| \leq |A_1 - A_2|$, as desired. □

Theorem 5.2.33. *For all $A_1, A_2 \in R_D$, $(Z < A_1) \rightarrow [(|A_2| < A_1)$ iff $(- A_1 < A_2 < A_1)]$.*

Proof. Suppose $(Z < A_1) \wedge (|A_2| < A_1)$, then, by Theorem 5.2.19, $(A_2 < Z) \vee (A_2 = Z) \vee (Z < A_2)$ and two cases arise:

Case 1. Suppose $A_2 < Z$, then, by Definition 5.1.03 and 5.1.09, $(A_2 < A_1) \wedge (- A_2 < A_1)$, hence, by Lemma 5.2.04 and Theorem 5.2.29, $- A_1 < A_2 < A_1$.

Case 2. Suppose $Z \leq A_2$, then, by Definition 5.1.09, $Z \leq A_2 < A_1$ and, by Theorem 5.2.30, $- A_1 < - A_2$, hence, by Lemma 5.2.04 and Theorem 5.2.19, $- A_1 < A_2 < A_1$.

In both cases, $- A_1 < A_2 < A_1$.

Therefore, $(|A_2| < A_1) \rightarrow (- A_1 < A_2 < A_1)$.

Suppose $(Z < A_1) \wedge (- A_1 < A_2 < A_1)$, then, by Theorem 5.2.19, $(A_2 < Z) \vee (A_2 = Z) \vee (Z < A_2)$ and two cases arise:

Case 1. Suppose $A_2 < Z$, then, by Lemma 5.2.04, $A_2 \cup - A_2 = - A_2$, and, by Theorem 5.2.30, $- A_2 < A_1$, hence, by Definition 5.1.09, $|A_2| < A_1$.

Case 2. Suppose $Z \leq A_2$, then, by Definition 5.1.09, $|A_2| < A_1$.

In both cases, $|A_2| < A_1$.

Therefore, $(- A_1 < A_2 < A_1) \rightarrow (|A_2| < A_1)$.

Therefore, $(Z < A_1) \rightarrow [(|A_2| < A_1)$ iff $(- A_1 < A_2 < A_1)]$, as desired. □

Therorem 5.2.34. *For all $A_1, A_2 \in R_D$, $|A_1| * |A_2| \leq |A_1 * A_2|$.*

Proof. Let $A_1, A_2 \in R_D$ be arbitrary, then, by Definition 5.1.10, eight cases arise, two of which are redundant, leaving six cases to consider:

Case 1. Suppose $(Z < A_1, A_2) \wedge \neg (K \vee L)$, then, by Definition 5.1.09 and 5.1.10:

$$|A_1| * |A_2| = A_1 * A_2;$$
$$= M \cup Z;$$
$$= |M \cup Z|;$$
$$= |A_1 * A_2|.$$

Case 2. Suppose $(Z < A_1, A_2) \wedge [(K \wedge \neg L) \vee (L \wedge \neg K)]$, then, by Definition 5.1.09 and 5.1.10, Lemma 5.2.04, and Theorem 5.2.05:

$$|A_1| * |A_2| = A_1 * A_2;$$
$$= - (|M| \cup Z);$$
$$< - [- (|M| \cup Z)];$$
$$= |A_1 * A_2|.$$

Case 3. Suppose $(A_1 < Z) \wedge (Z < A_2)$, then, by Definition 5.1.03, $A_1 \cup - A_1 = - A_1$ and, by Definition 5.1.09 and 5.1.10 and Lemma 5.2.04:

$$|A_1| * |A_2| = - A_1 * A_2;$$
$$= M \cup Z;$$
$$= - [- (M \cup Z)];$$
$$= |A_1 * A_2|.$$

This result remains unchanged in the case $(Z < A_1) \wedge (A_2 < Z)$.

Case 4. Suppose $(A_1, A_2 < Z) \wedge \neg (K \vee L)$, then, by Definition 5.1.03, $(A_1 \cup - A_1 = - A_1) \wedge (A_2 \cup - A_2 = - A_2)$ and, by Definition 5.1.09 and 5.1.10 and Theorem 5.2.05:

$$|A_1| * |A_2| = - A_1 * - A_2;$$
$$= M \cup Z;$$
$$= |A_1 * A_2|.$$

Case 5. Suppose $(A_1, A_2 < Z) \wedge [(K \wedge \neg L) \vee (L \wedge \neg K)]$, then, by Definition 5.1.03, $(A_1 \cup - A_1 = - A_1) \wedge (A_2 \cup - A_2 = - A_2)$ and, by Definition 5.1.09 and 5.1.10 and Theorem 5.2.05:

$$|A_1| * |A_2| = - A_1 * - A_2;$$
$$= - (|M| \cup Z);$$
$$< - [- (|M| \cup Z)];$$
$$= |A_1 * A_2|.$$

Case 6. Suppose $(A_1 = Z) \wedge (A_2 \neq Z)$, then, by Definition 5.1.05, 5.1.07, and 5.1.09 and Lemma 5.2.18:

$|A_1| * |A_2| = |Z| * A_2;$

$= Z * A_2;$

$= Z;$

$= |Z|;$

$= |Z * A_2|;$

$= |A_1 * A_2|.$

This result remains unchanged in the case $(A_1 \neq Z) \wedge (A_2 = Z)$.

In all six cases, $|A_1| * |A_2| \leq |A_1 * A_2|$.

Therefore, $|A_1| * |A_2| \leq |A_1 * A_2|$, as desired. □

Theorem 5.2.35. *For all $n \in Z$, $A \in R_D$, A^n is defined.*

Proof. We proceed by induction on n. Let P(x) be the property, "A^x is defined," then:

P(0). By Definition 5.1.16, $A^0 = 1\_0$.

Suppose P(n) is true, then A^n is defined and:

P(n + 1). By Definition 5.1.16, $A^{(n+1)} = A^n * A$, hence, by Definition 5.1.10 and Theorem 5.2.07, $A^{(n+1)}$ is defined.

P(–n). By Definition 5.1.16, $A^{-n} = (A^{-1})^n$, hence, by Theorem 5.2.26, A^{-n} is defined.

Therefore, $P(n) \rightarrow [P(n + 1) \wedge P(-n)]$ and, by the Principle of Induction on Z (reference [CD], Chapter 5, page 173), for all $n \in Z$, $A \in R_D$, A^n is defined, as desired. □

Lemma 5.2.36. *For all $A \in R_D^-$, A^n is negative if n is odd and positive if n is even.*

Proof. This is an immediate consequence of Definition 5.1.08, 5.1.10, and 5.1.16 and Theorem 5.2.22, as desired. □

Theorem 5.2.37. *For all $n \in Z$, $A_1, A_2 \in R_D$, $(A_1 * A_2)^n = A_1^n * A_2^n$.*

Proof. We proceed by induction on n. Let P(x) be the property, "$(A_1 * A_2)^x = A_1^x * A_2^x$," then:

P(0). By Definition 5.1.16 and Theorem 5.2.22, $(A_1 * A_2)^0 = 1\_0 = 1\_0 * 1\_0 = A_1^0 * A_2^0$.

Suppose P(n) is true, then $(A_1 * A_2)^n = A_1^n * A_2^n$ and:

P(n + 1). By Definition 5.1.16 and Theorem 5.2.09 and 5.2.11:

$(A_1 * A_2)^{(n+1)} = (A_1 * A_2)^n * (A_1 * A_2);$

$= A_1^n * A_2^n * (A_1 * A_2);$

$= A_1^n * [A_2^n * (A_1 * A_2)];$

$= A_1^n * (A_2^n * A_1) * A_2;$

$= A_1^n * (A_1 * A_2^n) * A_2;$

$= (A_1^n * A_1) * (A_2^n * A_2);$

$= A_1^{(n+1)} * A_2^{(n+1)}$.

P(– n). Given that $A_1, A_2 \in R_D$ are arbitrary, by Theorem 5.2.07 and 5.2.37:

$(A_1 * A_2)^{-n} = (A_1^{-1} * A_2^{-1})^n$;

$= (A_1^{-1})^n * (A_2^{-1})^n$;

$= A_1^{-n} * A_2^{-n}$.

