# Unique Relativistic Extension of the Pauli Hamiltonian

Steven Kenneth Kauffmann*

**Abstract**

Relativistic extension of the Pauli Hamiltonian is ostensibly achieved by minimal coupling of electromagnetism to the free-particle Dirac Hamiltonian. But the free-particle Pauli Hamiltonian is pathology-free in its nonrelativistic domain, while the free-particle Dirac Hamiltonian yields completely fixed particle speed which is greater than $c$, spin-orbit torque whose ratio to kinetic energy tends to infinity in the zero-momentum limit, and mega-violation of Newton's First Law in that limit. Furthermore, relativistic extension of the Pauli Hamiltonian is unique in principle because inertial frame hopping can keep the particle nonrelativistic. That extension is indeed readily achieved by upgrading the terms of the Pauli Hamiltonian's corresponding action to appropriate Lorentz invariants. The resulting relativistic Lagrangian yields a canonical momentum that can't be analytically inverted in general, but a physically-sensible successive-approximation scheme applies. For hydrogen and simpler systems approximation isn't needed, and the result, which includes spin-orbit coupling, is as transparently physically sensible as the relativistic Lorentz Hamiltonian is, a far cry from the Dirac Hamiltonian pathologies.

## Introduction

Although relativistic extension of the nonrelativistic Pauli Hamiltonian [1],

$$H(\mathbf{r},\mathbf{P},t,\vec{\sigma}) = \left(|\mathbf{P}-(e/c)\mathbf{A}|^2/(2m)\right) + e\phi - (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}), \tag{1}$$

is widely thought to be achieved by minimal coupling of the electromagnetic four-potential $A^\mu = (\phi, \mathbf{A})$ to the free-particle Dirac Hamiltonian [2], namely,

$$H^{(0)}_{\mathrm{Dirac}}(\mathbf{p},\vec{\alpha},\beta) = c\mathbf{p}\cdot\vec{\alpha} + mc^2\beta, \tag{2}$$

the fact that the free-particle Dirac Hamiltonian above is *linear* in the particle momentum $\mathbf{p}$ while the free-particle reduction of the Pauli Hamiltonian of Eq. (1), namely,

$$H^{(0)}(\mathbf{p}) = |\mathbf{p}|^2/(2m), \tag{3}$$

is *quadratic* in $\mathbf{p}$ causes those two free-particle Hamiltonians to have *completely incompatible implications*, even *notwithstanding* that the free-particle reduction of the Pauli Hamiltonian is valid *only* in the nonrelativistic limit $|\mathbf{p}| \ll mc$. For example, the free-particle reduction of the Pauli Hamiltonian given by Eq. (3) implies that,

$$\dot{\mathbf{r}} = \mathbf{p}/m, \tag{4a}$$

the familiar nonrelativistic free-particle *proportionality* of velocity to momentum, whereas the free-particle Dirac Hamiltonian of Eq. (2) implies that,

$$\dot{\mathbf{r}} = c\vec{\alpha}, \tag{4b}$$

namely that velocity *is completely independent of momentum*, which *flatly contradicts* the nonrelativistic free-particle *proportionality* of velocity to momentum given by Eq. (4a). This *contradiction* definitely *exists when* $|\mathbf{p}| \ll mc$, namely when Eq. (4a) *is unquestionably valid*. Furthermore, *the orthodox relativistic extension* of the free-particle Eq. (4a), namely,

$$\dot{\mathbf{r}} = \mathbf{p}/\left(m^2 + (|\mathbf{p}|^2/c^2)\right)^{\frac{1}{2}}. \tag{4c}$$

indicates that free-particle relativistic velocity is expected *to always be parallel to momentum*, certainly *never independent of momentum*, as the Eq. (4b) result of the free-particle Dirac Hamiltonian is.

In *addition*, because the three components $\alpha^1, \alpha^2$ and $\alpha^3$ of the Dirac vector $\vec{\alpha}$ satisfy $(\alpha^1)^2 = (\alpha^2)^2 = (\alpha^3)^2 = 1$, the Dirac free-particle result of Eq. (4b) implies that,

$$|\dot{\mathbf{r}}| = c\sqrt{3}. \tag{4d}$$

* Retired, American Physical Society Senior Life Member, E-mail: SKKauffmann@gmail.com

Thus any Dirac free particle *has a completely fixed speed which absurdly is greater than the limiting speed $c$ of special relativity*. In complete contrast, the *orthodox relativistic* Eq. (4c) *respects* $c$ as the relativistic limiting speed, and in *agreement* with physical experience *permits any speed* $|\dot{\mathbf{r}}|$ *which satisfies* $0 \le |\dot{\mathbf{r}}| < c$.

Clearly Dirac *must* have *grievously misapplied* the precepts of special relativity in the course of *developing* his free-particle Hamiltonian of Eq. (2). In his 1928 paper [3], Dirac took space-time symmetry to be the *essence* and *touchstone* of special relativity. Since the Schrödinger equation is *first-order* in the time derivative, Dirac asserted that the "relativistic principle" of *space-time symmetry* requires its Hamiltonian operator to be *first-order* in the space derivatives, namely *linear* in the momentum operators. Precisely *that* was Dirac's *motivation* for writing down *the problematic* "relativistic" free-particle Hamiltonian form of Eq. (2). Dirac then went on to work out what the algebraic properties of the components of $\vec{\alpha}$ and of $\beta$ must be in order to *force* his free-particle wave function—which of course satisfies the Schrödinger equation with his postulated "relativistic" Hamiltonian form of Eq. (2)—*to as well satisfy the free-particle Klein-Gordon equation*.

Dirac *failed to appreciate* that the idea of space-time symmetry in special relativity *arises merely colloquially from certain characteristics of the Lorentz transformation*, and that *rigid insistence on the space-time symmetry of theoretical entities* is in fact *subordinate to making theoretically appropriate choices of Lorentz-transformation representation for those entities*.

