On a Problem in Euler and Navier-Stokes Equations

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Abstract – A study respect to a problem found in the equations of Euler and Navier-Stokes, whose adequate treatment solves a centennial problem about the solution of these equations and a most correct modeling of fluid in movement.

Keywords – Euler equations, Navier-Stokes equations, Eulerian description, Lagrangian description, breakdown solutions, non-uniqueness, vector pressure.

1 – Introduction

This article is a better version of [1], which in turn was motived by my works on Lagrangian and Eulerian descriptions in Euler[2] and Navier-Stokes[3] equations, where I used for velocity's components the relation

$$(1.1) \qquad \begin{cases} \dfrac{\partial u_i}{\partial x_j} = 0, \ i \neq j, \\ \partial x_i = u_i \partial t \end{cases}$$

because the construction of the non-linear terms $u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}$ in these equations was based on the 2nd law of Newton, $F = ma$, making

$$(1.2) \qquad a = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt},$$

with

$$(1.3) \qquad \begin{cases} \dfrac{dx}{dt} = u_1 \\ \dfrac{dy}{dt} = u_2 \\ \dfrac{dz}{dt} = u_3 \end{cases}$$

I now realize that it is possible, or better said, it is necessary for a more appropriate modeling of fluids in motion, the simultaneous use of both velocities, in the Lagrangian and Eulerian descriptions, in the same equation (Euler equations or Navier-Stokes equations), what we will see in section 4. For while, we think in each description or formulation separate of the other, i.e., used exclusively, in an equation.

The equations (1.3), writing synthetically as $\frac{dx_i}{dt} = u_i$, with $x_1 \equiv x,\ x_2 \equiv y$, $x_3 \equiv z$, show us that the velocity's component u_i is dependent only of coordinate x_i, regardless of the values of others $x_j,\ j \neq i$, justifying the use of (1.1).

Following this idea, the original system for $n = 3$ spatial dimension and volumetric mass density $\rho = 1$,

$$(1.4)\qquad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu \nabla^2 u_1 + \frac{1}{3}\nu \nabla_1(\nabla \cdot u) + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = \nu \nabla^2 u_2 + \frac{1}{3}\nu \nabla_2(\nabla \cdot u) + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = \nu \nabla^2 u_3 + \frac{1}{3}\nu \nabla_3(\nabla \cdot u) + f_3 \end{cases}$$

can be transformed in

$$(1.5)\qquad \begin{cases} \frac{1}{u_1}\frac{\partial p}{\partial t} + \frac{Du_1}{Dt} = \nu(\nabla^2 u_1)|_t + \frac{1}{3}\nu(\nabla_1(\nabla \cdot u))|_t + f_1|_t \\ \frac{1}{u_2}\frac{\partial p}{\partial t} + \frac{Du_2}{Dt} = \nu(\nabla^2 u_2)|_t + \frac{1}{3}\nu(\nabla_2(\nabla \cdot u))|_t + f_2|_t \\ \frac{1}{u_3}\frac{\partial p}{\partial t} + \frac{Du_3}{Dt} = \nu(\nabla^2 u_3)|_t + \frac{1}{3}\nu(\nabla_3(\nabla \cdot u))|_t + f_3|_t \end{cases}$$

thus (1.4) and (1.5) are equivalent systems, according validity of (1.2) and (1.3), since that the partial derivatives of the pressure and velocities were correctly transformed to the variable time, using $\partial x = u_1 \partial t,\ \partial y = u_2 \partial t,\ \partial z = u_3 \partial t$. The nabla and Laplacian operators are considered calculated in Lagrangian formulation, i.e., in the variable time. Likewise for the calculation of $\frac{Du}{Dt}$, following (1.2), and external force f, using $x = x(t), y = y(t), z = z(t)$. The integration of the system (1.5) shows that anyone of its equations can be used for solve it, and the results must be equals each other, except for a constant of integration. Then this is a condition to the occurrence of solutions, if the velocity u and external force f are given and the pressure p must be calculated.

We use the following transformations (omitting the use of $|_t$, the calculation at time t of the position (x, y, z) of the moving particle):

$$(1.6.1)\qquad \frac{\partial u_i}{\partial x_j} = \begin{cases} \frac{\partial u_i/\partial t}{\partial x_i/\partial t} = \frac{1}{u_i}\frac{\partial u_i}{\partial t},\ i = j \\ 0,\ i \neq j \end{cases}$$

$$(1.6.2)\qquad \nabla \cdot u = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} = \sum_{j=1}^{3} \frac{1}{u_j}\frac{\partial u_j}{\partial t}$$

$$(1.6.3)\qquad \nabla_i(\nabla \cdot u) = \frac{\partial}{\partial x_i}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) = \frac{\partial}{\partial x_i}\frac{\partial u_i}{\partial x_i} = \frac{\partial/\partial t}{\partial x_i/\partial t}\frac{1}{u_i}\frac{\partial u_i}{\partial t}$$

$$= \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right]$$

and

(1.7.1) $$\frac{\partial^2 u_i}{\partial x_j^2} = \begin{cases} \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right], & i = j \\ 0, & i \neq j \end{cases}$$

(1.7.2) $$\nabla^2 u_i = \frac{\partial^2 u_i}{\partial x_i^2} = \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right]$$

and thus the system (1.5) can be integrated, finding the pressure p on the particle in motion.

From equations (1.5) to (1.7) it is possible to construct the Euler and Navier-Stokes equations in a new Lagrangian description from the respective Eulerian description. Although in the Eulerian description a position (x, y, z) refers to any position, generally adopted as fixed in time, when we want it to refer to a particle motion we arrive at this new Lagrangian description aforementioned. While in this Introduction the equations (1.5) to (1.7) lead to a new Lagrangian formulation of the Euler and Navier-Stokes equations, in section 4 and Conclusion we will see the respective modification of the Eulerian formulation, or a kind of mixed description.

Next, in section 2 we will deduce the equations of Euler, in section 3 we will deduce the equations of Navier-Stokes, the section 4 will show a new expression for the equations of Euler and Navier-Stokes, with the simultaneous use of the Eulerian and Lagrangian formulations (or a correction of the Eulerian formulation), and in the section 5 we will give examples of the need to use the new equations here deduced, rather than the traditional equations known.

The section 6 deals with the issue of breakdown solutions, section 7 on non-uniqueness of solutions, and section 8, finally, will be our conclusion.

Except for sections 2 and 3 we use mass density $\rho = 1$, otherwise if it is necessary replace the pressure p by p/ρ and the viscosity coefficient ν by ν/ρ. I believe that the new equations presented here really need to be accepted, and we will have exact solutions found faster for the various applications.

2 – Deduction of Euler equations

Many deductions of the Euler (and Navier-Stokes) equations start from the assumption that the pressure is a scalar magnitude, equal in all directions at the same point. Particularly I do not think this needs to be this way, or rather, I believe that the pressure can be a vector entity, rather than a scalar, so there is a vector pressure such that $p = (p_1,\ p_2,\ p_3)$, which would make it extraordinarily simple to solve the Euler and Navier-Stokes equations. Instead of using the gradient of p, the vector $\nabla p \equiv \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$, we should use the vector $\left(\frac{\partial p_1}{\partial x}, \frac{\partial p_2}{\partial y}, \frac{\partial p_3}{\partial z}\right)$, and then

(2.1) $$p_i = \int_{x_i^0}^{x_i} \left[-\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} \right) + f_i \right] dx_i + \theta_i(t),$$

for $i = 1, 2, 3$, solves the Euler equations, i.e., calculate the components of pressure given the velocity and an external force, conservative or not, and an "arbitrary" (well behaved, smooth, physically reasonable) function of time $\theta(t)$. This will be a pressure with independence of path, depending only of the initial and final points, $(x_1^0,\ x_2^0,\ x_3^0)$ and $(x_1,\ x_2,\ x_3)$ respectively. Without wanting to deepen this subject now, we will continue using scalar pressure, at least in general.

We will follow the deduction of Landau & Lifshitz[4] and as they we will use $\boldsymbol{v}$ to indicate velocity and bold characters for vectors. They *emphasize that* $\boldsymbol{v}(x, y, z, t)$ *is the velocity of the fluid at a given point* (x, y, z) *in space and at a given time* t*, i.e., it refers to fixed points in space and not to specific particles of the fluid; in the course of time, the latter move about in space. The same remarks apply to* ρ *and* p*.*

Let us considerer some volume in the fluid. The total force acting on this volume is equal to the integral (the minus signal indicates a compressive force)

$$-\oint p\, d\boldsymbol{f}$$

of the pressure, taken over the surface bounding the volume. Transforming it to a volume integral, we have

(2.2) $$-\oint p\, d\boldsymbol{f} = -\int \boldsymbol{grad}\, p\, dV.$$

Hence we see that the fluid surrounding any volume element dV *exerts on that element a force* $-dV\, \boldsymbol{grad}\, p$*. In other words, we can say that a force* $-\boldsymbol{grad}\, p$ *acts on unit volume of the fluid.*

See that an equality similar to Gauss's law was used with the previous acceptance of scalar pressure. The same equality, with equal reason, could be rewritten, using a vector pressure $\boldsymbol{p} = (p_1,\ p_2,\ p_3)$, as

(2.3) $$-\oint \boldsymbol{p}\, df = -\int \left(\frac{\partial p_1}{\partial x}, \frac{\partial p_2}{\partial y}, \frac{\partial p_3}{\partial z} \right) dV,$$

i.e., without assuming that $p_1 = p_2 = p_3 = p$ and with the convention that $\boldsymbol{p}$ is a resultant vector of pressures applied on a volume element $dV = dx\, dy\, dz$ centered at point (x, y, z) and time t.