Therefore, $P(n) \rightarrow [P(n + 1) \wedge P(-n)]$ and, by the Principle of Induction on Z (reference [CD], Chapter 5, page 173), for all $n \in Z$, $A_1, A_2 \in R_D$, $(A_1 * A_2)^n = A_1^n * A_2^n$, as desired. □

Theorem 5.2.38. *For all m, n ∈ Z, A ∈ R_D, $A^m * A^n = A^{(m+n)}$.*

Proof. We proceed by induction on n. Let P(x) be the property, "$A^m * A^x = A^{(m+x)}$," then:

P(0). By Definition 5.1.16 and Theorem 5.2.22, $A^m * A^0 = A^m * 1\_0 = A^m = A^{(m+0)}$.

Suppose P(n) is true, then $A^m * A^n = A^{(m+n)}$ and:

P(n + 1). By Definition 5.1.16 and Theorem 5.2.11:

$A^m * A^{(n+1)} = A^m * (A^n * A)$;

$= (A^m * A^n) * A$;

$= A^{(m+n)} * A$;

$= A^{[(m+n)+1]}$;

$= A^{[m+(n+1)]}$.

P(– n). This is an immediate consequence of the above argument together with the fact that, for all m, n ∈ Z, m – n = m + (– n) (reference [CD], Chapter 5, page 165).

Therefore, $P(n) \rightarrow [P(n + 1) \wedge P(-n)]$ and, by the Principle of Induction on Z (reference [CD], Chapter 5, page 173), for all m, n ∈ Z, A ∈ R_D, $A^m * A^n = A^{(m+n)}$, as desired. □

Theorem 5.2.39. *For all m, n ∈ Z, A ∈ R_D, $(A^m)^n = A^{(m*n)}$.*

Proof. We proceed by induction on n. Let P(x) be the property, "$(A^m)^x = A^{(m*x)}$," then:

P(0). By Definition 5.1.16, $(A^m)^0 = 1\_0 = A^0 = A^{(m*0)}$.

Suppose P(n) is true, then $(A^m)^n = A^{(m*n)}$ and:

P(n + 1). By Definition 5.1.16 and Theorem 5.2.38:

$(A^m)^{(n+1)} = (A^m)^n * A^m$;

$= A^{(m*n)} * A^m$;

$= A^{[(m*n)+m]}$;

$= A^{[m*(n+1)]}$.

P(– n). By Definition 5.1.16 and the fact that A ∈ R_D is arbitrary:

$(A^m)^{-n} = [(A^{-1})^m]^n$;

$= (A^{-1})^{(m * n)}$;

$= A^{-(m * n)}$;

$= A^{[m * (-n)]}$.

Therefore, $P(n) \rightarrow [P(n + 1) \wedge P(-n)]$ and, by the Principle of Induction on Z (reference [CD], Chapter 5, page 173), for all m, n ϵ Z, A ϵ R_D, $(A^m)^n = A^{(m * n)}$, as desired. □

Theorem 5.2.40. *(Q_D, <, +, *), where Q_D is defined in Definition 5.1.14, is ring isomorphic to (Q_Q, <, +, *).*

Proof. There is an obvious isomorphism, $f:Q_D \rightarrow Q_Q$, defined by f(A) = sup A. Let A_1, A_2 ϵ Q_D be arbitrary, then:

Addition. By Definition 5.1.04 and Theorem 4.2.31 and 4.2.32:

$f(A_1 + A_2) = \sup (A_1 + A_2)$;

$= \sup A_1 + \sup A_2$;

$= f(A_1) + f(A_2)$.

Multiplication. By Definition 5.1.10, eight cases arise, two of which are redundant, leaving six cases to consider:

Case 1. Suppose $(Z < A_1, A_2) \wedge \neg (K \vee L)$, then, by Definition 5.1.10 and Theorem 4.2.33:

$f(A_1 + A_2) = f(M \cup Z)$;

$= \sup (M \cup Z)$;

$= \sup A_1 * \sup A_2$;

$= f(A_1) * f(A_2)$.

Case 2. Suppose $(Z < A_1, A_2) \wedge [(K \wedge \neg L) \vee (L \wedge \neg K)]$, then, by Definition 4.1.04 and Theorem 3.2.29, $(A_1 * A_2) < Z$ and, by Definition 5.1.10:

$f(A_1 * A_2) = f[-(|M| \cup Z)]$;

$= \sup[-(|M| \cup Z)$;

$= \sup A_1 * \sup A_2$;

$= f(A_1) * f(A_2)$.

Case 3. Suppose $(A_1 < Z) \wedge (Z < A_2)$, then, by Definition 4.1.04 and Lemma 4.2.23, $(A_1 * A_2) < Z$ and, by Definition 5.1.10:

$f(A_1 * A_2) = f[-(M \cup Z)]$;

$= \sup[-(M \cup Z)$;

$= \sup A_1 * \sup A_2$;

$= f(A_1) * f(A_2)$.

This result remains unchanged in the case $(Z < A_1) \wedge (A_2 < Z)$.

Case 4. Suppose $(A_1, A_2 < Z) \wedge \neg (K \vee L)$, then, by Definition 4.1.04 and Lemma 4.2.24, $Z < (A_1 * A_2)$ and, by Definition 5.1.10:

$f(A_1 * A_2) = f(M \cup Z);$

$= \sup (M \cup Z);$

$= \sup A_1 * \sup A_2;$

$= f(A_1) * f(A_2).$

Case 5. Suppose $(A_1, A_2 < Z) \wedge [(K \wedge \neg L) \vee (L \wedge \neg K)]$, then, by Definition 4.1.04 and Theorem 3.2.29, $(A_1 * A_2) < Z$ and, by Definition 5.1.10:

$f(A_1 * A_2) = f[- (M \cup Z)];$

$= \sup[- (M \cup Z);$

$= \sup A_1 * \sup A_2;$

$= f(A_1) * f(A_2).$

Case 6. Suppose $(A_1 = Z) \wedge (A_2 \neq Z)$, then, by Definition 5.1.10:

$f(A_1 * A_2) = f(A_1 * Z);$

$= \sup (A_1 * Z);$

$= \sup A_1 * \sup Z;$

$= f(A_1) * f(A_2).$

By Theorem 5.2.09, this result remains unchanged in the case $(A_1 \neq Z) \wedge (A_2 = Z)$.

In all six cases, $f(A_1 * A_2) = f(A_1) * f(A_2)$.

Therefore, $f(A_1 * A_2) = f(A_1) * f(A_2)$.

Therefore, $(Q_D, <, +, *)$ is ring isomorphic to $(Q_Q, <, +, *)$, as desired. □

Theorem 5.2.41. *The set $(Q_D, <)$ of Definition 5.1.14 is dense in R_D.*

Proof. Let $A_1, A_2 \in R_D$ be arbitrary but such that $A_1 < A_2$, then, by Definition 5.1.03, $(A_1 - A_2 = \phi) \wedge (A_2 - A_1 \neq \phi)$. By Definition 5.1.01 and 5.1.02 and Theorem 4.2.48, there exists a_b/c_d $\in Q_Q$ such that (a_b/c_d $\in A_2 - A_1) \wedge \neg$ (sup A_1 = a_b/c_d). But then, by Theorem 5.2.40, a_b/c_d = sup A_i for some $A_i \in Q_D$ and, by Definition 5.1.03, $A_1 < A_i < A_2$, as desired. □

Corollary 5.2.42. *For all $A \in R_D$, there exists a_b $\in Z_Q$ such that sup $A <$ a_b.*

Proof. By Theorem 3.2.59, 4.2.29, 5.2.40, and 5.2.41, as desired. □

Corollary 5.2.43. *For all $A \in R_D^+$, there exists a_b $\in Z_Q^+$ such that 0_1/a_b < sup A.*

Proof. By Theorem 5.2.41, as desired. □

Theorem 5.2.44. *$(R_D, <)$ is dense in itself.*

Proof. Let $A_1, A_2 \in R_D$ be arbitrary but such that $A_1 < A_2$, then, by Definition 5.1.13 and Theorem 5.2.12, 5.2.13, and 5.2.27:

$(U + U) * A_1 = A_1 + A_1;$

$< A_1 + A_2;$

$< A_2 + A_2$;

$= A_2 * (U + U)$.