In fact, from the standpoint of selection of Lorentz-transformation representation, insistence on space-time symmetry is tantamount to selecting the invariant representation. Indeed, there *actually exists* an elaborate, highly sophisticated mathematical demonstration that the Dirac free-particle equation, which is the Schrödinger equation with the Dirac free-particle Hamiltonian of Eq. (2), *is form-invariant under Lorentz transformations*—we shall touch on that demonstration at slightly greater length further below.

But is *Lorentz form-invariance* theoretically appropriate for a Schrödinger equation? Scrutiny of the generic configuration-representation Schrödinger equation for a particle reveals a first-order time derivative on that equation's left-hand side (Dirac got *that much* right) *and an energy, namely the Hamiltonian operator,* on the Schrödinger equation's right-hand side. *It has long been established in special relativity that energy is the time component of a four-vector Lorentz representation for energy-momentum*. The time partial derivative operator on the left-hand side of the Schrödinger equation is as well *the time-component* of a four-vector representation of the Lorentz transformation *which is naturally constructed from that time partial-derivative operator* $(\partial/\partial t)$ *together with the spatial gradient operator* $\nabla$*, as follows*,

$$\partial^\mu \stackrel{\text{def}}{=} ((1/c)((\partial/\partial t), -\nabla).$$

Thus the generic configuration-representation Schrödinger equation for a particle is clearly *the time component of the four-vector system of equations which is given by*,

$$i\hbar\partial^\mu\psi = \widehat{P}^\mu\psi, \tag{5}$$

where,

$$\widehat{P}^\mu \stackrel{\text{def}}{=} (\widehat{H}/c,\, \widehat{\mathbf{P}}),$$

$\widehat{H}$ being the particle's Hamiltonian operator in configuration representation, and $\widehat{\mathbf{P}}$ being the particle's canonical three-momentum operator in configuration representation. The *three-vector equation subsystem of* Eq. (5), namely,

$$-i\hbar\nabla\psi = \widehat{\mathbf{P}}\psi,$$

in fact expresses *nothing more* than Schrödinger's familiar *quantization rule in configuration representation for the particle's canonical three-momentum operator*. Thus all four sub-equations of Eq. (5) *are familiar features of nonrelativistic quantum mechanics.*

But to represent *theoretically-coherent relativistic quantum mechanics* the four sub-equations of Eq. (5) must transform as a Lorentz-covariant four-vector. That will be the case if $\widehat{P}^\mu = (\widehat{H}/c,\, \widehat{\mathbf{P}})$ transforms as a Lorentz-covariant four-vector—which is assured if the particle's Lagrangian $L$, canonical three-momentum $\widehat{\mathbf{P}}$ and Hamiltonian $\widehat{H}$ *all follow from a Lorentz-invariant action.*

Notice, however, that if the time-component Schrödinger equation,

$$i\hbar(\partial\psi/\partial t) = \widehat{H}\psi.$$

of the Eq. (5) system *is Lorentz form-invariant*, four-vector Lorentz-covariance of that system is stymied!

Since Dirac's imposition of space-time symmetry on his free-particle time-component Schrödinger equation *made it Lorentz form-invariant*, that equation *can't* be the time component of a theoretically-coherent Lorentz-covariant four-vector relativistic Schrödinger-equation system, a fact which is of course borne out in spades by the relativistically absurd Eq. (4d) property of the Eq. (2) Dirac free-particle Hamiltonian, namely that $|\dot{\mathbf{r}}| = c\sqrt{3}$.

Dirac's 1928 work [3] indicates unfamiliarity with the concept of Lorentz-transformation representations, and the impression that special relativity *imposes space-time symmetry*—whose presence in the Dirac equation produces *violation* of special relativity!

The *hallmark property of a free particle*, whether nonrelativistic or relativistic, is,

$$\dot{\mathbf{p}} = \mathbf{0}, \tag{6a}$$

because a free particle experiences no force. For a nonrelativistic free particle, Eq. (6a) taken together with Eq. (4a) yields *Newton's First Law*, namely,

$$\ddot{\mathbf{r}} = \mathbf{0}. \tag{6b}$$

For *an orthodox relativistic* free particle, Eq. (6a) taken together with Eq. (4c) *still* yields *Newton's First Law* as given by Eq. (6b).

But for the *relativistically nonviable* Dirac free particle, *its velocity's complete independence of its momentum* in league with Dirac's specific algebraic properties for $\vec{\alpha}$ and $\beta$ implies a startlingly egregious *violation of Newton's First Law*. Specifically, Eq. (4b) yields that,

$$\ddot{\mathbf{r}} = c d\vec{\alpha}/dt, \tag{7a}$$

and from the Heisenberg equation of motion together with Dirac's algebraic properties for $\vec{\alpha}$ and $\beta$ one obtains,

$$d\vec{\alpha}/dt = (-i/\hbar)[\vec{\alpha}, H^{(0)}_{\mathrm{Dirac}}] = (-i/\hbar)[\vec{\alpha}, c\mathbf{p}\cdot\vec{\alpha} + mc^2\beta] = (2c/\hbar)(\mathbf{p}\times\vec{\sigma}) + (2mc^2/\hbar)(i\beta\vec{\alpha}), \tag{7b}$$

where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ is defined as follows in terms of $\vec{\alpha}$,

$$\vec{\sigma} \stackrel{\mathrm{def}}{=} (-i\alpha^2\alpha^3, -i\alpha^3\alpha^1, -i\alpha^1\alpha^2). \tag{7c}$$

From Dirac's algebraic properties for $\vec{\alpha}$ the three components $\sigma^1, \sigma^2$ and $\sigma^3$ of $\vec{\sigma}$ defined by Eq. (7c) are readily shown to have all of the algebraic properties of the Pauli matrices, namely that they are Hermitian, and also that,

$$(\sigma^1)^2 = (\sigma^2)^2 = (\sigma^3)^2 = 1,\ \ \sigma^2\sigma^3 = -\sigma^3\sigma^2 = i\sigma^1,\ \ \sigma^3\sigma^1 = -\sigma^1\sigma^3 = i\sigma^2,\ \ \sigma^1\sigma^2 = -\sigma^2\sigma^1 = i\sigma^3. \tag{7d}$$