Continuing Landau & Lifshitz, *we can now write the equation of motion of a volume element in the fluid by equating the force* $-\boldsymbol{grad}\, p$ *to the produt of the mass per unit volume* (ρ) *and the acceleration* $d\boldsymbol{v}/dt$*:*

(2.4) $$\rho\, d\boldsymbol{v}/dt = -\boldsymbol{grad}\, p.$$

The derivative $d\boldsymbol{v}/dt$ *which appears here denotes not the rate of change of the fluid velocity at a fixed point in space, but the rate of change of the velocity of a given fluid particle as it moves about in space. This derivative has to be expressed in terms of quantities referring to points fixed in space. To do so, we notice that the change* $d\boldsymbol{v}$ *in the velocity of the given fluid particle during the time* dt *is composed of two parts, namely the change during* dt *in the velocity at a point fixed in space, and the difference between the velocities (at the same instant) at two points* $d\boldsymbol{r}$ *apart, where* $d\boldsymbol{r}$ *is the distance moved by the given fluid particle during the time* dt*. The first part is* $(\partial\boldsymbol{v}/\partial t)dt$*, where the derivative* $\partial\boldsymbol{v}/\partial t$ *is taken for constant* x, y, z*, i.e., at the given point in space. The second part is*

(2.5) $$dx\frac{\partial \boldsymbol{v}}{\partial x} + dy\frac{\partial \boldsymbol{v}}{\partial y} + dz\frac{\partial \boldsymbol{v}}{\partial z} = (d\boldsymbol{r}\cdot\boldsymbol{grad})\boldsymbol{v}.$$

Thus

(2.6) $$d\boldsymbol{v} = (\partial\boldsymbol{v}/\partial t)dt + (d\boldsymbol{r}\cdot\boldsymbol{grad})\boldsymbol{v},$$

or, dividing both sides by dt*,*

(2.7) $$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{grad})\boldsymbol{v}.$$

Substituting this in (2.4)*, we find*

(2.8) $$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{grad})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{grad}\, p;$$

it was first obtained by L. Euler in 1755.

If the fluid is in a gravitational field, an additional force $\rho\boldsymbol{g}$*, where* $\boldsymbol{g}$ *is the acceleration due to gravity, acts on any unit volume. This force must be added to the right-side of equation* (2.4)*, so the equation* (2.8) *takes the form*

(2.9) $$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{grad})\boldsymbol{v} = -\frac{\boldsymbol{grad}\, p}{\rho} + \boldsymbol{g}.$$

Using the vector pressure, the correspondent to equation (2.9), with a generic density of external force $\boldsymbol{f}$ (not only gravitational), is

(2.10) $$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{grad})\boldsymbol{v} = -\frac{1}{\rho}\left(\frac{\partial p_1}{\partial x}, \frac{\partial p_2}{\partial y}, \frac{\partial p_3}{\partial z}\right) + \boldsymbol{f},$$

therefore a new kind of Euler's equation, and whose integration does not involve major difficulties.

It is interesting observe that Batchelor[5] wrote (chap. 3.3) "*The simple notion of a pressure acting equally in all directions is lost in most cases of a fluid in motion"*, thus shown that the imposition or acceptation of a pressure equal in the three rectangular coordinates is, in fact, something fragile, possibly not true in the nature, for fluids in motion.

3 – Deduction of Navier-Stokes equations

Among several deductions of the equations of Navier-Stokes, we will choose the one described in Richardson[6](1950), for its brevity, simplicity and understanding.

Richardson firstly makes his deduction of the Euler equations (*Acad. Berlin*, 1755),

(3.1) $$\begin{cases} \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \\ \frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{cases}$$

where the velocity of fluid is (U, V, W), the external force (on unit mass) is (X, Y, Z), the pressure is p and the volumetric density of mass is ρ.

The equations are constructed from the statement of Newton's Second Law of Motion, i.e., that the total force acting on a particle is the product of its mass and acceleration.

If x, y, z *are the rectilinear co-ordinates of a small cube of the material (density* ρ*) of volume* δv*,* $\ddot{x}, \ddot{y}, \ddot{z}$ *the components of its acceleration and* X, Y, Z *of forces on unit mass, let* X_p*,* Y_p*,* Z_p *be the components of the external forces acting normally on the three surfaces of area* δS *due to the differences of pressure (Fig. 1).*

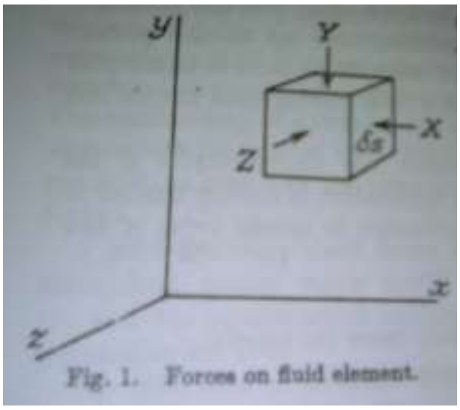

Fig. 1 – Forces on fluid element.

Setting aside the frictional forces for the moment (which resulting in Navier-Stokes equations), *we have these conditions of equilibrium:*

(3.2) $$\begin{cases} \rho\ddot{x}\delta v = X\rho\delta v + X_p\delta S \\ \rho\ddot{y}\delta v = Y\rho\delta v + Y_p\delta S \\ \rho\ddot{z}\delta v = Z\rho\delta v + Z_p\delta S \end{cases}$$

In place of $X_p,\ Y_p,\ Z_p$ *we shall insert the pressure gradients in the corresponding directions, i.e.*

(3.3) $$\begin{cases} X_p \cdot \delta S = \frac{\partial p}{\partial x} \cdot \delta v \\ Y_p \cdot \delta S = \frac{\partial p}{\partial y} \cdot \delta v \\ Z_p \cdot \delta S = \frac{\partial p}{\partial z} \cdot \delta v \end{cases}$$

For (3.3), in an ideal fluid, the pressure acts equally in all directions in the interior and at right angles to any surface presented to it. Then $X_p,\ Y_p,\ Z_p$ *are each derived from p, the mean hydrostatic pressure at the point in the fluid circumscribed by the cube.*

Substituting in (3.2) we get

(3.4) $$\begin{cases} \rho\ddot{x} = \rho X - \frac{\partial p}{\partial x} \\ \rho\ddot{y} = \rho Y - \frac{\partial p}{\partial y} \\ \rho\ddot{z} = \rho Z - \frac{\partial p}{\partial z} \end{cases}$$

These equations are not suited to direct application since the quantities x, y, z *appear in them at once as dependent and independent variables. There are two ways of adapting them to suit experimental observation. We can ask ourselves, "At a given point, what fluid occupies the element of space subsequently?" or, "Where does a given particle find itself as times goes on?" The first attitude* *corresponds to that of a fixed observer, the second to that of an observer who moves with the general velocity of the medium.*

Mathematically, the first question can be put thus: "What function of x, y, z *and* t *are the velocity components* $U(=\dot{x})$, $V(=\dot{y})$, $W(=\dot{z})$*?" We proceed to retain* x, y, z *as independent variables but eliminate their dependent aspects to obtain*

$$\frac{d^2x}{dt^2} = \frac{dU}{dt} = \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial U}{\partial z} \cdot \frac{dz}{dt}, \text{ etc.} \tag{3.5}$$

which with (3.4) resolve into the Eulerian equations (3.1).

Answering to the first question, Richardson says that *the second form of our question* ("Where does a given particle find itself as times goes on?") *can be translated thus: "What functions of time and place are those co-ordinates – let them be* a, b, c *– which characterize a given particle?" To answer this, we get rid of* x, y, z *as independent variables but retain them where dependent and arrive at the Lagrangian (Mem. Acad. (Berlin), 1781) form of the equations of motion:*

$$\begin{cases} \left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial a} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial a} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial a} + \frac{1}{\rho}\frac{\partial p}{\partial a} = 0 \\ \left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial b} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial b} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial b} + \frac{1}{\rho}\frac{\partial p}{\partial b} = 0 \\ \left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial c} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial c} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial c} + \frac{1}{\rho}\frac{\partial p}{\partial c} = 0 \end{cases} \tag{3.6}$$

As we known, *the form due to Euler is, however, more generally used.*

Now let us introduce the frictional forces. We define the coefficient of viscosity, η*, as the force per unit area of two parallel laminae of fluid unit distance apart, measured across the direction of flow. Thus, if* U *and* $U + \delta U$ *(Fig. 2) are the velocities (in the direction of* x*) at two planes* δy *apart, the force per unit area on the fluid in either plane is* $\eta \cdot \partial U/\partial y$*, i.e., the product of the coefficient of viscosity and the velocity gradient perpendicular to the direction of flow. If* A, B *and* C *are such laminae, each of area* S*,* A *exerts a force on* B *equal to* $-\eta \cdot \partial U/\partial y \cdot S$*;* C *exerts a force on* B *equal to* $\eta \cdot (\partial U/\partial y + \partial^2 U/\partial y^2 \cdot \delta y) \cdot S$*, so that the net force on* B *is*

$$\eta \cdot \frac{\partial^2 U}{\partial y^2} \cdot \delta y \cdot S = \frac{\eta}{\rho} \cdot \delta m \cdot \frac{\partial^2 U}{\partial y^2} = \eta \cdot \delta v \cdot \frac{\partial^2 U}{\partial y^2} \tag{3.7}$$

where δm *is the mass of fluid between* A *and* B and δv is the respective volume. *The factor* η/ρ*, written* ν*, which we shall often require, is called the kinematic (coefficient of) viscosity. (It should be noted that it is here assumed that* η *is constant for a given fluid, invariable with* $\partial U/\partial y$*,* but a more general proof also is made posteriorly in [6], here omitted.)

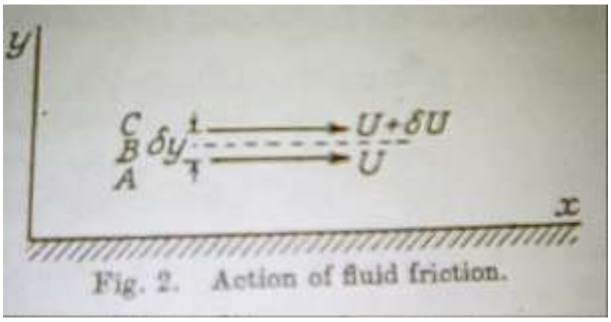

Fig. 2 – Action of fluid friction.