Hence, by Theorem 5.2.19, 5.2.20, 5.2.25, and 5.2.29, $A_1 < (A_1 + A_2)/ (U + U) < A_2$. Let A_3 = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ [a_b/c_d < $(A_1 + A_2)/ (U + U)$]}, as desired. □

Theorem 5.2.45, *R_D has neither least nor greatest elements.*

Proof. By Definition 5.1.01 and 5.1.02 and Theorem 4.2.55, as desired. □

Theorem 5.2.46. *R_D forms a field.*

Proof. By Theorem 5.2.08, 5.2.09, 5.2.10, 5.2.11, 5.2.12, 5.2.13, 5.2.14, 5.2.16, 5.2.20, and 5.2.25, as desired. □

Theorem 5.2.47. *(R_D, <, +, *) is Dedekind complete per Definition 5.1.17.*

Proof. Let S = {K_i|K_i ϵ R_D, i ϵ N} = ran K, for some index function K, be a Dedekind cut on R_D and let A ϵ R_D – S be arbitrary. Then, by Definition 5.1.01, (S ≠ ϕ) ^ (R_D – S ≠ ϕ), hence, A exists and is an upper bound of S. To demonstrate that US is a Dedekind cut on Q_Q, it will suffice to notice that:

1) Since, for every K_i ϵ S, K_i ϵ R_D, K_i is a Dedekind cut on Q_Q and since, for every A ϵ R_D – S, A ϵ R_D, A is a Dedekind cut on Q_Q;
2) Since (S ≠ ϕ) ^ (R_D – S ≠ ϕ), (US ≠ ϕ) ^ [U(R_D – S) ≠ ϕ];
3) Since, for every K_i ϵ S, A ϵ R_D – S, K_i < A, by Definition 5.1.03, for every a_b/c_d ϵ US, e_f/g_h ϵ U(R_D – S), a_b/c_d < e_f/g_h, from which immediately follows, (US) U [U(R_D – S)] = Q_Q;
4) Since, for every K_i ϵ S, K_i has no greatest element, US has no greatest element.

Finally, to demonstrate the existence of sup S, for every K_i ϵ S, K_i is contained in US and K_i is contained in A, hence, US is contained in A and sup S = US.

Therefore, (R_D, <, +, *) is Dedekind complete, as desired. □

Theorem 5.2.48. *(R_D, <, +, *) has the infimum property of Definition 5.1.18.*

Proof. Let S = {K_i|K_i ϵ R_D, i ϵ N} = ran K, for some index function K, be an arbitrary but non-empty subset of R_D and let A ϵ R_D be a lower bound of S, the existence of such an A being guaranteed by Theorem 5.3.45. To demonstrate that S has an infimum in R_D, it will suffice to notice that:

1) Since, for every K_i ϵ S, K_i ϵ R_D, by Definition 5.1.02, each K_i and the lower bound A are Dedekind cuts on Q_Q;
2) Since (S ≠ ϕ) ^ ¬ (A ϵ S), R_D – S ≠ ϕ, and, by Definition 5.1.01, (∩S ≠ ϕ) ^ [U(R_D – ∩S) ≠ ϕ];
3) Since every K_i ϵ S is a Dedekind cut, by Definition 5.1.01, for every a_b/c_d ϵ ∩S, e_f/g_h ϵ (US – ∩S), a_b/c_d < e_f/g_h and, since, for every k_p/m_q ϵ US, n_r/o_s ϵ U(R_D – S), k_p/m_q < n_r/o_s, by Theorem 4.2.30, for every a_b/c_d ϵ ∩S, n_r/o_s ϵ U(R_D – S), a_b/c_d < n_r/o_s, from which immediately follows, (∩S) U [U(R_D – ∩S)] = Q_Q;
4) Since, for every K_i ϵ S, K_i has no greatest element, ∩S has no greatest element.

By the last three statements and Definition 5.1.01, ∩S is a Dedekind cut on Q_Q and ∩S is contained in every K_i ϵ S, hence, ∩S is a lower bound for S. Suppose, for contradiction, that A is not contained in ∩S, then there exists e_f/g_h ϵ A such that ¬ (e_f/g_h ϵ ∩S), but e_f/g_h ϵ K_i for every K_i ϵ S, a contradiction. Therefore, A is contained in ∩S and inf S = ∩S, as desired. □

Theorem 5.2.49. *(R_D, <, +, *) forms a lattice complete field per Definition 5.1.19.*

Proof. By Theorem 5.2.46, 5.2.47, and 5.2.48 (reference [AJ], Chapter 7, page 84), as desired. □

Theorem 5.2.50. *(R_D, <, +, *) is the unique lattice completion of N_Q.*

Proof. Let (M, <, +, *) be a lattice complete field constructed set theoretically from N_Q. Then, by Theorem 5.2.40, M has a copy of Q_Q embedded in it, by Theorem 4.2.29, M has a copy of Z_Q embedded in it, and, by Theorem 3.2.22, M has the foundation N_Q embedded in it. Now, since M is lattice complete and constructed set theoretically, for every $a\_b \in M$, $a\_b = \sup D$ for some Dedekind cut D defined on the embedded copy of Q_Q, which is constructed from the embedded copy of Z_Q, which is constructed from the foundation N_Q. Let $K = \{\sup A \mid A \in R_D\}$ and define an index function, $f:K \rightarrow R_D$, by $f(a\_b) = A_{a\_b}$ iff $\sup A = a\_b$. Then there is a natural and obvious ring isomorphism, $g:R_D \rightarrow M$, defined by $g(A_{a\_b}) = a\_b$, as desired. □

Theorem 5.2.51. *Let $W = \{A \mid (A \in R_D) \wedge (\sup A = x\_y) \wedge (x = 0)\}$, then, for all $A \in R_D - W$, $\sup A = a\_b$, $(a\_b)^{1/2}$ has four distinct roots:*

1) $a^{1/2}\_[- a^{1/2} + (a + b)^{1/2}]$;
2) $a^{1/2}\_[- a^{1/2} - (a + b)^{1/2}]$;
3) $- a^{1/2}\_[a^{1/2} + (a + b)^{1/2}]$;
4) $- a^{1/2}\_[a^{1/2} - (a + b)^{1/2}]$.

Proof. It will suffice to demonstrate that each distinct root, multiplied by itself, is equal to a_b:

1) $a^{1/2}\_[- a^{1/2} + (a + b)^{1/2}] * a^{1/2}\_[- a^{1/2} + (a + b)^{1/2}] =$

$$(a^{1/2} * a^{1/2})\_([- a^{1/2} + (a + b)^{1/2}] * a^{1/2} + a^{1/2} * [- a^{1/2} + (a + b)^{1/2}] + [- a^{1/2} + (a + b)^{1/2}] * [- a^{1/2} + (a + b)^{1/2}]);$$

$$= a\_(- a + (a + b)^{1/2} * a^{1/2} - a + a^{1/2} * (a + b)^{1/2} + [- a^{1/2} + (a + b)^{1/2}] * (-a^{1/2}) + [- a^{1/2} + (a + b)^{1/2}] * (a + b)^{1/2});$$

$$= a\_(- a + a^{1/2} * (a + b)^{1/2} - a + a^{1/2} * (a + b)^{1/2} + a + (a + b)^{1/2} * (-a^{1/2}) + (- a^{1/2}) * (a + b)^{1/2} + a + b);$$

$$= a\_(- 2a + 2a^{1/2} * (a + b)^{1/2} + 2a - 2a^{1/2} * (a + b)^{1/2} + b);$$

$$= a\_b.$$

2) $a^{1/2}\_[- a^{1/2} - (a + b)^{1/2}] * a^{1/2}\_[- a^{1/2} - (a + b)^{1/2}] =$

$$(a^{1/2} * a^{1/2})\_([- a^{1/2} - (a + b)^{1/2}] * a^{1/2} + a^{1/2} * [- a^{1/2} - (a + b)^{1/2}] + [- a^{1/2} - (a + b)^{1/2}] * [- a^{1/2} - (a + b)^{1/2}]);$$