Eq. (7a) together with Eq. (7b) shows that the relativistically nonviable Dirac free particle *violates Newton's First Law*. To get some idea of *the extent* of that violation of Newton's First Law, we note from Eqs. (7a) and (7b) that for a *zero-momentum* ($\mathbf{p} = \mathbf{0}$) Dirac free particle,

$$\ddot{\mathbf{r}} = (2mc^3/\hbar)(i\beta\vec{\alpha}), \tag{7e}$$

which from Dirac's algebraic properties for $\vec{\alpha}$ and $\beta$ yields,

$$|\ddot{\mathbf{r}}| = 2\sqrt{3}mc^3/\hbar. \tag{7f}$$

Therefore a Dirac free particle of mass $m$ and *zero* momentum has *a spontaneous acceleration* magnitude $|\ddot{\mathbf{r}}|$ of the order of the particle's "Compton acceleration" $mc^3/\hbar$, which for the *electron* has the staggering value of roughly $10^{28}$ times the acceleration of gravity at the earth's surface! ("Compton acceleration", being *proportional to particle mass*, is of course proportionately *greater* for the muon.) This staggering *violation of Newton's First Law*, along with the physically impossible Eq. (4d), *flows from Dirac's imposition of space-time symmetry on the Schrödinger equation*.

As well as adhering to Newton's First Law, a *free* particle would *also* be expected to manifest *conservation of orbital angular momentum* $\mathbf{L}$, where,

$$\mathbf{L} \stackrel{\mathrm{def}}{=} \mathbf{r}\times\mathbf{p}. \tag{8a}$$

Since $\dot{\mathbf{p}} = \mathbf{0}$ for a free particle, it is consequently the case that for a free particle,

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p}, \tag{8b}$$

so from *both* of Eqs. (4a) and (4c) for $\dot{\mathbf{r}}$ we *indeed* see that orbital angular momentum is *conserved*, respectively, for *both* nonrelativistic and *orthodox relativistic* free particles. In fact it is *a well-established theorem* of special-relativistic dynamics that the *spin angular momentum* of a *free system* is *conserved* [4], so since the *total* angular momentum of a *free system* is conserved, *a free system's orbital angular momentum must be conserved as well in special-relativistic dynamics*.

However for the relativistically nonviable Dirac free particle, Eqs. (4b) and (8b) together imply relativistically unacceptable *nonconservation of orbital angular momentum* in the form,

$$\dot{\mathbf{L}} = c(\vec{\alpha} \times \mathbf{p}), \tag{8c}$$

which from Dirac's algebraic properties for $\vec{\alpha}$ implies that the *torque magnitude*,

$$|\dot{\mathbf{L}}| = \sqrt{2}c|\mathbf{p}|, \tag{8d}$$

is exerted on the Dirac free particle's orbital angular momentum. This *relativistically unacceptable* torque magnitude exerted on the Dirac free particle's orbital angular momentum *always exceeds the Dirac free particle's kinetic energy*, which is,

$$((mc^2)^2 + (c|\mathbf{p}|)^2)^{\frac{1}{2}} - mc^2 = (c|\mathbf{p}|)^2 / \left[((mc^2)^2 + (c|\mathbf{p}|)^2)^{\frac{1}{2}} + mc^2\right].$$

As a matter of fact, the *ratio* of the Eq. (8d) relativistically unacceptable *torque magnitude* $|\dot{\mathbf{L}}| = \sqrt{2}c|\mathbf{p}|$ exerted on the Dirac free particle's orbital angular momentum to that particle's *kinetic energy*, which is written just above, becomes *infinite* in the *extreme nonrelativistic limit* $|\mathbf{p}| \to 0$! Such is *yet another physically impossible consequence of Dirac's imposition of space-time symmetry on the Schrödinger equation*.

To complete this discussion of the nonconservation of the orbital angular momentum of the Dirac free particle, it of course must be pointed out that the Eq. (8c) torque $\dot{\mathbf{L}} = c(\vec{\alpha} \times \mathbf{p})$ exerted on that particle's orbital angular momentum *is matched by an equal and opposite torque exerted on that particle's spin angular momentum* $(\hbar/2)\vec{\sigma}$. That torque can be calculated by using Eq. (7c), followed by Eq. (7b); the *explicit insertion of* Eq. (7b) and the *ensuing* Dirac algebra *aren't* displayed below,

$$(\hbar/2)d\vec{\sigma}/dt = -i(\hbar/2)(\dot{\alpha}^2\alpha^3 + \alpha^2\dot{\alpha}^3, \dot{\alpha}^3\alpha^1 + \alpha^3\dot{\alpha}^1, \dot{\alpha}^1\alpha^2 + \alpha^1\dot{\alpha}^2) = c(\mathbf{p} \times \vec{\alpha}) = -\dot{\mathbf{L}}. \tag{8e}$$

Although the Dirac free particle thus does indeed conserve *total* angular momentum $\mathbf{J} = \mathbf{L} + (\hbar/2)\vec{\sigma}$, the fact that torque is exerted on its *spin* angular momentem *directly contradicts the theorem of special-relativistic dynamics that the spin angular momentum of a free system is conserved* [4].