In the general case, the total viscous force on an element of mass m *due to the component* U *will be*

$$\eta \cdot \delta v \cdot \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

written shortly $\nu m \nabla^2 U$. *This force must be added to those on the right-hand side of the equations we have already derived* (Euler equations), resulting in the *equations ascribed to Navier (Mem. Acad. Sci. (Paris), 1822) and Stokes (Camb. Trans., 1845),*

(3.8)
$$\begin{cases} \dfrac{\partial U}{\partial t} + U\dfrac{\partial U}{\partial x} + V\dfrac{\partial U}{\partial y} + W\dfrac{\partial U}{\partial z} = X - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\,\nabla^2 U \\ \dfrac{\partial V}{\partial t} + U\dfrac{\partial V}{\partial x} + V\dfrac{\partial V}{\partial y} + W\dfrac{\partial V}{\partial z} = Y - \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\,\nabla^2 V \\ \dfrac{\partial W}{\partial t} + U\dfrac{\partial W}{\partial x} + V\dfrac{\partial W}{\partial y} + W\dfrac{\partial W}{\partial z} = Z - \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + \nu\,\nabla^2 W \end{cases}$$

with $\nu = \eta/\rho$ the (kinematic) viscosity coefficient.

Confirming the difficulty of the Lagrangian description of the Euler and Navier-stokes equations, based on [7], the Navier-Stokes equations without external force and with volumetric mass density $\rho = 1$ are, describing the velocity as (u_1, u_2, u_3) and the spatial coordinates as (x_1, x_2, x_3),

(3.9.1)
$$\frac{\partial^2 X_i}{\partial t^2} = -\sum_{j=1}^{3} \frac{\partial A_j}{\partial x_i}\frac{\partial p}{\partial a_j} + \\ + \nu \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\left(\frac{\partial^2 A_l}{\partial x_k \partial x_k}\frac{\partial u_i}{\partial a_l} + \frac{\partial A_j}{\partial x_k}\frac{\partial A_l}{\partial x_k}\frac{\partial^2 u_i}{\partial a_j \partial a_l}\right),$$

(3.9.2)
$$\frac{\partial A_j}{\partial x_i} \equiv \frac{\partial}{\partial x_i} X_j(x_n, t)|_{x_n = X_n(a_m, s|t)},$$

where a_m is the label given to the fluid particle at time s. Its position and velocity at time t are, respectively, $X_n(a_m, s|t)$ and $u_n(a_m, s|t)$. The respective deduction of these equations we will omit, but the reader can consult [7] for more details.

4 – A new form of Euler and Navier-Stokes equations

The Eulerian (equations (3.1) and (3.8)) and Lagrangian (equations (3.6) and (3.9)) forms are not the unique possible equations for description of fluids. Other equation for modeling of fluids is possible, based on them, with the great advantage of linearity. It is what we will show in this section.

The system (1.3), for the sake of mathematical rigor, needs to be replaced by

$$(4.1) \qquad \begin{cases} \frac{dx}{dt} = u_1(t) \\ \frac{dy}{dt} = u_2(t) \\ \frac{dz}{dt} = u_3(t) \end{cases}$$

emphasizing that the velocity components that appear as the time derivative of the coordinate (x, y, z) are legitimate functions of time, i.e., can be considered as representative of the Lagrangian description, $u_i(t)$, unlike the derivatives of u_i in $\frac{\partial u_i}{\partial t}, \frac{\partial u_i}{\partial x_j}$, $\nabla \cdot u$ and $\nabla^2 u_i$, that are in the Eulerian description, function of (x, y, z, t).

Representing the Eulerian velocity and respective components with the letter E indicated as upper index, and the corresponding Lagrangian components with the letter L, the system (1.4) is rewritten as

(4.2)

$$\begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z} = \nu \nabla^2 u_1^E + \frac{1}{3} \nu \nabla_1 (\nabla \cdot u^E) + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z} = \nu \nabla^2 u_2^E + \frac{1}{3} \nu \nabla_2 (\nabla \cdot u^E) + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z} = \nu \nabla^2 u_3^E + \frac{1}{3} \nu \nabla_3 (\nabla \cdot u^E) + f_3 \end{cases}$$

being the pressure p and external force f implicitly defined in the Eulerian description. A more concise notation for (4.2) is simply, for $i = 1, 2, 3$,

$$(4.3) \quad \frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \alpha_1 \frac{\partial u_i}{\partial x} + \alpha_2 \frac{\partial u_i}{\partial y} + \alpha_3 \frac{\partial u_i}{\partial z} = \nu \nabla^2 u_i + \frac{1}{3} \nu \nabla_i (\nabla \cdot u) + f_i,$$

where p, f_i, u and u_i are in Eulerian description and $\alpha_i = \alpha_i(t)$ in Lagrangian description, i.e., $\alpha_i = \frac{dx_i}{dt}$, with the radius vector $r = (x_1, x_2, x_3) \equiv (x, y, z)$ function of time and indicating a motion of a specific particle of fluid starting from position $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$.

The equations (4.2) and (4.3) shows us that the nonlinear form disappear, facilitating the obtaining of its solutions, transforming when $\nabla \cdot u = 0$ into a linear and second-order partial differential equation of elliptic type, already well-studied[8]. If $\nu = 0$ (Euler equations) we have equations of first order, obviously, which is also widely studied[9]. We realize that for each possible value of α_i it is possible to obtain different values of u_i, and reciprocally, i.e., there is not an one-one correspondence between α_i and u_i, thus it is convenient choose more easy time functions for the $\alpha_i(t)$, provided that compatible with the physical problem to be studied.

Nevertheless, even though it is very interesting to study other mathematical solutions for the original system (1.4) or the new system (4.2), I understand that the final conclusion made in [2] and [3] remains valid: it is possible to exist velocities in the Eulerian formulation that do not correspond to a real movement of particles of a fluid, according to the Lagrangian formulation. When I wrote this the first time I did not have the equations (4.2) and (4.3), deduced later in [1], but if it is true (as it is) that we should have (1.3) and (4.1) for a motion of fluid particle, then x_i and its respective velocity u_i are closely related, and the initial use of (1.1) in section 1 is valid. This is an excellent question to be examined with examples, which we will see in the next section.

But even when the relationship (1.1) is not required, a general solution for the new Euler equations ($\nu = 0$)

(4.4) $$\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \alpha_1 \frac{\partial u_i}{\partial x} + \alpha_2 \frac{\partial u_i}{\partial y} + \alpha_3 \frac{\partial u_i}{\partial z} = f_i$$

or

(4.5) $$\frac{\partial p}{\partial x_i} + \frac{Du_i}{Dt} = f_i,$$

in the case which the pressure p and external force $f = (f_1,\ f_2,\ f_3)$ are given and the velocity $u = (u_1,\ u_2,\ u_3)$ is calculated, is

(4.6) $$u_i = u_i^0 + \left(\int_0^t \left(f_i - \frac{\partial p}{\partial x_i}\right)|_L \, dt\right)|_E,$$

using

(4.7) $$\frac{Du_i}{Dt} = \frac{Du_i^E}{Dt} = \left(f_i - \frac{\partial p}{\partial x_i}\right)|_L.$$

u_i^0 is the component i of the initial velocity u^0, $|_L$ represents the use of transformation from Eulerian description to Lagrangian description and $|_E$ represents the inverse transformation used in $|_L$, returning to Eulerian description. We use implicitly $u_i^0 = (u_i^0|_L)|_E$ as well as $u_i = (u_i|_L)|_E$.

So here we conclude that the new Euler equations have a natural physical solution when the pressure and external force are given (or chosen) and the integration in (4.6) is possible, for $i = 1, 2, 3$, solution which varies with the specific movement of particles that is used. Boundary conditions must be in accordance with the solution (4.6) and it is also necessary substitute (4.6) in (4.4) for verification of possible conditions to be obeyed by each u_i^0 and α_i.

In special, when $\left(f_i - \frac{\partial p}{\partial x_i}\right)|_L$ is a function without temporal dependence, a constant function, the solution (4.6) is

(4.8) $$u_i = u_i^0 + \left(f_i - \frac{\partial p}{\partial x_i}\right)|_L\, t,$$

which is an exact solution and it is relatively fast and easy to simulate computationally. Substituting (4.8) in (4.4) we have

(4.9) $$\alpha_1 \frac{\partial u_i^0}{\partial x} + \alpha_2 \frac{\partial u_i^0}{\partial y} + \alpha_3 \frac{\partial u_i^0}{\partial z} = 0,$$

then a condition to be obeyed in this case.

We will see in section 8, Conclusion, an even better form of these equations, where we use

(4.10) $$\frac{D\alpha}{Dt} = \left(\frac{\partial u^E}{\partial t} + \alpha_1 \frac{\partial u^E}{\partial x} + \alpha_2 \frac{\partial u^E}{\partial y} + \alpha_3 \frac{\partial u^E}{\partial z}\right)|_t\, .$$

5 – Verification of physically reasonable solutions

§ 1

Of a point of view purely mathematical, it is not necessary to have the adoption of (1.1). It is possible forgotten that the Euler and Navier-Stokes equations have something relation with motion of fluids, liquids or gases, and accept that they are just equations of high level and difficulty of Pure Mathematics, but in this section we want to keep the bond or link between theses equations and the motion of fluids, and thus the use of (1.1) is born and can be used, as we will see.

If a particle (or some volume) of fluid has the movement governed according to the position vector $r = (x, y, z)$, with a temporal dependence $x = x(t),\ y = y(t),\ z = z(t)$, then the respective velocity of this particle (or volume) of fluid is $u = \frac{dr}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$, also, *a priori*, dependent of time (except if all three derivatives are equal to constant).

The first equation of (1.1),

$$\frac{\partial u_i}{\partial x_j} = 0,\ i \neq j, \tag{5.1.1}$$

is valid when we intend to follow the movement of a particle (or group of particles in a small volume) because in a mechanical movement we have by definition

$$u_i = \frac{dx_i}{dt}, \tag{5.1.2}$$

i.e., the component i of velocity is dependent only of component i of position, which is obvious, then we have $\frac{\partial u_i}{\partial x_j} = 0$ if $i \neq j$, according we saw in section 1.

From equation (5.1.2) we conclude that $dx_i = u_i dt$, or

$$\partial x_i = u_i \partial t, \tag{5.1.3}$$

the second equation of (1.1).