$$= a\_(- a - (a + b)^{1/2} * a^{1/2} - a - a^{1/2} * (a + b)^{1/2} + [- a^{1/2} - (a + b)^{1/2}] * (-a^{1/2}) + [- a^{1/2} - (a + b)^{1/2}] * [- (a + b)^{1/2}]);$$

$$= a\_(- a - a^{1/2} * (a + b)^{1/2} - a - a^{1/2} * (a + b)^{1/2} + a - (a + b)^{1/2} * (-a^{1/2}) + a^{1/2} * (a + b)^{1/2} + a + b);$$

$$= a\_(- 2a - 2a^{1/2} * (a + b)^{1/2} + 2a + 2a^{1/2} * (a + b)^{1/2} + b);$$

$$= a\_b.$$

3) $(- a^{1/2}\_[a^{1/2} + (a + b)^{1/2}]) * (- a^{1/2}\_[a^{1/2} + (a + b)^{1/2}]) =$

$$[- a^{1/2} * (- a^{1/2})]\_([a^{1/2} + (a + b)^{1/2}] * (- a^{1/2}) - a^{1/2} * [a^{1/2} + (a + b)^{1/2}] + [a^{1/2} + (a + b)^{1/2}] * [a^{1/2} + (a + b)^{1/2}]);$$

$$= a\_(- a + (a + b)^{1/2} * (- a^{1/2}) - a - a^{1/2} * (a + b)^{1/2} + [a^{1/2} + (a + b)^{1/2}] * a^{1/2} + [a^{1/2} + (a + b)^{1/2}] * (a + b)^{1/2});$$

$$= a\_(- a - a^{1/2} * (a + b)^{1/2} - a - a^{1/2} * (a + b)^{1/2} + a + (a + b)^{1/2} * a^{1/2} + a^{1/2} * (a + b)^{1/2} + a + b);$$

$$= a\_(- 2a - 2a^{1/2} * (a + b)^{1/2} + 2a + 2a^{1/2} * (a + b)^{1/2} + b);$$

$$= a\_b.$$

4) $(- a^{1/2}\_[a^{1/2} - (a + b)^{1/2}]) * (- a^{1/2}\_[a^{1/2} - (a + b)^{1/2}]) =$

$[- a^{1/2} * (- a^{1/2})]\_([a^{1/2} - (a + b)^{1/2}] * (- a^{1/2}) - a^{1/2} * [a^{1/2} - (a + b)^{1/2}] + [a^{1/2} - (a + b)^{1/2}] * [a^{1/2} - (a + b)^{1/2}]);$

$= a\_(- a - (a + b)^{1/2} * (- a^{1/2}) - a + a^{1/2} * (a + b)^{1/2} + [a^{1/2} - (a + b)^{1/2}] * a^{1/2} + [a^{1/2} - (a + b)^{1/2}] * [- (a + b)^{1/2}]);$

$= a\_(- a + a^{1/2} * (a + b)^{1/2} - a + a^{1/2} * (a + b)^{1/2} + a - (a + b)^{1/2} * a^{1/2} - a^{1/2} * (a + b)^{1/2} + a + b);$

$= a\_(- 2a + 2a^{1/2} * (a + b)^{1/2} + 2a - 2a^{1/2} * (a + b)^{1/2} + b);$

$= a\_b.$

Therefore, $(a\_b)^{1/2}$ has four distinct roots, as desired. □

Theorem 5.2.52. *R_D is uncountable.*

Proof. In extending N_Q to Z_Q, because of Definition 2.1.13, 2.1.15, and 3.1.01, we not only extend positive q-naturals to negative q-integers, we also extend the q-components of each q-integer into the negative – i.e. for any a_b ϵ Z_Q, (b < 0) ᵛ (b = 0) ᵛ (0 < b), hence, we can think of Z_Q as Z x Z with the lexicographic order. This carries through to the q-reals, however, due to geometrical considerations (or, for that matter, Theorem 5.2.51 immediately above), we find that the q-components of the q-reals can also take complex values and, in addition, per Lemma 4.2.57, there are certain q-rationals which cannot be represented as elements of R x C. Let W = {a_b/c_d|(a_b/c_d ϵ Q_Q) ^ (a ≠ 0) ^ (c = 0)}, then we can think of R_Q, as defined in Definition 5.1.15, as (R x C) U W and there is a bijection, f:[(R x C) U W] → R_Q, defined by f(a_b) = a_b, if (a, b) ϵ R x C, and f(a_b/c_d) = a_b/c_d, if a_b/c_d ϵ W. One can view the set W as Z_Q x ({0} x Z), hence, by Theorem 3.2.58, W is countable (reference [HJ], Chapter 4, pages 75 – 77). But then, the set R x C is uncountable, hence, (R x C) U W is uncountable (reference [HJ], Chapter 5, pages 98 – 99) and R_Q is uncountable. But then, by Definition 5.1.15, R_D is uncountable, as desired. □

6. Q-Complex. We develop the q-complex in the standard way, as ordered pairs of q-real numbers.

6.1. Definitions. We define our mathematical entities using standard terminology.

Definition 6.1.01. Let C_Q = {(a_b, c_d)|(a_b, c_d) ϵ R_Q x R_Q}.

Definition 6.1.02. The operation "+" (addition) on C_Q is defined by:

(a_b, c_d) + (e_f, g_h) = (a_b + e_f, c_d + g_h).

Definition 6.1.03. The operation "*" (multiplication) on C_Q is defined by:

(a_b, c_d) * (e_f, g_h) = (a_b * e_f - c_d * g_h, a_b * g_h + c_d * e_f).

Definition 6.1.04. The members of C_Q, subject to Definition 6.1.02 and 6.1.02, will be called q_complex numbers.

Definition 6.1.05. Denote q-complex numbers of the form (a_b, 0_0) by a_b.

Definition 6.1.06. Denote the q-complex number (0_0, 1_0) by i.

Definition 6.1.07. The q-complex number a_b - (c_d)i is called the q-complex conjugate of the q-complex number a_b + (c_d)i.

Definition 6.1.08. In the q-complex number a_b + (c_d)i, a_b is the q-real coefficient and c_d is the q-imaginary coefficient. If c_d ≠ 0_0, then the q-complex number is called q-imaginary. If a_b = 0_0, then the q-complex number is called pure q-imaginary.

Definition 6.1.09. For all a_b + (c_d)i ϵ C_Q, its negative is - (a_b) - (c_d)i.

Definition 6.1.10. For all a_b + (c_d)i ϵ C_Q, a_b + (c_d)i ≠ 0_0 + (0_0)i = 0_0:

$[a\_b + (c\_d)i]^{-1} = (a\_b/[(a\_b)^2 + (c\_d)^2]) – (c\_d/[(a\_b)^2 + (c\_d)^2])i$.

Definition 6.1.11. The operation "-" (subtraction) on C_Q is defined by:

(a_b, c_d) - (e_f, g_h) = (a_b, c_d) + [- (e_f) - (g_h)].

Definition 6.1.12. The operation "÷" (division) on C_Q is defined by:

$(a\_b, c\_d) \div (e\_f, g\_h) = (a\_b, c\_d) * (e\_f + g\_h)^{-1}$.

6.2. Arguments. We demonstrate our arguments using the standard methods and terminology of mathematical logic and ZFC/AFA or generalizations thereof. Specific to the current work, we generalize the Principle of Induction to the Principle of Q-Induction, we reproduce certain arguments, verbatim, from reference [HJ], and utilize results from references [HJ] and [CD].