Notwithstanding a widespread belief to the contrary, the Dirac free-particle Hamiltonian of Eq. (2) minimally coupled to the electromagnetic four-potential $A^\mu$ no more implies the nonrelativistic Pauli Hamiltonian when $|\mathbf{p}| \ll mc$ than the Dirac free-particle Hamiltonian *itself* when $|\mathbf{p}| \ll mc$ implies the nonrelativistic free-particle Hamiltonian of Eq. (3) *to which the Pauli Hamiltonian of course reduces when* $A^\mu = 0$; indeed we have clearly seen that the Dirac free-particle *speed* is *completely fixed* to the superluminal value $c\sqrt{3}$ *regardless* of the value of $|\mathbf{p}|$, *in utter and complete disagreement with nonrelativistic particle speed* $|\mathbf{p}|/m$, *especially when* $|\mathbf{p}| \ll mc$!

Since the widely believed "theorem" to the effect that the Dirac free-particle Hamiltonian of Eq. (2) minimally coupled to the electromagnetic four-potential $A^\mu$ implies the nonrelativistic Pauli Hamiltonian when $|\mathbf{p}| \ll mc$ clearly falls flat on its face when $A^\mu = 0$, of course *it isn't a theorem at all*!

Naturally it is of interest to understand *what* the august purveyors [5, 6] of a "theorem" which certainly *isn't* a theorem *overlooked*.

The *blunder* common to *all* of the "demonstrations" that the minimally electromagnetically coupled Dirac Hamiltonian reduces to the Pauli Hamiltonian when $|\mathbf{p}| \ll mc$ is *an insufficiently scrutinized assumption* that under that *exclusionary* circumstance *all* of the *remaining* Dirac-equation solutions have energy $E$ which satisfies the inequality $|E - mc^2| \ll mc^2$, an assumption *which overlooks the unavoidably-present negative-energy solutions of the Dirac equation*!

Indeed, given the fact that the Dirac equation *unavoidably* has *a plethora of solutions which aren't shared by the Pauli equation*, no "theorem" about the Dirac Hamiltonian approaching the Pauli Hamiltonian when $|\mathbf{p}| \ll mc$ *should ever have been entertained*, much less spuriously "proved".

Unfortunately that false "theorem" encouraged an unjustifiably sanguine view of the validity of the Dirac equation, as the following quotes [7] reveal,

[The emergence of the Pauli equation] "gives us confidence that we are on the right track in accepting [the Dirac free-particle equation] "and [its minimal coupling to $A^\mu$] "as a starting point in constructing a relativistic electron theory."

"Fortified by this successful nonrelativistic reduction of the Dirac equation, we go on and establish Lorentz covariance of the Dirac theory, as required by special relativity."

The *false* impression of "successful nonrelativistic reduction of the Dirac equation" thus inspired the proposal and "proof" of yet *another* false "theorem", namely "Lorentz covariance of the Dirac theory".

Clearly, if the Dirac theory *were* Lorentz covariant, *it wouldn't generate the egregiously unphysical absurdity that every free Dirac particle has the fixed speed* $c\sqrt{3}$. Nor would it *violate* the special-relativistic *theorem* that free systems *conserve their spin angular momentum* [4], nor would a free Dirac particle *staggeringly contravene Newton's First Law*.

So far as a "proof" of the Lorentz covariance of a Schrödinger-equation theory is concerned, it would be legitimate *only* if it successfully grappled with the question of whether that Schrödinger-equation's corresponding four-equation system of Eq. (5) *is a Lorentz-covariant four-vector*.

The "proof" actually proffered however paid *no attention whatsoever* to the issue of whether the Dirac equation's corresponding *four-equation system of* Eq. (5) is a Lorentz-covariant four-vector. *Instead*, as was mentioned in the discussion following Eq. (4d), it developed an elaborately constructed mapping of the Lorentz transformations onto Dirac's four-component spinor wave functions that leaves *form-invariant* the Dirac free-particle time-component Schrödinger equation which follows from the Eq. (2) Dirac Hamilton [8].

That the Dirac free-particle time-component Schrödinger equation *by itself is form-invariant under the Lorentz transformations* of course is as good a demonstration as there could possibly be that the corresponding *four-equation system of* Eq. (5) *is precluded from transforming as a Lorentz four-vector*!

In other words, what the august purveyors of "proofs" billed as the establishment of the Lorentz covariance of the Dirac theory [7, 8] *elaborately demonstrates precisely the opposite*! What in fact was accomplished was to produce *yet another addition* to a continually-growing list of demonstrations that the Dirac theory *egregiously violates physical Lorentz covariance*.

We have seen that there is an extensive catalogue of ways in which the space-time symmetric Dirac free-particle time-component Schrödinger equation corresponding to the Hamiltonian of Eq. (2) egregiously violates relativistic dynamical principles. Thus there is no theoretical justification for the widely held belief that relativistic extension of the nonrelativistic Pauli Hamiltonian of Eq. (1) is achieved by minimal coupling of the free-particle Dirac Hamiltonian of Eq. (2) to the electromagnetic four-potential $A^\mu$.

The nonrelativistic Pauli Hamiltonian of Eq. (1) must, from a physical standpoint, have a *unique* relativistic extension because an appropriate instantaneous Lorentz boost can always render the Pauli particle nonrelativistic—albeit that boost *alters* the electromagnetic four-potential $A^\mu = (\phi, \mathbf{A})$ and magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ with which the Pauli particle *interacts*. As was mooted in the discussion below Eq. (5), such a relativistic extension of the nonrelativistic Pauli Hamiltonian can be assured if, inter alia, the Pauli particle's *relativistic* Lagrangian $L_{\mathrm{rel}}$, canonical three-momentum $\mathbf{P}_{\mathrm{rel}}$ and Hamiltonian $H_{\mathrm{rel}}$ *all follow from a Lorentz-invariant action* $S_{\mathrm{rel}}$. Therefore *seamless* relativistic extension of the nonrelativistic Pauli Hamiltonian $H$ of Eq. (1) *requires perusal of its corresponding nonrelativistic action* $S$, which must be extended to the *unique* Lorentz-invariant action $S_{\mathrm{rel}}$ *that reduces to* $S$ *in the nonrelativistic limit*. This program is carried out in the next section.