Thus we emphasize that if it is not necessary to have some particle or group of particles in the elementary volume $dV = dx\,dy\,dz$ in position (x, y, z) at time t then the use of (1.1), or (5.1.1) and (5.1.3), can be ignored, and we will have a problem purely mathematical.

Even if there is some bond or link between the coordinates, as $x^2 + y^2 + z^2 = R^2$ and $x\dot{x} + y\dot{y} + z\dot{z} = 0$ in a circular motion of constant radius R, the relation (5.1.2) is still true, by definition, and we do not need despise (5.1.1), a calculation facilitator, except if the external force is intrinsically dependent of the more than one spatial coordinate in at least one of the three orthogonal directions and we have $\nabla p \neq f$.

Then, what can be done when it is indispensable to use a determined relation between x, y and z, for example, when the particles need to be moving on a specific surface or manifold as $z = g(x, y)$? We try to first solve the equations using each variable in isolation, following (5.1.1), and at the end we use the dependence $z = g(x, y)$, i.e., the final solution for velocity will be

$$\begin{cases} u_1 = \varphi_1(x, t) \\ u_2 = \varphi_2(y, t) \\ u_3 = \varphi_3(z, t) = \varphi_3(g(x, y), t) = h(x, y, t) \end{cases} \tag{5.1.4}$$

and so we have indeed, in final consequence, $\frac{\partial u_3}{\partial z} = 0$. Obviously, if such procedure is not mathematically possible for some situation or configuration, we should abandon the use of (5.1.1) in this specific case.

We will check now the use of the relations (4.1),

$$(5.1.5)\qquad \begin{cases} \frac{dx}{dt} = u_1(t) \\ \frac{dy}{dt} = u_2(t) \\ \frac{dz}{dt} = u_3(t) \end{cases}$$

origin of the fundamental difference between the traditional equations and the new equations presented here. In fact, when we use and distinguish in a same equation the Eulerian u^E and Lagrangian u^L velocities the use of (1.1) is of secondary importance.

§ 2

Be the example 1

$$(5.2.1)\qquad \begin{cases} x = x_0 + t,\ \frac{dx}{dt} = 1 = u_1^L,\ \frac{Du_1^L}{Dt} = 0 \\ y = y_0 + 2t,\ \frac{dy}{dt} = 2 = u_2^L,\ \frac{Du_2^L}{Dt} = 0 \\ z = z_0 + 3t,\ \frac{dz}{dt} = 3 = u_3^L,\ \frac{Du_3^L}{Dt} = 0 \end{cases}$$

in fact a movement of total acceleration equal to zero, $\frac{Du_1^L}{Dt} = \frac{Du_2^L}{Dt} = \frac{Du_3^L}{Dt} = 0$, each particle starting from a generic initial position (x_0, y_0, z_0).

Suppose that the introduction of external force, internal frictional forces and internal pressure generated a solution for velocity, in the Eulerian formulation, such that, for example,

$$(5.2.2)\qquad \begin{cases} u_1^E = x,\ \frac{Du_1^E}{Dt} = \frac{Dx}{Dt} = \frac{D(x_0+t)}{Dt} = 1 \\ u_2^E = y,\ \frac{Du_2^E}{Dt} = \frac{Dy}{Dt} = \frac{D(y_0+2t)}{Dt} = 2 \\ u_3^E = z,\ \frac{Du_3^E}{Dt} = \frac{Dz}{Dt} = \frac{D(z_0+3t)}{Dt} = 3 \end{cases}$$

The acceleration as used in the Euler and Navier-Stokes equations is

$$(5.2.3)\qquad \begin{cases} \frac{Du_1^E}{Dt} = \frac{\partial u_1^E}{\partial t} + u_1^E \frac{\partial u_1^E}{\partial x} + u_2^E \frac{\partial u_1^E}{\partial y} + u_3^E \frac{\partial u_1^E}{\partial z} = x,\ x(t) = t \not\equiv 1 \\ \frac{Du_2^E}{Dt} = \frac{\partial u_2^E}{\partial t} + u_1^E \frac{\partial u_2^E}{\partial x} + u_2^E \frac{\partial u_2^E}{\partial y} + u_3^E \frac{\partial u_2^E}{\partial z} = y,\ y(t) = 2t \not\equiv 2 \\ \frac{Du_3^E}{Dt} = \frac{\partial u_3^E}{\partial t} + u_1^E \frac{\partial u_3^E}{\partial x} + u_2^E \frac{\partial u_3^E}{\partial y} + u_3^E \frac{\partial u_3^E}{\partial z} = z,\ z(t) = 3t \not\equiv 3 \end{cases}$$

i.e., the use of the expression according to the traditional Euler and Navier-Stokes equations generates a wrong value for the value of the acceleration $\frac{Du^E}{Dt}$.

By other side, using the correct form of the new Euler and Navier-Stokes equations, according (4.2), we have

$$(5.2.4) \qquad \begin{cases} \frac{Du_1^E}{Dt} = \frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z} = 1 \\ \frac{Du_2^E}{Dt} = \frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z} = 2 \\ \frac{Du_3^E}{Dt} = \frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z} = 3 \end{cases}$$

therefore the correct and expected result conform (5.2.2) for the acceleration $\frac{Du^E}{Dt}$, but with the disagreement $\frac{Du^E}{Dt} \neq \frac{Du^L}{Dt}$.

For that to be $\frac{Du^E}{Dt} = \frac{Du^L}{Dt}$ for all time and position it is necessary too, by a logical necessity of consistency between both velocities, that

$$(5.2.5) \qquad u^E(x(t), y(t), z(t), t) = u^L(t),$$

so, from (5.2.1)

$$(5.2.6) \qquad \begin{cases} u_1^E = 1,\ \frac{\partial u_1^E}{\partial t} = \frac{\partial u_1^E}{\partial x_j} = 0 \\ u_2^E = 2,\ \frac{\partial u_2^E}{\partial t} = \frac{\partial u_2^E}{\partial x_j} = 0 \\ u_3^E = 3,\ \frac{\partial u_3^E}{\partial t} = \frac{\partial u_3^E}{\partial x_j} = 0 \end{cases}$$

and now $\frac{Du^E}{Dt} = \frac{Du^L}{Dt} = 0$.

§ 3

Be now the example 2

$$(5.3.1) \qquad \begin{cases} x = x_0 + u_0 t + f\frac{t^2}{2},\ \frac{dx}{dt} = u_0 + ft = u_1^L,\ \frac{Du_1^L}{Dt} = f \\ y = y_0 + v_0 t + g\frac{t^2}{2},\ \frac{dy}{dt} = v_0 + gt = u_2^L,\ \frac{Du_2^L}{Dt} = g \\ z = z_0 + w_0 t + h\frac{t^2}{2},\ \frac{dz}{dt} = w_0 + ht = u_3^L,\ \frac{Du_3^L}{Dt} = h \end{cases}$$

for constants $x_0, y_0, z_0,\ u_0, v_0, w_0,\ f, g, h,$ a movement of constant acceleration (f, g, h).

Suppose again that the introduction of external force, internal frictional forces and internal pressure generated a solution for velocity, in the Eulerian formulation, such that, for example,

$$(5.3.2)\qquad \begin{cases} u_1^E = u_0 + ft, \ \frac{Du_1^E}{Dt} = f \\ u_2^E = v_0 + gt, \ \frac{Du_2^E}{Dt} = g \\ u_3^E = w_0 + ht, \ \frac{Du_3^E}{Dt} = h \end{cases}$$

without dependence of spatial position and with $u^E = u^L$.

The acceleration as used in the Euler and Navier-Stokes equations is

$$(5.3.3)\qquad \begin{cases} \frac{Du_1^E}{Dt} = \frac{\partial u_1^E}{\partial t} + u_1^E \frac{\partial u_1^E}{\partial x} + u_2^E \frac{\partial u_1^E}{\partial y} + u_3^E \frac{\partial u_1^E}{\partial z} = \frac{\partial u_1^E}{\partial t} = f \\ \frac{Du_2^E}{Dt} = \frac{\partial u_2^E}{\partial t} + u_1^E \frac{\partial u_2^E}{\partial x} + u_2^E \frac{\partial u_2^E}{\partial y} + u_3^E \frac{\partial u_2^E}{\partial z} = \frac{\partial u_2^E}{\partial t} = g \\ \frac{Du_3^E}{Dt} = \frac{\partial u_3^E}{\partial t} + u_1^E \frac{\partial u_3^E}{\partial x} + u_2^E \frac{\partial u_3^E}{\partial y} + u_3^E \frac{\partial u_3^E}{\partial z} = \frac{\partial u_3^E}{\partial t} = h \end{cases}$$

i.e., this time the use of the expression according to the traditional Euler and Navier-Stokes equations generates a correct value for the acceleration $\frac{Du^E}{Dt}$ because there is no dependence of position.

Besides this, using the correct form of the new Euler and Navier-Stokes equations, according (4.2), we have

$$(5.3.4)\qquad \begin{cases} \frac{Du_1^E}{Dt} = \frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z} = \frac{\partial u_1^E}{\partial t} = f \\ \frac{Du_2^E}{Dt} = \frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z} = \frac{\partial u_2^E}{\partial t} = g \\ \frac{Du_3^E}{Dt} = \frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z} = \frac{\partial u_3^E}{\partial t} = h \end{cases}$$

therefore the correct and expected result conform (5.3.2) for the acceleration $\frac{Du^E}{Dt}$, this time with the agreement $\frac{Du^E}{Dt} = \frac{Du^L}{Dt}$.