Theorem 6.2.01. *The set C_Q of Definition 6.1.01 exists.*

Proof. By Definition 5.1.15 and Theorem 5.2.01, R_Q exists, hence, by Definition 6.1.01, the Axiom of Power Set, the definition of ordered pair, and the definition of Cartesian product, R_Q x R_Q = C_Q exists, as desired. □

Theorem 6.2.02. *The set C_Q is closed under the arithmetical operation addition.*

Proof. This is an immediate consequence of Definition 5.1.15, 6.1.01, and 6.1.02 and Theorem 5.2.02, as desired. □

Theorem 6.2.03. *For all (a_b, c_d), (e_f, g_h) ϵ C_Q, (a_b, c_d) + (e_f, g_h) = (e_f, g_h) + (a_b, c_d).*

Proof. This is an immediate consequence of Definition 5.1.15 and 6.1.02 and Theorem 5.2.08. Therefore, addition on C_Q is commutative, as desired. □

Theorem 6.2.04. *For all (a_b, c_d), (e_f, g_h), (i_j, k_l) ϵ C_Q, [(a_b, c_d) + (e_f, g_h)] + (i_j, k_l) = (a_b, c_d) + [(e_f, g_h) + (i_j, k_l)].*

Proof. This is an immediate consequence of Definition 5.1.15 and 6.1.02 and Theorem 5.2.10. Therefore, addition on C_Q is associative, as desired. □

Theorem 6.2.05. *The set C_Q is closed under the arithmetical operation multiplication.*

Proof. This is an immediate consequence of Definition 5.1.15, 6.1.01, and 6.1.03 and Theorem 5.2.02, 5.2.07, and 5.2.23, as desired. □

Theorem 6.2.06. *For all (a_b, c_d), (e_f, g_h) ϵ C_Q, (a_b, c_d) * (e_f, g_h) = (e_f, g_h) * (a_b, c_d).*

Proof. This is an immediate consequence of Definition 5.1.15 and 6.1.03 and Theorem 5.2.09. Therefore, multiplication on C_Q is commutative, as desired. □

Theorem 6.2.07. *For all (a_b, c_d), (e_f, g_h), (i_j, k_l) ϵ C_Q, [(a_b, c_d) * (e_f, g_h)] * (i_j, k_l) = (a_b, c_d) * [(e_f, g_h) * (i_j, k_l)].*

Proof. Letting A = [(a_b, c_d) * (e_f, g_h)] * (i_j, k_l), by Definition 5.1.15 and 6.1.03 and Theorem 5.2.08, 5.2.10, 5.2.12, and 5.2.13, and Lemma 5.2.22:

A = (a_b * e_f - c_d * g_h, a_b * g_h + c_d * e_f) * (i_j, k_l);

= {[(a_b * e_f - c_d * g_h) * i_j - (a_b * g_h + c_d * e_f) * k_l], [(a_b * e_f - c_d * g_h) * k_l + (a_b * g_h + c_d * e_f) * i_j]};

= [(a_b * e_f * i_j - c_d * g_h * i_j - a_b * g_h * k_l - c_d * e_f * k_l), (a_b * e_f * k_l - c_d * g_h * k_l + a_b * g_h * i_j + c_d * e_f * i_j)];

= {[a_b * e_f * i_j + (- c_d * g_h * i_j - a_b * g_h * k_l) - c_d * e_f * k_l], [a_b * e_f * k_l + (- c_d * g_h * k_l + a_b * g_h * i_j) + c_d * e_f * i_j]};

= {[a_b * e_f * i_j - a_b * g_h * k_l + (- c_d * g_h * i_j - c_d * e_f * k_l)], [a_b * e_f * k_l + a_b * g_h * i_j + (- c_d * g_h * k_l + c_d * e_f * i_j)]};

= [(a_b * e_f * i_j - a_b * g_h * k_l - c_d * e_f * k_l - c_d * g_h * i_j)], [a_b * e_f * k_l + a_b * g_h * i_j + c_d * e_f * i_j - c_d * g_h * k_l)];

= {[a_b * (e_f * i_j - g_h * k_l) - c_d * (e_f * k_l + g_h * i_j)], [a_b * (e_f * k_l + g_h * i_j) + c_d * (e_f * i_j - g_h * k_l)]};

= (a_b, c_d) * (e_f * i_j - g_h * k_l, e_f * k_l + g_h * i_j);

= (a_b, c_d) * [(e_f, g_h) * (i_j, k_l)].

Therefore, multiplication on C_Q is associative, as desired. □

Theorem 6.2.08. *For all (a_b, c_d), (e_f, g_h), (i_j, k_l) ϵ C_Q, (a_b, c_d) * [(e_f, g_h) + (i_j, k_l)] = (a_b, c_d) * (e_f, g_h) + (a_b, c_d) * (i_j, k_l).*

Proof. Letting A = (a_b, c_d) * [(e_f, g_h) + (i_j, k_l)], by Definition 5.1.15, 6.1.02, and 6.1.03 and Theorem 5.2.08, 5.2.10, and 5.2.12:

A = (a_b, c_d) * (e_f + i_j, g_h + k_l);

= {[a_b * (e_f + i_j) - c_d * (g_h + k_l)], [a_b * (g_h + k_l) + c_d * (e_f + i_j)]};

= [(a_b * e_f + a_b * i_j - c_d * g_h - c_d * k_l), (a_b * g_h + a_b * k_l + c_d * e_f + c_d * i_j)];

= {[a_b * e_f + (a_b * i_j - c_d * g_h) - c_d * k_l], [a_b * g_h + (a_b * k_l + c_d * e_f) + c_d * i_j]};

= {[(a_b * e_f - c_d * g_h) + (a_b * i_j - c_d * k_l)], [(a_b * g_h + c_d * e_f) + (a_b * k_l + c_d * i_j)]};

= (a_b, c_d) * (e_f, g_h) + (a_b, c_d) * (i_j, k_l).

Therefore, multiplication is left distributive over addition on C_Q, as desired. □

Theorem 6.2.09. *For all (a_b, c_d), (e_f, g_h), (i_j, k_l) ϵ C_Q, [(e_f, g_h) + (i_j, k_l)] * (a_b, c_d) = (e_f, g_h) * (a_b, c_d) + (i_j, k_l) * (a_b, c_d).*

Proof. Letting A = [(e_f, g_h) + (i_j, k_l)] * (a_b, c_d), by Definition 5.1.15, 6.1.02, and 6.1.03 and Theorem 5.2.08, 5.2.10, and 5.2.13:

A = (e_f + i_j, g_h + k_l) * (a_b, c_d);

= {[(e_f + i_j) * a_b - (g_h + k_l) * c_d], [(e_f + i_j) * c_d + (g_h + k_l) * a_b]};

= [(e_f * a_b + i_j * a_b - g_h * c_d - k_l * c_d), (e_f * c_d + i_j * c_d + g_h * a_b + k_l * a_b)];

= {[e_f * a_b + (i_j * a_b - g_h * c_d) - k_l * c_d], [e_f * c_d + (i_j * c_d + g_h * a_b) + k_l * a_b]};

= {[(e_f * a_b - g_h * c_d) + (i_j * a_b - k_l * c_d)], [(e_f * c_d + g_h * a_b) + (i_j * c_d + k_l * a_b)]};

= (e_f, g_h) * (a_b, c_d) + (i_j, k_l) * (a_b, c_d).

Therefore, multiplication is right distributive over addition on C_Q, as desired. □

Theorem 6.2.10. *For all (a_b, c_d) ϵ C_Q, (a_b, c_d) + (0_0, 0_0) = (a_b, c_d) and (0_0, 0_0) is unique.*

Proof. This is an immediate consequence of Definition 5.1.15 and 6.1.02, Theorem 5.2.14, and Corollary 5.2.15. Therefore, (0_0, 0_0) is the unique additive identity on C_Q, as desired. □

Theorem 6.2.11. *For all (a_b, c_d) ϵ C_Q, (a_b, c_d) * (1_0, 0_0) = (a_b, c_d) and (1_0, 0_0) is unique.*

Proof. This is an immediate consequence of Definition 5.1.15 and 6.1.03, Theorem 5.2.14, 5.2.16, and 5.2.20, and Corollary 5.2.15, 5.2.17, and 5.2.21. Therefore, (1_0, 0_0) is the unique multiplicative identity on C_Q, as desired. □

Theorem 6.2.12. *For all (a_b, c_d) ϵ C_Q, (a_b, c_d) + [- (a_b), - (c_d)] = (0_0, 0_0) and [- (a_b), - (c_d)] is unique.*

Proof. This is an immediate consequence of Definition 5.1.15 and 6.1.02, Theorem 5.2.16, and Corollary 5.2.17. Therefore, [- (a_b), - (c_d)] is the unique additive inverse of (a_b, c_d) on C_Q, as desired. □