## Action-based unique relativistic extension of the Pauli Hamiltonian

In preparation for the relativistic extension of the nonrelativistic Pauli Hamiltonian of Eq. (1), we add to it the particle's rest-mass energy $mc^2$,

$$H = mc^2 + \left(|\mathbf{P} - (e/c)\mathbf{A}|^2/(2m)\right) + e\phi - (e\hbar/(2mc))(\vec{\sigma} \cdot \mathbf{B}). \tag{9a}$$

Note that the addition of such a constant term to a Hamiltonian in no way changes the quantum Heisenberg or classical Hamiltonian equations of motion.

To obtain the action $S$ which corresponds to the Hamiltonian $H$ of Eq. (9a), we first work out the Lagrangian $L$ which corresponds to that Hamiltonian $H$. The conversion of such a particle Hamiltonian to a particle Lagrangian requires swapping the Hamiltonian's dependence on the canonical three-momentum $\mathbf{P}$ for the Lagrangian's dependence on the particle's three-velocity $\dot{\mathbf{r}}$. We obtain that particle three-velocity $\dot{\mathbf{r}}$ from the Heisenberg equation of motion (or alternatively, in this case, from the equivalent classical Hamiltonian equation of motion),

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r}, H] = \nabla_{\mathbf{P}} H = (\mathbf{P} - (e/c)\mathbf{A})/m. \tag{9b}$$

We now *invert* the relation of Eq. (9b) between particle velocity $\dot{\mathbf{r}}$ and canonical momentum $\mathbf{P}$ to read,

$$\mathbf{P} = m\dot{\mathbf{r}} + (e/c)\mathbf{A}, \tag{9c}$$

and insert it into the well-known relationship of the Lagrangian to the Hamiltonian, namely,

$$L = \dot{\mathbf{r}} \cdot \mathbf{P} - H\Big|_{\mathbf{P}=m\dot{\mathbf{r}}+(e/c)\mathbf{A}} = -mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 - e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}), \tag{9d}$$

from which we immediately obtain the action,

$$S = \int L dt = \int \left[-mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 - e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})\right] dt. \tag{9e}$$

We shall treat this action $S$ as a sum of three terms, and shall work out the Lorentz-invariant upgrade of each of those three terms invidually. The first term of $S$ which we tackle is that of the free particle,

$$S^0 = \int (-mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2)dt. \tag{10a}$$

The first step toward finding the Lorentz-invariant upgrade of $S^0$ is to change the integration variable from time $t$ to Lorentz-invariant light-cone proper time, where the square of the infinitesimal Lorentz-invariant light-cone proper-time interval is given by,

$$(d\tau)^2 = (dt)^2 - (|d\mathbf{r}|^2/c^2), \tag{10b}$$

which implies that,

$$d\tau/dt = (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}, \tag{10c}$$

and from this it of course follows that,

$$dt/d\tau = (1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}. \tag{10d}$$

Thus we can write $S^0$ as,

$$S^0 = \int (-mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2)(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} d\tau = \int (-mc^2)(1 + O(|\dot{\mathbf{r}}/c|^4)d\tau. \tag{10e}$$

Therefore in the nonrelativistic regime where $|\dot{\mathbf{r}}| \ll c$ the nonrelativistic free-particle action $S^0$ is very well approximated by its Lorentz-invariant upgrade, which we can read off from Eq. (10e) to be simply,

$$S^0_{\text{rel}} = \int (-mc^2)d\tau. \tag{10f}$$

Eq. (10f), by use of Eq. (10c) can of course also be expressed as,

$$S^0_{\text{rel}} = \int (-mc^2)(1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} dt. \tag{10g}$$

We next tackle the part of the action $S$ which encompasses the interaction of the particle's charge $e$ with the electromagnetic potential $A^\mu = (\phi, \mathbf{A})$,

$$S^e = \int (-e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}))dt. \tag{11a}$$

Finding the Lorentz-invariant upgrade of $S^e$ is facilitated by *starting* from the $\dot{\mathbf{r}} \to \mathbf{0}$ *static limit* of $S^e$, namely by starting from,

$$S^e_{\text{static}} = \int (-e\phi)dt. \tag{11b}$$

We carry out the Lorentz-invariant upgrade of $S^e_{\mathrm{static}}$ by replacing the differential $dt$ in Eq. (11b) by the Lorentz-invariant differential $d\tau$ and upgrading the $\dot{\mathbf{r}} = \mathbf{0}$ static-limit potential energy $e\phi$ to a dynamic Lorentz-invariant function of $\dot{\mathbf{r}}$. To do so we first rewrite the static potential energy $e\phi$ as the faux Lorentz invariant,

$$e\phi = eU_\mu(\dot{\mathbf{r}} = \mathbf{0})A^\mu, \tag{11c}$$

that has the faux Lorentz-covariant constituent,

$$U_\mu(\dot{\mathbf{r}} = \mathbf{0}) = \delta^0_\mu. \tag{11d}$$

which is valid *only* in the particle's rest frame where the particle's velocity $\dot{\mathbf{r}} = \mathbf{0}$. To upgrade the static faux Lorentz-covariant $U_\mu(\dot{\mathbf{r}} = \mathbf{0})$ to a dynamic true Lorentz-covariant entity $U_\mu(\dot{\mathbf{r}})$, we Lorentz-boost it from the particle's rest frame to the inertial frame where the particle has velocity $\dot{\mathbf{r}}$,

$$U_\mu(\dot{\mathbf{r}}) = U_\alpha(\dot{\mathbf{r}} = \mathbf{0})\Lambda^\alpha_\mu(\dot{\mathbf{r}}) = \delta^0_\alpha\Lambda^\alpha_\mu(\dot{\mathbf{r}}) = \Lambda^0_\mu(\dot{\mathbf{r}}). \tag{11e}$$