§ 4

We will next use the solution (4.6) of (4.5),

$$(5.4.1)\qquad u_i = u_i^0 + \left(\int_0^t \left(f_i - \frac{\partial p}{\partial x_i}\right)|_L \, dt\right)|_E,$$

solution of the new Euler equations, for the special and easier case that $f_i = \frac{\partial p}{\partial x_i}$, i.e., the external force is conservative, a gradient field, being the pressure its respective potential, and

(5.4.2) $$u_i = u_i^E = u_i^0,\ \frac{Du_i^E}{Dt} = \frac{\partial u_i^E}{\partial t} = 0,$$

and with

(5.4.3) $$\begin{cases} x = x_0 e^{-t},\ \frac{dx}{dt} = -x_0 e^{-t} = u_1^L,\ \frac{Du_1^L}{Dt} = x_0 e^{-t} \\ y = y_0 e^{-t},\ \frac{dy}{dt} = -y_0 e^{-t} = u_2^L,\ \frac{Du_2^L}{Dt} = y_0 e^{-t} \\ z = z_0 e^{-t},\ \frac{dz}{dt} = -z_0 e^{-t} = u_3^L,\ \frac{Du_3^L}{Dt} = z_0 e^{-t} \end{cases}$$

for constants x_0, y_0, z_0, a movement of contraction from (x_0, y_0, z_0) to $(0,0,0)$, with $\frac{Du^L}{Dt} = (x_0, y_0, z_0)e^{-t} = (x(t), y(t), z(t))$.

The acceleration as used in the traditional Euler and Navier-Stokes equations is

(5.4.4)

$$\begin{cases} \frac{Du_1^E}{Dt} = \left(\frac{\partial u_1^E}{\partial t} + u_1^E \frac{\partial u_1^E}{\partial x} + u_2^E \frac{\partial u_1^E}{\partial y} + u_3^E \frac{\partial u_1^E}{\partial z}\right)\Big|_t = \left(u_1^0 \frac{\partial u_1^0}{\partial x} + u_2^0 \frac{\partial u_1^0}{\partial y} + u_3^0 \frac{\partial u_1^0}{\partial z}\right)\Big|_t \\ \frac{Du_2^E}{Dt} = \left(\frac{\partial u_2^E}{\partial t} + u_1^E \frac{\partial u_2^E}{\partial x} + u_2^E \frac{\partial u_2^E}{\partial y} + u_3^E \frac{\partial u_2^E}{\partial z}\right)\Big|_t = \left(u_1^0 \frac{\partial u_2^0}{\partial x} + u_2^0 \frac{\partial u_2^0}{\partial y} + u_3^0 \frac{\partial u_2^0}{\partial z}\right)\Big|_t \\ \frac{Du_3^E}{Dt} = \left(\frac{\partial u_3^E}{\partial t} + u_1^E \frac{\partial u_3^E}{\partial x} + u_2^E \frac{\partial u_3^E}{\partial y} + u_3^E \frac{\partial u_3^E}{\partial z}\right)\Big|_t = \left(u_1^0 \frac{\partial u_3^0}{\partial x} + u_2^0 \frac{\partial u_3^0}{\partial y} + u_3^0 \frac{\partial u_3^0}{\partial z}\right)\Big|_t \end{cases}$$

which shows us the possibility of being valid $\frac{Du_i^E}{Dt} \neq 0$ with $\frac{\partial u_i^E}{\partial t} = 0$.

Being necessary in this case that $\frac{Du_i^E}{Dt} = \frac{\partial u_i^E}{\partial t} = 0$, for $i = 1,2,3$, we have

(5.4.5) $$\begin{cases} u_1^0 \frac{\partial u_1^0}{\partial x} + u_2^0 \frac{\partial u_1^0}{\partial y} + u_3^0 \frac{\partial u_1^0}{\partial z} = 0 \\ u_1^0 \frac{\partial u_2^0}{\partial x} + u_2^0 \frac{\partial u_2^0}{\partial y} + u_3^0 \frac{\partial u_2^0}{\partial z} = 0 \\ u_1^0 \frac{\partial u_3^0}{\partial x} + u_2^0 \frac{\partial u_3^0}{\partial y} + u_3^0 \frac{\partial u_3^0}{\partial z} = 0 \end{cases}$$

which is valid, for example, for initial velocities such that

(5.4.6) $$u_i^0 = k_i \phi_i(ax + by + cz),$$

with

(5.4.7) $$k_1 \phi_1 a + k_2 \phi_2 b + k_3 \phi_3 c = 0,$$

k_i, a, b, c real numbers, $\phi_i: \mathbb{R} \to \mathbb{R}$ differentiable functions, for $i = 1, 2, 3$. If the condition of incompressibility $\nabla \cdot u = \nabla \cdot u^0 = 0$ is required in the resolution of a given problem then it is also necessary that

(5.4.8) $$k_1 \phi_1' a + \ k_2 \phi_2' b + k_3 \phi_3' c = 0,$$

always satisfied when (5.4.7) is true.

With the correct form of the new Euler and Navier-Stokes equations we have, using (5.4.2),

(5.4.9)

$$\begin{cases} \frac{Du_1^E}{Dt} = \left(\frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z}\right)\Big|_t = \left(u_1^L \frac{\partial u_1^0}{\partial x} + u_2^L \frac{\partial u_1^0}{\partial y} + u_3^L \frac{\partial u_1^0}{\partial z}\right)\Big|_t = 0 \\ \frac{Du_2^E}{Dt} = \left(\frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z}\right)\Big|_t = \left(u_1^L \frac{\partial u_2^0}{\partial x} + u_2^L \frac{\partial u_2^0}{\partial y} + u_3^L \frac{\partial u_2^0}{\partial z}\right)\Big|_t = 0 \\ \frac{Du_3^E}{Dt} = \left(\frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z}\right)\Big|_t = \left(u_1^L \frac{\partial u_3^0}{\partial x} + u_2^L \frac{\partial u_3^0}{\partial y} + u_3^L \frac{\partial u_3^0}{\partial z}\right)\Big|_t = 0 \end{cases}$$

which also has by solution, for example,

(5.4.10) $$u_i^0 = k_i \phi_i (ax + by + cz),$$

supposing $\phi_i: \mathbb{R} \to \mathbb{R}$ differentiable functions and k_i, a, b, c real numbers, for $i = 1, 2, 3$, but this time with

(5.4.11) $$a\, u_1^L(t) + b\, u_2^L(t) + c\, u_3^L(t) = 0,$$

or equivalently

(5.4.12.1) $$u_1^L(t) = -\frac{1}{a}\left(b\, u_2^L(t) + c\, u_3^L(t)\right),\ a \neq 0,$$

(5.4.12.2) $$u_2^L(t) = -\frac{1}{b}\left(a\, u_1^L(t) + c\, u_3^L(t)\right),\ b \neq 0,$$

(5.4.12.3) $$u_3^L(t) = -\frac{1}{c}\left(a\, u_1^L(t) + b\, u_2^L(t)\right),\ c \neq 0,$$

for all $t \geq 0$, or all ${\phi_i}'$ are constants. For that $\nabla \cdot u = \nabla \cdot u^0 = 0$ it is necessary also be valid (5.4.8) or all ϕ_i need be constant.

According to the solution (5.4.10) and for the chosen movement given by (5.4.3), the condition (5.4.11) imposes that

(5.4.13.1) $$x_0 = -\frac{1}{a}(b\, y_0 + c\, z_0),$$

(5.4.13.2) $\quad y_0 = -\frac{1}{b}(a\,x_0 + c\,z_0),$

(5.4.13.3) $\quad z_0 = -\frac{1}{c}(a\,x_0 + b\,y_0),$

respectively if $a \neq 0,\ b \neq 0,\ c \neq 0,$ therefore each initial position of a specific particle or group of particles need to obey the previous condition, in this case: initial positions on a plane for each family of coefficients (a, b, c).

Note that in this way the Lagrangian solution is which governs the movement of fluids, or rather, explains what happens in the fluid, with respect to velocity. We can choose many different ϕ functions for Eulerian solution of u^E, but the individual motion of the particles or group of particles is the same with each prefixed choice of u^L. Thus, it is unnecessary to choose complicated initial velocities in the Eulerian formulation when the movement in the Lagrangian formulation is simpler, at least when the external force is a conservative field.

As made in § 2, by a logical necessity of consistency between both velocities and for that $\frac{Du^E}{Dt} = \frac{Du^L}{Dt}$ for all time and position it is necessary too that

(5.4.14) $\quad u^E(x(t), y(t), z(t), t) = u^L(t),$

so, from (5.4.3) we have

(5.4.15)
$$\begin{cases} u_1^E = -x,\ \frac{\partial u_1^E}{\partial t} = \frac{\partial u_1^E}{\partial y} = \frac{\partial u_1^E}{\partial z} = 0,\ \frac{\partial u_1^E}{\partial x} = -1 \\ u_2^E = -y,\ \frac{\partial u_2^E}{\partial t} = \frac{\partial u_2^E}{\partial x} = \frac{\partial u_2^E}{\partial z} = 0,\ \frac{\partial u_2^E}{\partial y} = -1 \\ u_3^E = -z,\ \frac{\partial u_3^E}{\partial t} = \frac{\partial u_3^E}{\partial x} = \frac{\partial u_3^E}{\partial y} = 0,\ \frac{\partial u_3^E}{\partial z} = -1 \end{cases}$$

and now $\frac{Du^E}{Dt}\big|_t = \frac{Du^L}{Dt} = \big(x(t), y(t), z(t)\big)$, but it is a compressible motion, with $\nabla \cdot u^E = -3$.

§ 5

In this present case we will analyze the same Lagrangian solution in (5.4.3), but now with time dependent Eulerian solution, i.e., with some or all $\frac{\partial u_i^E}{\partial t} \neq 0$. Again with $\nabla p = f$ and $\frac{Du^E}{Dt} = 0$, the Lagrangian solution is

(5.5.1)
$$\begin{cases} x = x_0 e^{-t},\ \frac{dx}{dt} = -x_0 e^{-t} = u_1^L,\ \frac{Du_1^L}{Dt} = x_0 e^{-t} \\ y = y_0 e^{-t},\ \frac{dy}{dt} = -y_0 e^{-t} = u_2^L,\ \frac{Du_2^L}{Dt} = y_0 e^{-t} \\ z = z_0 e^{-t},\ \frac{dz}{dt} = -z_0 e^{-t} = u_3^L,\ \frac{Du_3^L}{Dt} = z_0 e^{-t} \end{cases}$$

for constants x_0, y_0, z_0, a movement of contraction from (x_0, y_0, z_0) to $(0,0,0)$, with $\frac{Du^L}{Dt} = (x_0, y_0, z_0)e^{-t} = (x(t), y(t), z(t))$.