Theorem 6.2.13. *For all (a_b, c_d) ϵ C_Q – {(0_0, 0_0)}, the unique multiplicative inverse of (a_b, c_d) is [(a_b/[(a_b)^2 + (c_d)^2]), (- (c_d)/[(a_b)^2 + (c_d)^2])].*

Proof. Letting A = (a_b, c_d) * [(a_b/[(a_b)^2 + (c_d)^2]), (- (c_d)/[(a_b)^2 + (c_d)^2])], by Definition 5.1.15, 6.1.03, and 6.1.12, Lemma 5.2.04 and 5.2.22, and Theorem 5.2.08, 5.2.09, 5.2.16, and 5.2.25:

A = {[a_b * (a_b/[(a_b)^2 + (c_d)^2])] - [c_d * (- (c_d)/[(a_b)^2 + (c_d)^2])], [a_b * (- (c_d)/[(a_b)^2 + (c_d)^2]) + c_d * (a_b/[(a_b)^2 + (c_d)^2])]};

= {([(a_b)^2 + (c_d)^2]/[(a_b)^2 + (c_d)^2]), ([- (a_b * c_d) + (c_d * a_b)]/[(a_b)^2 + (c_d)^2])};

= {([(a_b)^2 + (c_d)^2]/[(a_b)^2 + (c_d)^2]), ([(a_b * c_d) + (- [a_b * c_d])]/[(a_b)^2 + (c_d)^2])};

= (1_0, 0_0).

Let (e_f, g_h) ϵ C_Q be such that, for all (a_b, c_d) ϵ C_Q – {(0_0, 0_0)}, (a_b, c_d) * (e_f, g_h) = (1_0, 0_0). Then, by Definition 6.1.03, (a_b * e_f - c_d * g_h = 1_0) ^ (a_b * g_h + c_d * e_f = 0_0), hence, by Definition 5.1.12, 5.1.13, and 5.1.15, e_f = - (a_b * g_h)/c_d, for c_d ≠ 0_0. By substitution, Theorem 5.2.09 and 5.2.11 and Lemma 5.2.22:

a_b * [- (a_b * g_h)/c_d] - c_d * g_h = - [(a_b)^2 * g_h]/c_d - [(c_d)^2 * g_h]/c_d;

= (- (g_h) * [(a_b)^2 + (c_d)^2])/c_d;

= 1_0.

Hence, by Definition 5.1.12 and 5.1.13, g_h = - (c_d)/[(a_b)^2 + (c_d)^2]. By substitution and Theorem 5.2.20 and 5.2.25:

e_f = - [a_b * (- (c_d)/[(a_b)^2 + (c_d)^2])]/c_d;

= {(a_b * c_d)/[(a_b)^2 + (c_d)^2]} * (1_0/c_d);

= a_b/[(a_b)^2 + (c_d)^2].

Suppose c_d = 0_0, then, by Definition 6.1.03, Theorem 5.2.14, and Lemma 5.2.18, (a_b * e_f = 1_0) ^ (a_b * g_h = 0_0) and, by substitution, (a_b * e_f = (a_b)^2/(a_b)^2 = 1_0) ^ (a_b * g_h = 0_0/(a_b)^2 = 0_0).

Therefore, [(a_b/[(a_b)^2 + (c_d)^2]), (- (c_d)/[(a_b)^2 + (c_d)^2])] is the unique multiplicative inverse of (a_b, c_d), as desired. □

Theorem 6.2.14. *Let S = {(a_b, 0_0)|(a_b, 0_0) ϵ C_Q}, then (S, <, +, *) is ring isomorphic to (R_Q, <, +, *).*

Proof. There is an obvious isomorphism, f:S → R_Q, defined by f[(a_b, 0_0)] = a_b. Let (a_b, 0_0), (c_d, 0_0) ϵ S be arbitrary, then:

Addition. By Definition 6.1.02 and Theorem 5.2.14:

f[(a_b, 0_0) + (c_d, 0_0)] = f[(a_b + c_d, 0_0 + 0_0)];

= f[(a_b + c_d, 0_0)];

= a_b + c_d;

= f[(a_b, 0_0)] + f[(c_d, 0_0)].

Multiplication. By Definition 6.1.03, Theorem 5.2.14, and Lemma 5.2.18 and 5.2.22:

f[(a_b, 0_0) * (c_d, 0_0)] = f[(a_b * c_d - 0_0 * 0_0, a_b * 0_0 + c_d * 0_0)];

= f[(a_b * c_d - 0_0, 0_0 + 0_0)];

= f[(a_b * c_d, 0_0)];

= a_b * c_d;

= f[(a_b, 0_0)] * f[(c_d, 0_0)].

Therefore, (S, <, +, *) is ring isomorphic to (R_Q, <, +, *), as desired. □

Lemma 6.2.15. *i^2 = (- 1_0, 0_0) = - 1_0, where i is that of Definition 6.1.06.*

Proof. By Definition 5.1.15, 6.1.03 and 6.1.05, Theorem 5.2.14 and 5.2.20, and Lemma 5.2.18 and 5.2.22:

(0_0, 1_0) * (0_0, 1_0) = (0_0 * 0_0 - 1_0 * 1_0, 0_0 * 1_0 + 1_0 * 0_0);

= (0_0 - 1_0, 0_0 + 0_0);

= (- 1_0, 0_0);

= - 1_0, as desired. □

Theorem 6.2.16. *For all (a_b, c_d) ϵ C_Q, (a_b, c_d) = a_b + (c_d) i.*

Proof. By Definition 6.1.02, 6.1.03, and 6.1.05, Theorem 5.2.14 and 5.2.20, and Lemma 5.2.18 and 5.2.22:

(a_b, c_d) = (a_b, 0_0) + (0_0, c_d);

= (a_b, 0_0) + (c_d * 0_0 - 0_0 * 1_0, c_d * 1_0 + 0_0 * 0_0);

= (a_b, 0_0) + (c_d, 0_0) * (0_0, 1_0);

= a_b + (c_d) i, as desired. □

Corollary 6.2.17. *Q-Complex numbers in a_b + (c_d) i form may be added and multiplied as polynomials in i, by replacing i^2, wherever it occurs, by -1_0.*

Proof. Let (a_b, c_d), (e_f, g_h) ϵ C_Q be arbitrary, then:

Addition. By Definition 6.1.02 and Theorem 5.2.13 and 6.2.16:

(a_b, c_d) + (e_f, g_h) = (a_b + e_f, c_d + g_h);

= (a_b + e_f) + (c_d + g_h) i;

= [a_b + (c_d) i] + [e_f + (g_h) i].

Multiplication. By Definition 6.1.03 and Theorem 5.2.08, 5.2.09, 5.2.10, 5.2.13, and 6.2.16:

(a_b, c_d) * (e_f, g_h) = (a_b * e_f - c_d * g_h, a_b * g_h + c_d * e_f);

= (a_b * e_f - c_d * g_h) + (a_b * g_h + c_d * e_f) i;

= [a_b * e_f + (c_d * g_h) i^2] + [(a_b * g_h) i + (c_d * e_f) i];

= a_b * e_f + {(c_d * g_h) i^2 + [(a_b * g_h) i + (c_d * e_f) i]};

= a_b * e_f + {[(a_b * g_h) i + (c_d * e_f) i] + (c_d * g_h) i^2};

= a_b * e_f + (a_b * g_h) i + (c_d * e_f) i + (c_d * g_h) i^2;

= [a_b + (c_d) i] * [e_f + (g_h) i].