Therefore *the dynamic Lorentz-invariant upgrade of the static potential energy $e\phi$* is,

$$eU_\mu(\dot{\mathbf{r}})A^\mu = e\Lambda^0_\mu(\dot{\mathbf{r}})A^\mu = e\gamma(\dot{\mathbf{r}})\,(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}), \tag{11f}$$

where,

$$\gamma(\dot{\mathbf{r}}) = (1 - (|\dot{\mathbf{r}}|^2/c^2))^{-\frac{1}{2}} = dt/d\tau. \tag{11g}$$

Thus the Lorentz-invariant upgrade of,

$$S^e_{\mathrm{static}} = \int(-e\phi)dt,$$

is,

$$S^e_{\mathrm{rel}} = \int(-eU_\mu(\dot{\mathbf{r}})A^\mu)d\tau = \int(-e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}))dt. \tag{11h}$$

We see from Eqs. (11h) and (11a) that,

$$S^e_{\mathrm{rel}} = S^e, \tag{11i}$$

so it turns out that the nonrelativistic action $S^e$ of Eq. (11a) *actually was already equal to its Lorentz-invariant upgrade $S^e_{\mathrm{rel}}$!*

Finally we tackle the part of the action $S$ that encompasses the interaction of the particle's spin with the magnetic field,

$$S^{\vec{\sigma}} = \int(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})dt. \tag{12a}$$

Again we replace the differential $dt$ by the Lorentz-invariant differential $d\tau$ and upgrade the static potential energy $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$, which is valid in the $\dot{\mathbf{r}} = \mathbf{0}$ particle rest frame, to a dynamic Lorentz-invariant function of $\dot{\mathbf{r}}$. Preliminary to the upgrading of the static potential energy $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$, we write it as,

$$-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}) = -(e\hbar/(2mc))(\vec{\sigma}\cdot(\nabla\times\mathbf{A})) = (e\hbar/(2mc))\epsilon_{ijk}\sigma^i(\partial^j A^k). \tag{12b}$$

This representation of the static potential energy can be rewritten as the faux Lorentz invariant,

$$(e\hbar/(2mc))\epsilon_{ijk}\sigma^i(\partial^j A^k) = (e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0})(\partial^\mu A^\nu), \tag{12c}$$

that has the faux Lorentz-covariant constituent,

$$\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0}) = \begin{cases} 0 & \text{if } \mu = 0 \text{ or } \nu = 0, \\ \epsilon_{ijk}\sigma^i & \text{if } \mu = j \text{ and } \nu = k,\ j,k = 1,2,3, \end{cases} \tag{12d}$$

which is valid *only* in the particle's rest frame where the particle's velocity $\dot{\mathbf{r}} = \mathbf{0}$. Note that $\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0})$ is *antisymmetric* under the interchange of its two indices $\mu$ and $\nu$. To upgrade the static faux Lorentz-covariant $\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0})$ to a dynamic true Lorentz-covariant entity $\sigma_{\mu\nu}(\dot{\mathbf{r}})$, we Lorentz-boost it from the particle's rest frame to the inertial frame where the particle has velocity $\dot{\mathbf{r}}$,

$$\sigma_{\mu\nu}(\dot{\mathbf{r}}) = \sigma_{\alpha\beta}(\dot{\mathbf{r}} = \mathbf{0})\Lambda^\alpha_\mu(\dot{\mathbf{r}})\Lambda^\beta_\nu(\dot{\mathbf{r}}) = \epsilon_{ijk}\sigma^i\Lambda^j_\mu(\dot{\mathbf{r}})\Lambda^k_\nu(\dot{\mathbf{r}}). \tag{12e}$$

It is apparent from Eq. (12e) that the Lorentz-covariant second-rank tensor $\sigma_{\mu\nu}(\dot{\mathbf{r}})$ is *also* antisymmetric under the interchange of its two indices $\mu$ and $\nu$. From Eqs. (12b) through (12e) it is clear that *the dynamic Lorentz-invariant upgrade of the static potential energy* $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$ is,

$$(e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^\mu A^\nu) = (e\hbar/(2mc))\epsilon_{ijk}\sigma^i\Lambda^j_\mu(\dot{\mathbf{r}})\Lambda^k_\nu(\dot{\mathbf{r}})(\partial^\mu A^\nu) = \\ (e\hbar/(2mc))\left(\vec{\sigma}\cdot[(\mathbf{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu)\times(\mathbf{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu)]\right), \tag{12f}$$

where,

$$(\mathbf{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu)^j \stackrel{\text{def}}{=} \Lambda^j_\mu(\dot{\mathbf{r}})\partial^\mu \text{ and } (\mathbf{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu)^k \stackrel{\text{def}}{=} \Lambda^k_\nu(\dot{\mathbf{r}})A^\nu. \tag{12g}$$

The space components of the Lorentz boost of the four-vector partial-derivative operator,

$$\partial^\mu = ((1/c)(\partial/\partial t), -\nabla),$$

from the rest frame of the particle to the inertial frame in which the particle has velocity $\dot{\mathbf{r}}$ are given by,

$$(\mathbf{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu) = -\nabla - (\gamma(\dot{\mathbf{r}})-1)|\dot{\mathbf{r}}|^{-2}\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\nabla) - \gamma(\dot{\mathbf{r}})(\dot{\mathbf{r}}/c)(1/c)(\partial/\partial t), \tag{12h}$$

and the space components of the *same* Lorentz boost of the electromagnetic four-vector potential,

$$A^\mu = (\phi, \mathbf{A}),$$

are given by,

$$(\mathbf{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu) = \mathbf{A} + (\gamma(\dot{\mathbf{r}})-1)|\dot{\mathbf{r}}|^{-2}\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{A}) - \gamma(\dot{\mathbf{r}})(\dot{\mathbf{r}}/c)\phi. \tag{12i}$$