We have in this case for Eulerian representation in the traditional meaning

$$(5.5.2)\qquad \begin{cases} \frac{Du_1^E}{Dt} = \left(\frac{\partial u_1^E}{\partial t} + u_1^E \frac{\partial u_1^E}{\partial x} + u_2^E \frac{\partial u_1^E}{\partial y} + u_3^E \frac{\partial u_1^E}{\partial z}\right)|_t = 0 \\ \frac{Du_2^E}{Dt} = \left(\frac{\partial u_2^E}{\partial t} + u_1^E \frac{\partial u_2^E}{\partial x} + u_2^E \frac{\partial u_2^E}{\partial y} + u_3^E \frac{\partial u_2^E}{\partial z}\right)|_t = 0 \\ \frac{Du_3^E}{Dt} = \left(\frac{\partial u_3^E}{\partial t} + u_1^E \frac{\partial u_3^E}{\partial x} + u_2^E \frac{\partial u_3^E}{\partial y} + u_3^E \frac{\partial u_3^E}{\partial z}\right)|_t = 0 \end{cases}$$

Choosing for respective solution

$$(5.5.3)\qquad u_i^E = k_i \phi_i(ax + by + cz + dt),$$

with $\phi_i : \mathbb{R} \to \mathbb{R}$ differentiable functions and k_i, a, b, c real numbers, for $i = 1,2,3$, we have

$$(5.5.4)\qquad k_1 \phi_1 a + k_2 \phi_2 b + k_3 \phi_3 c + d = 0,$$

otherwise all ϕ_i are constants. If the condition of incompressibility $\nabla \cdot u = \nabla \cdot u^0 = 0$ is required in the resolution of a given problem then it is also necessary that

$$(5.5.5)\qquad k_1 \phi_1' a + k_2 \phi_2' b + k_3 \phi_3' c = 0,$$

always satisfied when (5.5.4) is true.

With the correct form of the new Euler and Navier-Stokes equations we have

$$(5.5.6)\qquad \begin{cases} \frac{Du_1^E}{Dt} = \left(\frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z}\right)|_t = 0 \\ \frac{Du_2^E}{Dt} = \left(\frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z}\right)|_t = 0 \\ \frac{Du_3^E}{Dt} = \left(\frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z}\right)|_t = 0 \end{cases}$$

which also has by solution, for example,

$$(5.5.7)\qquad u_i^E = k_i \phi_i(ax + by + cz + dt),$$

for $\phi_i : \mathbb{R} \to \mathbb{R}$ differentiable functions, k_i, a, b, c real numbers, $i = 1,2,3$, but this time with

$$(5.5.8)\qquad a\, u_1^L(t) + b\, u_2^L(t) + c\, u_3^L(t) + d = 0,$$

or equivalently

(5.5.9.1) $\quad u_1^L(t) = -\frac{1}{a}(b\, u_2^L(t) + c\, u_3^L(t) + d),\ \ a \neq 0,$

(5.5.9.2) $\quad u_2^L(t) = -\frac{1}{b}(a\, u_1^L(t) + c\, u_3^L(t) + d),\ \ b \neq 0,$

(5.5.9.3) $\quad u_3^L(t) = -\frac{1}{c}(a\, u_1^L(t) + b\, u_2^L(t) + d),\ \ c \neq 0,$

for all $t \geq 0$, or all $\phi_i{}'$ are constants. For that $\nabla \cdot u = \nabla \cdot u^0 = 0$ it is necessary also be valid (5.5.5) or all ϕ_i need be constant.

According to the solution (5.5.7) and for the chosen movement given by (5.5.1), the condition (5.5.8) imposes that

(5.5.10.1) $\quad x_0 = -\frac{1}{a}(b\, y_0 + c\, z_0 - d),$

(5.5.10.2) $\quad y_0 = -\frac{1}{b}(a\, x_0 + c\, z_0 - d),$

(5.5.10.3) $\quad z_0 = -\frac{1}{c}(a\, x_0 + b\, y_0 - d),$

respectively if $a \neq 0,\ b \neq 0,\ c \neq 0,$ therefore each initial position of a specific particle or group of particles needs to obey the previous condition, in this case: initial positions on a plane for each family of coefficients (a, b, c, d).

Note that a solution in the Lagrangian description may correspond to two (or even more) solutions in the Eulerian description, for example, a steady state solution as well as a non-steady state solution, as can be seen by comparing the solutions in § 4 and § 5, so it is convenient to look for, or pre-define, simpler formats for Eulerian solutions.

On the other hand, as we have already said, for to have logical consistency between both velocities, it is necessary that

(5.5.11) $\quad u^E(x(t), y(t), z(t), t) = u^L(t)$

and $\frac{Du^E}{Dt}|_t = \frac{Du^L}{Dt}$ for all time $t \geq 0$, and we came back to the solution obtained in (5.4.15), a steady state solution, i.e.,

(5.5.12)
$$\begin{cases} u_1^E = -x,\ \frac{\partial u_1^E}{\partial t} = \frac{\partial u_1^E}{\partial y} = \frac{\partial u_1^E}{\partial z} = 0,\ \frac{\partial u_1^E}{\partial x} = -1 \\ u_2^E = -y,\ \frac{\partial u_2^E}{\partial t} = \frac{\partial u_2^E}{\partial x} = \frac{\partial u_2^E}{\partial z} = 0,\ \frac{\partial u_2^E}{\partial y} = -1 \\ u_3^E = -z,\ \frac{\partial u_3^E}{\partial t} = \frac{\partial u_3^E}{\partial x} = \frac{\partial u_3^E}{\partial y} = 0,\ \frac{\partial u_3^E}{\partial z} = -1 \end{cases}$$

a compressible motion with $\nabla \cdot u^E = -3$ and $\frac{Du^E}{Dt}|_t = \frac{Du^L}{Dt} = \big(x(t), y(t), z(t)\big)$.

§ 6

Lastly, we will see the new Navier-Stokes equations. As the Lagrangian description governs the movement of particles or group of particles, while the Eulerian description is a kind of complicating of the real (or approximate, say) behavior of fluids, at least when the external force is conservative and the pressure is its potential ($\nabla p = f$), we will try an Eulerian solution for velocity using (1.1), i.e., given $u^L = (u_1^L,\ u_2^L,\ u_3^L)$ we will use the form

(5.6.1) $$u_i^E = u_i^E(x_i, t) = \phi_i(x_i)\varphi_i(t)$$

in the equation

(5.6.2) $$\frac{Du_i^E}{Dt} = \nu\,\nabla^2 u_i^E + \frac{1}{3}\nu\frac{\partial}{\partial x_i}(\nabla\cdot u^E),$$

with

(5.6.3) $$\frac{Du_i^E}{Dt} = \frac{\partial u_i^E}{\partial t} + u_1^L\frac{\partial u_i^E}{\partial x} + u_2^L\frac{\partial u_i^E}{\partial y} + u_3^L\frac{\partial u_i^E}{\partial z}$$

and $\nabla\cdot u^E$ without specific value, thus

(5.6.4) $$\phi_i(x_i)\varphi_i'(t) + u_i^L(t)\phi_i'(x_i)\varphi_i(t) = \frac{4}{3}\nu\,\phi_i{}''(x_i)\varphi_i(t),$$

an ordinary differential equation, for $i = 1,2,3$, supposing ϕ_i and φ_i differentiable and continuous functions how much is needed.

By the superposition principle we can also add solutions,

(5.6.5) $$u_i^E = u_i^E(x_i, t) = \sum_{j=1}^{\infty} u_{ij}^E(x_i, t) = \sum_{j=1}^{\infty}\phi_{ij}(x_i)\varphi_{ij}(t),$$

and then

(5.6.6) $$\phi_{ij}(x_i)\varphi_{ij}'(t) + u_i^L(t)\phi_{ij}'(x_i)\varphi_{ij}(t) = \frac{4}{3}\nu\,\phi_{ij}{}''(x_i)\varphi_{ij}(t),$$

but the better use of (1.1) is when we give completely the Lagrangian and Eulerian solutions for velocity (i.e., a choose obeying the required initial and boundary conditions as well as the compressibility condition) and the external force is conservative, such that,

(5.6.7) $$\begin{cases} p = \int_L \left(-\frac{Du^E}{Dt} + \nu\,\nabla^2 u^E + \frac{1}{3}\nu\,\nabla(\nabla\cdot u^E) + f\right)\cdot dl \\ u_i^E = u_i^E(x_i, t) \end{cases}$$

for $i = 1, 2, 3$, i.e., the pressure is the unique function which we do not have *a priori* and need be calculated, while the choose components of velocities have the necessity to be logically consistent with the problem in question. In section 6 we will see again this solution.

We now will make the Eulerian solution even easier than (5.6.1) by removing the dependence of time,

(5.6.8) $$u_i^E = u_i^E(x_i) = \phi_i(x_i),$$

with

(5.6.9) $$\frac{Du_i^E}{Dt} = u_1^L \frac{\partial u_i^E}{\partial x} + u_2^L \frac{\partial u_i^E}{\partial y} + u_3^L \frac{\partial u_i^E}{\partial z} = \nu\, \nabla^2 u_i^E + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u^E),$$

$\nabla \cdot u^E$ with free value, and so

(5.6.10) $$u_i^L(t)\phi_i'(x_i) = \frac{4}{3}\nu\, \phi_i''(x_i)$$

or

(5.6.11) $$u_i^L(t) = \frac{4}{3}\nu \frac{\phi_i''(x_i)}{\phi_i'(x_i)} = c_i,$$

a spatial solution which obviously cannot varies in time and for this reason it is necessary that the function $u_i^L(t)$ is a real constant c_i. The solution is exponential in relation to coordinate x_i:

(5.6.12) $$u_i^E = \phi_i(x_i) = k_i e^{3c_i\, x_i/4\nu},$$

which in fact solves (5.6.9) for $k_i,\ c_i,\ \nu > 0$ real constants.