Therefore, q-complex numbers in a_b + (c_d) i form can be added and multiplied as polynomials in i, as desired. □

Corollary 6.2.18. *For all (a_b, c_d) ϵ C_Q, (a_b, c_d) * [a_b, - (c_d)] = (x_y, 0_0), for some x_y ϵ R_Q.*

Proof. Let (a_b, c_d) ϵ C_Q be arbitrary, then, by Definition 5.1.16 and Theorem 5.2.02, 5.2.07, 5.2.09, 5.2.13, 5.2.16, and 6.2.17 and Lemma 5.2.04:

(a_b, c_d) * [a_b, - (c_d)] = [a_b + (c_d) i] * [a_b - (c_d) i];

= [a_b + (c_d) i] * a_b - [a_b + (c_d) i] * (c_d) i;

= $(a\_b)^2$ + a_b * (c_d) i - [a_b * (c_d) i] - $(c\_d)^2$ i^2;

= $(a\_b)^2$ + $(c\_d)^2$;

= [$(a\_b)^2$ + $(c\_d)^2$, 0_0], as desired. □

Theorem 6.2.19. *C_Q forms a field.*

Proof. This is an immediate consequence of Theorem 6.2.03, 6.2.04, 6.2.06, 6.2.07, 6.2.08, 6.2.09, 6.2.10, 6.2.11, 6.2.12, and 6.2.13, as desired. □

Corollary 6.2.20. *C_Q is not an ordered field.*

Proof. The proof, that (0_0, 1_0) = i violates the ordered field axiom, can be found in reference [CD], Chapter 11, pages 391 – 392, as desired. □

7. Closing Remarks. We hate to use [LK] as a foil because it's such a nice paper, but a large part of what makes it so nice is the stimulation it provides regarding the philosophical question: When is a field truly complete? In his conclusion, Krapp states that, formally, a field is complete relative to some completion axiom but we find this more than a bit ambiguous and philosophically inadequate. In our assessment, one can productively distinguish between two distinct types of field completion: lattice completion and operational completion.

The completion axiom defined in [AJ], Chapter 7, page 84, subsumes all of those discussed in [LK] and it is, simply but rigorously, lattice completion: A field F is complete iff every non-empty subset of F which is bounded below has an infimum in F and every non-empty subset of F which is bounded above has a supremum in F. This rather eloquently captures the essence of field completeness; however, as Krapp points out, this notion of completeness is somewhat philosophically compromised by the existence of the hyper-reals! In section 4 of [LK], Krapp demonstrates a non-standard approach to constructing the completion of the standard rationals and early on assumes the existence of the transfinite which is added to the standard rationals. The existence of the transfinite is consistent with the axioms of the algebraic structure in question, hence, its introduction is formally justified; what we take issue with, is the manner in which the transfinite is introduced into the construction.

Krapp introduces the transfinite into the standard rationals rather than the standard naturals and, while this may arguably be an acceptable expediency with regards to an economical exposition such as [LK], it doesn't seem to us foundationally sound; formally, if one wishes to introduce the transfinite into one's construction, it would seem to us, that one should do so at the foundational level. In this manner, it is the foundation of the construction which distinguishes the standard reals from the non-standard hyper-reals and from this immediately follows: a field is lattice complete relative to some foundation. In the case of the standard reals, one could say they are naturally lattice complete; in the case of the hyper-reals, they are naturally transfinite lattice complete;[DV] in the case of the present work, the q-reals are q-naturally lattice complete; and in the event the Q-Universe is extended to the Hyper-Q-Universe, the hyper-q-reals would be q-naturally transfinite lattice complete.

The other field completion we would distinguish, operational completion, is intuitively simple: A field F is operationally complete iff it is fully closed under the foundational operations and their derivatives, where foundational operations are those included in the foundational structure. The only pure fitness criteria driving the evolution of our constructions is operational closure; it is the search for operational closure which informs each and every extension in our constructions. The taking of roots is nothing more or less than highly constrained division and it is the taking of roots which makes the standard complex field operationally complete and the standard real field not so complete.

So when is a field truly complete? If by truly one means absolutely, considering the existence of an uncountably many non-standard foundational structures, [SR] one is highly inclined to say never; however, if by truly one means relative to some foundational structure, then it is lattice complete when it forms a complete lattice and operationally complete when it is fully closed under the foundational operations and their derivatives. This seems to us a reasonable and non-ambiguous answer to the question. Is it adequate? The present work motivates this foundational distinction and seems to us a formidable argument that it is adequate; the developments outlined in the final section of this paper would seem to provide a bulwark against any argument to the contrary.

Now, what formal justification is there for calling the standard model standard? In our assessment there is no formal justification, rather, the justification is social and historical, and we suggest the practice should be discontinued. We further suggest that the standard model be designated the zeroth-order model or zeroth-order Universe and we justify this suggestion, formally, in the final section of this paper. Historically and sociologically, the zeroth-order Universe is called standard simply because it represents the "cropse of our seedfather;" it is the original evolution to completion, both lattice and operational, and, as such, it set the standard. The Hyper-Universe and the Q-Universe are both simply generalizations of this standard and neither could exist without it. Based on a historical and cognitive perspective, [LN] one is inclined to doubt the possibility of any other standard evolving; it seems that out environment places rather formidable constraints on our cognition. Of course, one can't but wonder at the myriad of possible constructions accessible to future artilects, whose minds may or may not be so constrained. As for the present author, he is rather firmly in agreement with [WV] in that intelligence is open-ended and, being a Humanist[RH] and a Constructivist, [AR] he has no qualms with either historical or sociological justifications; however, social constructs evolve and occasionally novel developments justify a re-assessment of those social and historical justifications (reference [BW], Chapters 6 and 8).

7.1. Do other recursive arithmetical structures exist? The attentive reader will have noticed the answer to this question in both the abstract and the table of contents: countably many. As you will see, these arithmetical structures are all related in a manner which formally justifies the zeroth-order terminology.

In their paper, "Origin of Complex Amplitudes and Feynman's Rules," [GKS] Philop Goyal, Kevin Knuth, and John Skilling develop an abstract formalism which allows them to represent typical quantum experiments algebraically. They use this formalism together with a single assumption they call the pair postulate (Bohr's complementarity) to derive Feynman's rules demonstrating along the way that complementarity induces the complex field requirement of Quantum Theory quite naturally. We were aware of this important work while engaged in the present work and, while developing the section on the q-complex field, it occurred to us that perhaps the Goyal, Knuth, Skilling approach to reconstructing the quantum formalism could be generalized to simple quantum entangled systems using the relevant subset of the q-complex field, those q-complex numbers of the form $a_1\_a_2 + a_3\_a_4$ i, where $0 \le a_i \le 1$. After notifying one of the paper's authors, we returned to the project at hand but we kept thinking about this idea of more faithfully modelling simple entangled systems with the q-complex field; specifically, we were thinking about modelling more complex entangled systems. Almost as a lark, we thought to ourself, "Could one possibly extend the Q-Universe out to a Universe constructed on ω^4?"

0) (1 * a = a) ^ (0 * a = 0);
1) (1_0 * a_b = a_b) ^ (0_0 * a_b = 0_0);
2) 1_0_0_0 * a_b_c_d = (1_0 * a_b)_(0_0 * a_b + 1_0 * c_d + 0_0 * c_d) = a_b_c_d;
3) 1_0_0_0_0_0_0_0 * a_b_c_d_e_f_g_h = (1_0_0_0 * a_b_c_d)_(0_0_0_0 * a_b_c_d + 1_0_0_0 * e_f_g_h + 0_0_0_0 * e_f_g_h) = a_b_c_d_e_f_g_h;

.

.

.

Not only can one do that, one can go further still and it would seem the foundational relations/operations are recusive – at least as defined on those domains less than the Church-Kleene ordinal[CK]!

Essentially, what we have here is a countable subsumption hierarchy of Universes constructed on foundations which conform to the geometric sequence $\{a_n\} = \{1, 2, 4, \dots, 2^n, \dots\}$, where $n \in N$; a countable hierarchy of fully nested lattice complete fields, all of which are extensions of recursive arithmetics. This is the formal justification for the zeroth-order designation. In the geometric sequence: $2^0 = 1$, $\omega^1 = \omega = N$, and the constant non-logical symbols of the foundational structure, (N, <, +, *, 0, 1), are single digit; $2^1 = 2$, $\omega^2 = N_Q$, and the constant non-logical symbols of the foundational structure, (N_Q, <, +, *, 0_0, 1_0), are double digit; $2^2 = 4$, $\omega^4 = N_4$, and the constant non-logical symbols of the foundational structure, (N_4, <, +, *, 0_0_0_0, 1_0_0_0), are quadruple digit; and so on to ω^ω, whose constant symbols are infinite digit.