Using Eqs. (12h) and (12i) one can, with tedious effort, verify that,

$$(\mathbf{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu)\times(\mathbf{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu) = -(\nabla\times\mathbf{A}) - \\ (\gamma(\dot{\mathbf{r}})-1)|\dot{\mathbf{r}}|^{-2}\left[\nabla\times(\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{A})) + (\dot{\mathbf{r}}\cdot\nabla)(\dot{\mathbf{r}}\times\mathbf{A})\right] - \gamma(\dot{\mathbf{r}})\left[(\dot{\mathbf{r}}/c)\times(\dot{\mathbf{A}}/c) - \nabla\times((\dot{\mathbf{r}}/c)\phi)\right] = \\ -(\nabla\times\mathbf{A}) - (\gamma(\dot{\mathbf{r}})-1)|\dot{\mathbf{r}}|^{-2}\left[\dot{\mathbf{r}}\times[-\nabla(\dot{\mathbf{r}}\cdot\mathbf{A}) + (\dot{\mathbf{r}}\cdot\nabla)\mathbf{A}]\right] + \gamma(\dot{\mathbf{r}})\left[(\dot{\mathbf{r}}/c)\times\left[-\nabla\phi - (\dot{\mathbf{A}}/c)\right]\right] = \\ -(\nabla\times\mathbf{A}) - (\gamma(\dot{\mathbf{r}})-1)|\dot{\mathbf{r}}|^{-2}\left[\dot{\mathbf{r}}\times[-\dot{\mathbf{r}}\times(\nabla\times\mathbf{A})]\right] + \gamma(\dot{\mathbf{r}})\left[(\dot{\mathbf{r}}/c)\times\left[-\nabla\phi - (\dot{\mathbf{A}}/c)\right]\right] = \\ -\mathbf{B} - (\gamma(\dot{\mathbf{r}})-1)|\dot{\mathbf{r}}|^{-2}\left[|\dot{\mathbf{r}}|^2\mathbf{B} - \dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{B})\right] + \gamma(\dot{\mathbf{r}})((\dot{\mathbf{r}}/c)\times\mathbf{E}) = \\ -\gamma(\dot{\mathbf{r}})\mathbf{B} + (\gamma(\dot{\mathbf{r}})-1)|\dot{\mathbf{r}}|^{-2}\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{B}) - \gamma(\dot{\mathbf{r}})(\mathbf{E}\times(\dot{\mathbf{r}}/c)). \tag{12j}$$

From Eqs. (12f) and (12j) one sees that the dynamic Lorentz-invariant upgrade of the static potential energy $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$ is,

$$(e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^\mu A^\nu) = (e\hbar/(2mc))\left(\vec{\sigma}\cdot[(\mathbf{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu)\times(\mathbf{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu)]\right) = \\ -(e\hbar/(2mc))\left[\gamma(\dot{\mathbf{r}})(\vec{\sigma}\cdot\mathbf{B}) - (\gamma(\dot{\mathbf{r}})-1)|\dot{\mathbf{r}}|^{-2}(\vec{\sigma}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{B}) + \gamma(\dot{\mathbf{r}})(\vec{\sigma}\cdot(\mathbf{E}\times(\dot{\mathbf{r}}/c)))\right], \tag{12k}$$

and thus the Lorentz-invariant upgrade of the Eq. (12a) spin contribution to the action, namely,

$$S^{\vec{\sigma}} = \int (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})dt.$$

comes out to be,

$$S^{\vec{\sigma}}_{\text{rel}} = -\int (e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^\mu A^\nu)d\tau = \\ \int (e\hbar/(2mc))\left[(\vec{\sigma}\cdot\mathbf{B}) - (1-(\gamma(\dot{\mathbf{r}}))^{-1})|\dot{\mathbf{r}}|^{-2}(\vec{\sigma}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{B}) + (\vec{\sigma}\cdot(\mathbf{E}\times(\dot{\mathbf{r}}/c)))\right]dt = \\ \int (e\hbar/(2mc))\left[(\vec{\sigma}\cdot\mathbf{B}) - (1+(\gamma(\dot{\mathbf{r}}))^{-1})^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\vec{\sigma}\times\mathbf{E})\cdot(\dot{\mathbf{r}}/c)\right]dt, \tag{12l}$$

as we see by using Eq. (12k) and the fact that,

$$\gamma(\dot{\mathbf{r}}) = (1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} = dt/d\tau.$$

In the last step of Eq. (12l), we have furthermore interchanged the "dot" $\cdot$ with the "cross" $\times$ in the triple scalar product,

$$(\vec{\sigma}\cdot(\mathbf{E}\times(\dot{\mathbf{r}}/c))),$$

and have as well applied the identity,

$$(1-(\gamma(\dot{\mathbf{r}}))^{-1})|\dot{\mathbf{r}}|^{-2} = (1+(\gamma(\dot{\mathbf{r}}))^{-1})^{-1}c^{-2}.$$

We are now in a position to write down the Lorentz-invariant upgrade $S_{\mathrm{rel}}$ of the nonrelativistic Pauli action $S$ of Eq. (9e),

$$\begin{gathered} S_{\mathrm{rel}} = S^0_{\mathrm{rel}} + S^e_{\mathrm{rel}} + S^{\vec{\sigma}}_{\mathrm{rel}} = \int \left[-mc^2 - eU_\mu(\dot{\mathbf{r}})A^\mu - (e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^\mu A^\nu)\right] d\tau = \\ \int \Bigg[-mc^2(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} - e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + \\ (e\hbar/(2mc))\left((\vec{\sigma}\cdot\mathbf{B}) - \left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E})\right)\Bigg] dt \end{gathered} \tag{13a}$$

From this Lorentz-invariant upgrade of the nonrelativistic Pauli action we can immediately write down the relativistic Pauli Lagrangian,

$$\begin{gathered} L_{\mathrm{rel}} = -mc^2(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} - e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + \\ (e\hbar/(2mc))\left((\vec{\sigma}\cdot\mathbf{B}) - \left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E})\right), \end{gathered} \tag{13b}$$

where, of course,

$$\mathbf{B} = \nabla\times\mathbf{A} \ \text{ and } \ \mathbf{E} = -\nabla\phi - (\dot{\mathbf{A}}/c). \tag{13c}$$