Note that although (5.6.12) is a spatially unlimited function for $x_i \to +\infty$ if $k_i \neq 0$ and $c_i > 0$, the respective Lagrangian solution $u_i^L(t) = c_i$, which indicates a motion of constant velocity, is well behaved, smooth and limited, for all position and all $t \geq 0$. Then this is another case (as in § 2) in that we have a regular motion in the time in Lagrangian description but with possibility of an unlimited solution in Eulerian description. By other side, if $k_i \neq 0$ and $c_i < 0$ the respective component u_i^E decreases with position for $x_i > 0$ and it is unlimited for $x_i \to -\infty$, which also is not compatible with the respective motion of those particles or group of particles, but nevertheless it is a possible solution in Eulerian description.

Also note that in each of the examples in this section, we had initially in general $u^L(t) \neq u^E(x, y, z, t)$, except if $t = 0$ and $x = x_0,\ y = y_0,\ z = z_0$ is the initial position, or some specific set of positions (x, y, z) and (x_0, y_0, z_0) at time t (in special, $x = x(t, x_0),\ y = y(t, y_0),\ z = z(t, z_0)$ according defined in the respective Lagrangian description) or if u^E is not dependent of position (as in § 3), so by the chain rule the correct form of the total acceleration $\frac{Du^E}{Dt}$ in a particle of fluid (or elementary volume dV or group of particles) is

(5.6.13) $$\frac{Du^E}{Dt} = \frac{\partial u^E}{\partial t} + u_1^L \frac{\partial u^E}{\partial x} + u_2^L \frac{\partial u^E}{\partial y} + u_3^L \frac{\partial u^E}{\partial z},$$

because we have in general

(5.6.14) $$u_1^L \frac{\partial u^E}{\partial x} + u_2^L \frac{\partial u^E}{\partial y} + u_3^L \frac{\partial u^E}{\partial z} \neq u_1^E \frac{\partial u^E}{\partial x} + u_2^E \frac{\partial u^E}{\partial y} + u_3^E \frac{\partial u^E}{\partial z}.$$

We are using implicitly the initial position $(x_0,\ y_0,\ z_0)$ in the Lagrangian description $u^L(t)$ as constant, although it has the same meaning as in $u^L(t, x_0, y_0, z_0)$.

In the last example of this § 6 for that

(5.6.15) $$u^E(x(t), y(t), z(t), t) = u^L(t) = \frac{d}{dt}(x(t), y(t), z(t))$$

and $\frac{Du^E}{Dt}|_t = \frac{Du^L}{Dt}$ for all $t \geq 0$ it is necessary to have, for $t = 0$,

(5.6.16) $$u_i^E(x_0, y_0, z_0, t = 0) = u_i^L(0) = c_i$$

and then, from (5.6.11) and (5.6.12),

(5.6.17) $$k_i = c_i e^{-3c_i x_i^0 / 4\nu}$$

and

(5.6.18) $$u_i^E = c_i e^{3c_i(x_i - x_i^0)/4\nu}$$

where $(x_0, y_0, z_0) \equiv (x_1^0, x_2^0, x_3^0)$ is the respective initial velocity, a motion of constant velocity $c = (c_1, c_2, c_3)$ for each particle or group of particles in Lagrangian description, without compressibility along time, but an exponential function in Eulerian description and with $\nabla \cdot u^E \neq 0$.

Also thinking about other time values, $t > 0$, we cannot accept this solution, and then the unique possible solution here is

(5.6.19) $$u_i^E\big(x_i(t)\big) = u_i^L(t) = c_i,$$

thus

(5.6.20) $$x_i = x_i^0$$

and so, no movement,

(5.6.21) $\qquad c_i = 0.$

The conclusion in this case is that it is necessary to have time dependence in the velocity u^E.

6 – The question of the breakdown solutions

Without passing through the Lagrangian formulation, given a velocity $u(x, y, z, t)$ at least two times differentiable with respect to spatial coordinates and one respect to time and an integrable external force $f(x, y, z, t)$, perhaps the better expression for the solution of the equation (1.4) is

(6.1)
$$p(x, y, z, t) = \int_L \ S \cdot dl + \theta(t) = \sum_{i=1}^{3} \int_{P_i^0}^{P_i} S_i dx_i + \theta(t),$$
$$S = (S_1,\ S_2,\ S_3),$$
$$S_i = -\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right) + \nu(\nabla^2 u_i) + \frac{1}{3}\nu\big(\nabla_i(\nabla \cdot u)\big) + f_i,$$

supposing possible the integrations and that the vector $S = -\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right] + \nu\nabla^2 u + \frac{1}{3}\nu\nabla(\nabla \cdot u) + f$ is a gradient function, where it is necessary that

(6.2)
$$\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}.$$

This is the development of the solution of (1.4) for the specific path L going parallely (or perpendicularly) to axes X, Y and Z from $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ to $(x_1, x_2, x_3) \equiv (x, y, z)$, since that the solution (6.1) is valid for any piecewise smooth path L. We can choose $P_1^0 = (x_0, y_0, z_0)$, $P_2^0 = (x, y_0, z_0)$, $P_3^0 = (x, y, z_0)$ for the origin points and $P_1 = (x, y_0, z_0)$, $P_2 = (x, y, z_0)$, $P_3 = (x, y, z)$ for the destination points. $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$ or adjustable for some given condition. Again we have seen that the system of Navier-Stokes equations has no unique solution, only given initial conditions, supposing that there is some solution. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

The remark given for the system (1.5), when used in (1.4), leads us to the following conclusion: the integration of the system (1.4), confronting with (1.5), shows that, except for a constant or free term of integration, respectively $A(y, z, t)$, $B(x, z, t)$ and $C(x, y, t)$, anyone of its equations can be used for solve it, and the results must be equals each other, if the velocity u and external force f are given and the pressure p must be calculated. Then again this is a condition to the occurrence of solutions, otherwise there is not any solution, which shows to us the possibility of existence of "breakdown" solutions, as defined in [10].

By other side, using the first condition (1.1), $\frac{\partial u_i}{\partial x_j} = 0$ if $i \neq j$, due to Lagrangian formulation, where $u_i = \frac{dx_i}{dt}$, the original system (1.4) is simplified as

$$\text{(6.3)} \qquad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = \frac{4}{3} \nu \frac{\partial^2 u_1}{\partial x^2} + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial y} = \frac{4}{3} \nu \frac{\partial^2 u_2}{\partial y^2} + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_3 \frac{\partial u_3}{\partial z} = \frac{4}{3} \nu \frac{\partial^2 u_3}{\partial z^2} + f_3 \end{cases}$$

where u_i is a function only of the respective x_i and t, but not x_j if $j \neq i$. When it is required the incompressibility condition, $\nabla \cdot u = \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right) = 0$, then the constant $\frac{4}{3}$ in (6.3) should be replaced by 1.

If the external force has potential, $f = \nabla V$, then the system (6.3) has solution

$$\text{(6.4)} \qquad \begin{aligned} p &= \sum_{i=1}^{3} \int_{P_i^0}^{P_i} \left[-\left(\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i} \right) + \frac{4}{3} \nu \frac{\partial^2 u_i}{\partial x_i^2} + f_i \right] dx_i + \theta(t) \\ &= V + \sum_{i=1}^{3} \int_{x_i^0}^{x_i} \left[-\left(\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i} \right) + \frac{4}{3} \nu \frac{\partial^2 u_i}{\partial x_i^2} \right] dx_i + \theta(t), \end{aligned}$$

$V = \int_L f \cdot dl$, which although similar to (6.1) has the solubility guaranteed by the special functional dependence of the components of the vector u, i.e., $u_i = u_i(x_i, t)$, with $\frac{\partial u_i}{\partial x_j} = 0$ if $i \neq j$, supposing u, its derivatives and f integrable vectors. In this case the vector S described in (6.1) is always a gradient function, i.e., the relation (6.2) is satisfied. Note that if f is not an irrotational or gradient vector, i.e., if it does not have a potential, then the system (6.3), with $u_i = u_i(x_i, t)$, it has no solution, the case of "breakdown" solution in [10].

When the incompressibility condition is imposed ($\nabla \cdot u = 0$) we have, using (1.1), a small variety of possible solutions for velocity, of the form

$$\text{(6.5)} \qquad u_i(x_i, t) = A_i(t) x_i + B_i(t),$$

$A_i, B_i \in C^\infty([0, \infty))$, with

$$\text{(6.6)} \qquad A_1(t) + A_2(t) + A_3(t) = 0,$$

if the coordinates $x_1,\ x_2,\ x_3$ are independent of each other. In this case it is valid $\nabla^2 u = 0$, i.e., the system of equations has a solution for velocity independent of viscosity coefficient, equal to Euler equations, and except when $u = 0$ (for some or all $t \geq 0$) we have always $\int_{\mathbb{R}^3} |u|^2\, dxdydz \to \infty$, the occurrence of unbounded or unlimited energy, which is not difficult to see.

Another class of solutions S for velocity gives more possibility for the construction of the components of velocity u_i, but maintains a bond between $x_1,\ x_2,\ x_3$ and t such that

(6.7) $\quad S = \{(u_1, u_2, u_3);\ u_i \in C^1(\mathbb{R} \times \mathbb{R}_0^+),\ (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times \mathbb{R}_0^+,\ \nabla \cdot u = 0\},$

where $\mathbb{R}_0^+ = [0, \infty)$, and there is a scalar function φ_3 with $x_3 = \varphi_3(x_1, x_2, t)$ or similarly $x_1 = \varphi_1(x_2, x_3, t)$ or $x_2 = \varphi_2(x_1, x_3, t)$. The dependence between $x_1,\ x_2,\ x_3$ and t is necessary for that $\nabla \cdot u = 0$ in these points (x_1, x_2, x_3) at each time t, forming a surface or manifold which is the domain of the solutions and which varies in time.