Of course addition and multiplication increase in complexity as $n \to \omega$ and factoring large integers in a large n Universe can be rather difficult, involving the solving of a somewhat complex system of "nested" Diophantine equations (see the appendix). But surprisingly, determining lexicographic order does not really increase much in complexity. To see this, consider ω^8:

a_b_c_d_e_f_g_h < i_j_k_l_m_n_o_p iff (a_b_c_d < i_j_k_l) ∨ [(a_b_c_d = i_j_k_l) ^ (e_f_g_h < m_n_o_p)];

a_b_c_d < i_j_k_l iff (a_b < i_j) ∨ [(a_b = i_j) ^ (c_d < k_l)];

a_b < i_j iff (a < i) ∨ [(a = i) ^ (b < j)].

Determining order begins with the zeroth-order and continues systematically in the hierarchic order; immediately upon discovering an inequality, the determination is made.

So now, is it not conceivable that one can use complex numbers from the $\omega\text{^}(2^n)$ Universe to faithfully model a quantum entangled system involving 2^n wave/particle pairs? $\{2^n\} = \{1, 2, 4, \ldots, 16, \ldots, 256, \ldots, 65{,}536, \ldots\}$ . . .

So perhaps, in the interests of formal consistency, the completion terminology should reflect the hierarchic structure as well. One could say: the standard reals are zeroth-order lattice complete; the standard hyper-reals are transfinite zeroth-order lattice complete; the q-reals are first-order lattice complete; the hyper-q-reals would be transfinite first-order lattice complete; and so on to ω^{th}-order lattice complete. This would seem to demonstrate, in a robust way, philosophical adequacy.

8. Appendix. To get a feel for the complexity involved, we will demonstrate the simplest situation from ω^8. Letting $A = a_1\_a_2\_a_3\_a_4\_a_5\_a_6\_a_7\_a_8$:

$A = b_1\_b_2\_b_3\_b_4\_b_5\_b_6\_b_7\_b_8 * c_1\_c_2\_c_3\_c_4\_c_5\_c_6\_c_7\_c_8$;

$= (b_1\_b_2\_b_3\_b_4 * c_1\_c_2\_c_3\_c_4)\_[(b_5\_b_6\_b_7\_b_8 * c_1\_c_2\_c_3\_c_4) + (b_1\_b_2\_b_3\_b_4 * c_5\_c_6\_c_7\_c_8) + (b_5\_b_6\_b_7\_b_8 * c_5\_c_6\_c_7\_c_8)]$;

$= (b_1\_b_2 * c_1\_c_2)\_[(b_3\_b_4 * c_1\_c_2) + (b_1\_b_2 * c_3\_c_4) + (b_3\_b_4 * c_3\_c_4)]\_[\{(b_5\_b_6 * c_1\_c_2)\_[b_7\_b_8 * c_1\_c_2 + b_7\_b_8 * c_3\_c_4]\} + \{(b_1\_b_2 * c_5\_c_6)\_[b_3\_b_4 * c_5\_c_6 + b_1\_b_2 * c_7\_c_8 + b_3\_b_4 * c_7\_c_8)]\} + \{(b_5\_b_6 * c_5\_c_6)\_[b_7\_b_8 * c_5\_c_6 + b_5\_b_6 * c_7\_c_8 + b_7\_b_8 * c_7\_c_8)]\}]$;

$= (b_1 c_1)\_(b_2 c_1 + b_1c_2 + b_2c_2)\_\{(b_3 c_1)\_(b_4c_1 + b_3c_2 + b_4c_2) + (b_1c_3)\_(b_2c_3 + b_1c_4 + b_2c_4) + (b_3c_3)\_(b_4c_3 + b_3c_4 + b_4c_4)\}\_\{(b_5 c_1)\_(b_6c_1 + b_5c_2 + b_6c_2)\_[(b_7 c_1)\_(b_8c_1 + b_7c_2 + b_8 c_2) + (b_5c_3)\_(b_6c_3 + b_5c_4 + b_6c_4) + (b_7c_3)\_(b_8c_3 + b_7c_4 + b_8c_4)]\} + \{(b_1c_5)\_(b_2c_5 + b_1c_6 + b_2c_6)\_[(b_3c_5)\_(b_4c_5 + b_3c_6 + b_4c_6) + (b_1c_7)\_(b_2c_7 + b_1c_8 + b_2c_8) + (b_3 c_7)\_(b_4c_7 + b_3c_8 + b_4c_8)]\} + \{(b_5c_5)\_(b_6c_5 + b_5c_6 + b_6c_6)\_[(b_7c_5)\_(b_8c_5 + b_7c_6 + b_8c_6) + (b_5c_7)\_(b_6c_7 + b_5c_8+ b_6c_8) + (b_7c_7)\_(b_8c_7 + b_7c_8+ b_8c_8)]\}$;

$= (b_1 c_1)\_(b_2 c_1 + b_1c_2 + b_2c_2)\_[b_1c_3 + b_3(c_1 + c_3)]\_[b_2c_3 + b_1c_4 + b_2c_4 + b_3c_2 + b_4(c_1 + c_2) + b_3c_4 + b_4(c_3 + c_4)]\_[b_5 c_1 + b_1c_5 + b_5c_5]\_[b_5 c_2 + b_6(c_1 + c_2) + b_1c_6 + b_2(c_5 + c_6) + b_5c_6 + b_6(c_5 + c_6)]\_\{[b_5c_3 + b_7(c_1 + c_3)]\_[b_7c_2 + b_8(c_1 + c_2) + b_5c_4 + b_6(c_3 + c_4) + b_7c_4 + b_8(c_3 + c_4)] + [b_1c_7 + b_3(c_5 + c_7)]\_[b_3c_6 + b_4(c_5 + c_6) + b_1c_8 + b_2(c_7 + c_8) + b_3c_8 + b_4(c_7 + c_8)] + [b_5c_7 + b_7(c_5 + c_7)]\_[b_7c_6 + b_8(c_5 + c_6) + b_5c_8 + b_6(c_7 + c_8) + b_7c_8 + b_8(c_7 + c_8)]\}$;

$= (b_1 c_1)\_[b_1c_2 + b_2(c_1 + c_2)]\_[b_1c_3 + b_3(c_1 + c_3)]\_[b_2c_3 + b_3c_2 + (b_1 + b_2 + b_3)c_4 + b_4(c_1 + c_2 + c_3 + c_4)]\_[b_1c_5 + b_5(c_1 + c_5)]\_[b_1c_6 + b_2(c_5 + c_6) + b_5(c_2 + c_6) + b_6(c_1 + c_2 + c_5 + c_6)]\_[b_5c_3 + b_3(c_5 + c_7) + b_7(c_1 + c_3 + c_5 + c_7) + (b_1 + b_5)c_7]\_[b_1c_8 + b_2(c_7 + c_8) + b_5(c_4 + c_8) + b_7(c_2 + c_4) + b_4(c_5 + c_6 + c_7 + c_8) + b_6(c_3 + c_4 + c_7 + c_8) + (b_7 + b_3)(c_6 + c_8) + b_8(c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8)]$.

From which we see that:

$a_1 = b_1c_1$;

$a_2 = b_1c_2 + b_2(c_1 + c_2)$;

$a_3 = b_1c_3 + b_3(c_1 + c_3)$;

$a_4 = b_2c_3 + b_3c_2 + (b_1 + b_2 + b_3)c_4 + b_4(c_1 + c_2 + c_3 + c_4)$;

$a_5 = b_1c_5 + b_5(c_1 + c_5)$;

$a_6 = b_1c_6 + b_2(c_5 + c_6) + b_5(c_2 + c_6) + b_6(c_1 + c_2 + c_5 + c_6)$;

$a_7 = b_5c_3 + b_3(c_5 + c_7) + b_7(c_1 + c_3 + c_5 + c_7) + (b_1 + b_5)c_7$;

$a_8 = b_1c_8 + b_2(c_7 + c_8) + b_5(c_4 + c_8) + b_7(c_2 + c_4) + b_4(c_5 + c_6 + c_7 + c_8) + b_6(c_3 + c_4 + c_7 + c_8) + (b_7 + b_3)(c_6 + c_8) + b_8(c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8)$.

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