Being correct that (1.1) and (4.1) can be used, which we saw in section 5, the solution (6.4) for pressure can therefore be replaced by

$$\begin{aligned} p &= \sum_{i=1}^{3} \int_{P_i^0}^{P_i} \left[-\left(\frac{\partial u_i}{\partial t} + \alpha_i \frac{\partial u_i}{\partial x_i} \right) + \frac{4}{3} \nu \frac{\partial^2 u_i}{\partial x_i^2} + f_i \right] dx_i + \theta(t) \\ &= V + \sum_{i=1}^{3} \int_{x_i^0}^{x_i} \left[-\left(\frac{\partial u_i}{\partial t} + \alpha_i \frac{\partial u_i}{\partial x_i} \right) + \frac{4}{3} \nu \frac{\partial^2 u_i}{\partial x_i^2} \right] dx_i + \theta(t) \\ &= V + \sum_{i=1}^{3} \left[p_i(x_i, t) - p_i\left(x_i^0, t\right) \right] + \theta(t), \end{aligned} \tag{6.8}$$

where $\alpha_i = \alpha_i(t)$ is the component i of the velocity in Lagrangian description of a particle of fluid in motion, $u_i = u_i(x_i,\ t)$ is the component i of the velocity in Eulerian description, $p_i(x_i, t) = \int_{x_i^0}^{x_i} \left[-\left(\frac{\partial u_i}{\partial t} + \alpha_i \frac{\partial u_i}{\partial x_i} \right) + \frac{4}{3} \nu \frac{\partial^2 u_i}{\partial x_i^2} \right] dx_i$ and the other meanings already given previously in this article. As we have already seen, when it is required the incompressibility condition then the constant $\frac{4}{3}$ in (6.8) should be replaced by 1 and the general solution (6.5) for velocity with the condition (6.6) remains valid, if the coordinates $x_1,\ x_2,\ x_3$ are independent of each other, as well as (6.7) with possible dependence between $x_1,\ x_2,\ x_3$ and t.

In section 8, Conclusion, we will see other cases of breakdown solution, when the Euler and Navier-Stokes equations have no solution.

7 – The non-uniqueness of solutions

The new equations presented here have clearly non-unique solutions (when there is at least one solution) in the following sense:

1) For the same initial Eulerian velocity, indicated as u^0, we can propose different velocities in the Lagrangian description, u^L, to compose the new equations, also with possibility of collisions between the particles belonging to the different movements described by each u^L. This can result in a rather chaotic Eulerian solution for velocity, in fact many velocities for a same point, and consequently also for the pressure, if it has not previously been chosen.

2) When we analyze the uniqueness of solutions (u^E, p) bearing in mind that the Lagrangian velocity u^L is predetermined, if only the initial velocity u^0 is given we have the non uniqueness of the pair (u^E, p) because we can construct many possible and different velocities u^E, as $u^E = \varphi(t)u^0 + \tau(t), \varphi(0) = 1, \tau(0) = 0$, $\varphi: [0, \infty) \to \mathbb{R}$, $\tau: [0, \infty) \to \mathbb{R}^3$, all smooth functions, and the pressure will be given by (6.8), where we are supposing the use of (1.1), i.e., $u_i^E = u_i^E(x_i, t)$, with $\frac{\partial u_i^E}{\partial x_j} = 0$ if $i \neq j$. Note that in this case we have $\nabla \times u^E = 0$ and the equation has solution, again with many possible pressures.

3) If is given a boundary condition of type $u^E|_{\partial S} = u^\partial$ (Dirichlet condition), with $u^\partial \in C^\infty(\mathbb{R}^3 \times [0, \infty))$ and $u^\partial(x, y, z, t = 0) = u^0$, then we can use the solution for velocity as $u^E = u^\partial$ and also we have the non uniqueness of the pair (u^E, p), because for the pressure to be unique it needs to be known the values of $p_1(x_0, t)$, $p_2(y_0, t)$, $p_3(z_0, t)$, i.e., the pressure is dependent of the values of x_0, y_0, z_0, and moreover $\theta(t)$, according (6.8). Naturally, the velocities u^∂ and u^0 must, themselves, obey to the new equations of Euler and Navier-Stokes, u^∂ for $t \geq 0$ and u^0 for $t = 0$. Note that in our convention the functions $p_1(x_0, t)$, $p_2(y_0, t)$, $p_3(z_0, t)$ denote the pressure value in a generic time $t \geq 0$, respectively at the positions (x_0, y, z), (x, y_0, z), (x, y, z_0), where (x_0, y_0, z_0) is the initial position. In this condition we have $\theta(t = 0) = 0$.

8 – Conclusion

In fact we saw two problems in Euler and Navier-Stokes equations, not only one:

1) the pressure is (or may be) a vector, which was viewed briefly in sections 2 and 3 during the deductions of these equations;

2) the nonlinear characteristic of these equations is not correct for modeling of motion of fluids, because the use of chain rule in $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$ implies that $u_1 = \frac{dx}{dt}$, $u_2 = \frac{dy}{dt}$ and $u_3 = \frac{dz}{dt}$ are time functions only, without spatial dependence, which we viewed in section 4.

We propose a new form for the Euler ($\nu = 0$) and Navier-Stokes equations, where there is the simultaneous use of Euler and Lagrangian descriptions in a same equation, i.e., for $i = 1, 2, 3$,

$$(8.1) \quad \frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \alpha_1 \frac{\partial u_i}{\partial x} + \alpha_2 \frac{\partial u_i}{\partial y} + \alpha_3 \frac{\partial u_i}{\partial z} = \nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i,$$

where p, f_i, u and u_i are in Eulerian description and $\alpha_i = \alpha_i(t)$ in Lagrangian description, i.e., $\alpha_i = \frac{dx_i}{dt}$, according equation (4.3). Of this manner the nonlinear form of these equations disappear, replacing it by linear equations, a second-order equation of elliptic type if $\nu > 0$ or first order equation if $\nu = 0$.

Obviously, using the vector nature of pressure the equation (8.1) needs to be modified to

$$(8.2) \quad \frac{\partial p_i}{\partial x_i} + \frac{\partial u_i}{\partial t} + \alpha_1 \frac{\partial u_i}{\partial x} + \alpha_2 \frac{\partial u_i}{\partial y} + \alpha_3 \frac{\partial u_i}{\partial z} = \nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i.$$

In (8.1) it is still necessary to have a resultant conservative field, a gradient vector, specifically for the integrable vector $S = (S_1,\ S_2,\ S_3)$, with

$$(8.3) \quad S_i = \left(\nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i\right) - \left(\frac{\partial u_i}{\partial t} + \alpha_1 \frac{\partial u_i}{\partial x} + \alpha_2 \frac{\partial u_i}{\partial y} + \alpha_3 \frac{\partial u_i}{\partial z}\right),$$

whereas in equation (8.2) this is no longer necessary.

In section 4 we conclude that the new Euler equations have a natural physical solution when the pressure and external force are given (or chosen) and the integration in (4.6), which is the mentioned solution,

$$(8.4) \qquad u_i = u_i^0 + \left(\int_0^t \left(f_i - \frac{\partial p}{\partial x_i}\right)|_L \, dt\right)|_E,$$

is possible, for $i = 1, 2, 3$, in general a non unique solution varying with the transformations indicated as $|_L$ and $|_E$. Beside this, boundary conditions must be in accordance with this solution, as well as it is necessary the verification of possible conditions to be obeyed by each u_i^0 and $\alpha_i(t)$, substituting the solution in the equation, for that the mentioned solution effectively satisfies the equation of a mathematical point of view.

The functions α describe the velocity of the particles of the fluid over time, so the importance of them can be considered greater than that of velocity u, that is, it is convenient to choose initial velocities u^0 as simple as possible that are compatible with the selected movement described by the α functions, in special: $u^0(x_0, y_0, z_0) = \alpha(t = 0, x_0, y_0, z_0)$. Without the compromise of the equality in time of the Eulerian and Lagrangian descriptions, it is even possible that different velocities u, for example $u' \neq u''$, correspond to the same motion described by α, and we have $div\, u' = 0$ and $div\, u'' \neq 0$. So, seems that the incompressibility condition is not of priority importance for the description of motion of fluids. Note that similarly to what we have already said in section 5, we use implicitly the initial position (x_0, y_0, z_0) in the function $\alpha(t)$ as constant, although it has the same meaning as in $\alpha(t, x_0, y_0, z_0)$. Other constant parameters also can be included, of course: R, θ_0, ω, ν, ρ, etc., able to describe a very large class of motions.

It is also possible an easier form for the Euler ($\nu = 0$) and Navier-Stokes equations, that is

(8.5) $$\frac{\partial p_i}{\partial x_i} + \frac{D\alpha_i}{Dt} = \nu\nabla^2 u_i + \frac{1}{3}\nu\frac{\partial}{\partial x_i}(\nabla\cdot u) + f_i,$$

where we can substitute p_i by p if $p_1 = p_2 = p_3 = p$ is scalar pressure. Here $\frac{D\alpha_i}{Dt}$ is, in fact, a function only of time (and possibly constant parameters), without explicit dependence of x, y, z. The new forms for these equations are most didactic, because they can remind us of the need to be valid

(8.6) $$u^E(x(t), y(t), z(t), t) = u^L(t) = \alpha(t) = \frac{d}{dt}(x(t), y(t), z(t))$$

and

(8.7) $$\frac{Du^E}{Dt}|_t = \frac{Du^L}{Dt} = \frac{D\alpha}{Dt} = \left(\frac{\partial u^E}{\partial t} + \alpha_1\frac{\partial u^E}{\partial x} + \alpha_2\frac{\partial u^E}{\partial y} + \alpha_3\frac{\partial u^E}{\partial z}\right)|_t$$

when we analyze a fluid motion, a physical system, not only the solution of a problem purely mathematical, without application.

Now, to solve the equations of Navier-Stokes, and especially the Euler equations, is no more difficult than solve the traditional equations of mathematical physics, as heat equation, wave equation, Laplace and Poisson equations, etc., all of them linear differential equations. Despite this, in case of scalar pressure, if $\nu = 0$ and the external force is non conservative there is no solution for Euler equations, as well as if the initial velocity is gradient ($u^0 = \nabla\phi^0,\ \nabla\times u^0 = 0$) and the external force is non conservative, which leads us to the case of breakdown solution described in [10], when the pressure is a scalar function, because is not possible the calculation of pressure, according rule (6.2) viewed in section 6.

Note that the application of a non conservative force in fluid is naturally possible and there will always be some movement, even starting from rest. So that this is not a paradoxical situation it seems certain that the pressure in this case cannot be scalar, but rather vector, and thus the equations returns to solution in all cases (assuming all derivatives are possible, etc.). It is as indicated in (8.2) and (8.5). With the use of vector pressure the conditions mentioned for systems (1.4) and (1.5) also becomes unnecessary.

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