From Eq. (13b) we calculate *the relativistic Pauli Lagrangian's corresponding canonical momentum*,

$$\begin{gathered} \mathbf{P} = \nabla_{\dot{\mathbf{r}}} L_{\mathrm{rel}} = m\dot{\mathbf{r}}(1-|\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + (e/c)\mathbf{A} + (e\hbar/(2mc^2))(\vec{\sigma}\times\mathbf{E}) - \\ (e\hbar/(2mc^2))\left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}\Big[\vec{\sigma}((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))\mathbf{B} + \\ \left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\dot{\mathbf{r}}/c)(1-|\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})\Big]. \end{gathered} \tag{13d}$$

The *last three terms of* Eq. (13d), which all arise from *the relativistic distortion of the magnetic field* **B**, unfortunately *preclude solving analytically* for the particle's *velocity* $\dot{\mathbf{r}}$ in terms of the system's *canonical momentum* **P**. For that reason we cannot in general *analytically* parlay the relativistic Pauli system's *energy* $E_{\mathrm{rel}}$, namely,

$$E_{\mathrm{rel}} = \dot{\mathbf{r}}\cdot\mathbf{P} - L_{\mathrm{rel}}, \tag{13e}$$

into its relativistic Pauli *Hamiltonian* $H_{\mathrm{rel}}(\mathbf{r},\vec{\sigma},\mathbf{P},t)$. However we see from Eq. (13d) that the three offending terms which arise from the relativistic distortion of the magnetic field **B** are all *higher-order corrections in powers of* $|\dot{\mathbf{r}}/c|$, so we can easily rewrite Eq. (13d) as *a successive-approximation scheme* for the desired inversion of the canonical momentum **P** that is *consonant* with *the systematic carrying out of relativistic corrections*. The scheme *is considerably more transparent*, however, after all occurrences of the particle velocity $\dot{\mathbf{r}}$ on the right-hand side of Eq. (13d) (and as well on the right-hand side of Eq. (13e)) are *replaced* by occurrences *of the free-particle momentum* **p**, which is,

$$\begin{gathered} \mathbf{p} \stackrel{\mathrm{def}}{=} m\dot{\mathbf{r}}(1-|\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}, \ \text{ and implies,} \\ (\dot{\mathbf{r}}/c)(1-|\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} = \mathbf{p}/(mc), \quad (\dot{\mathbf{r}}/c) = \mathbf{p}(m^2c^2+|\mathbf{p}|^2)^{-\frac{1}{2}}, \quad (1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} = mc(m^2c^2+|\mathbf{p}|^2)^{-\frac{1}{2}}. \end{gathered} \tag{13f}$$

Using Eq. (13f) to eliminate all occurrences of the particle velocity $\dot{\mathbf{r}}$ on the right-hand side of Eq. (13d) in favor of the free-particle momentum $\mathbf{p}$ yields,

$$\begin{gathered}\mathbf{P} = \mathbf{p} + (e/c)\mathbf{A} + (e\hbar/(2mc^2))(\vec{\sigma}\times\mathbf{E}) - \\ (e\hbar/(2mc^2))\left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}\times \\ \left[\vec{\sigma}(\mathbf{p}\cdot\mathbf{B}) + (\vec{\sigma}\cdot\mathbf{p})\mathbf{B} + \left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}(\mathbf{p}/(mc))(\vec{\sigma}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B})\right].\end{gathered} \tag{13g}$$

Eq. (13g) can now be readily rewritten *as a successive approximation scheme for the resolution of the free-particle momentum* $\mathbf{p}$ *in terms of the canonical momentum* $\mathbf{P}$,

$$\begin{gathered}\mathbf{p} = \mathbf{P} - (e/c)\mathbf{A} - (e\hbar/(2mc^2))(\vec{\sigma}\times\mathbf{E}) + \\ (e\hbar/(2mc^2))\left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}\times \\ \left[\vec{\sigma}(\mathbf{p}\cdot\mathbf{B}) + (\vec{\sigma}\cdot\mathbf{p})\mathbf{B} + \left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}(\mathbf{p}/(mc))(\vec{\sigma}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B})\right].\end{gathered} \tag{13h}$$

In order for these successive approximations to $\mathbf{p}$ in terms of $\mathbf{P}$ to be able to produce successive approximations to the relativistic Pauli *Hamiltonian* $H_{\mathrm{rel}}$, we must *also* banish all occurrences of the particle velocity $\dot{\mathbf{r}}$ in the system's *energy* $E_{\mathrm{rel}}$, which is given on the right-hand side of Eq. (13e), in favor of the free-particle momentum $\mathbf{p}$.

We shall, however, *first* calculate that relativistic Pauli energy $E_{\mathrm{rel}} = \dot{\mathbf{r}}\cdot\mathbf{P} - L_{\mathrm{rel}}$ of Eq. (13e) *entirely in terms of* $\dot{\mathbf{r}}$ by using the $L_{\mathrm{rel}}$ which is given by Eq. (13b) and the $\mathbf{P}$ which is given by Eq. (13d), and *then* use the relations given in Eq. (13f) to eliminate $\dot{\mathbf{r}}$ from $E_{\mathrm{rel}}$ in favor of $\mathbf{p}$.

From Eq. (13b) we obtain that,

$$\begin{gathered}-L_{\mathrm{rel}} = mc^2(1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} + e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) - \\ (e\hbar/(2mc))\Big((\vec{\sigma}\cdot\mathbf{B}) - \left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E})\Big),\end{gathered} \tag{13i}$$

and from Eq. (13d) we obtain that,

$$\begin{gathered}\dot{\mathbf{r}}\cdot\mathbf{P} = m|\dot{\mathbf{r}}|^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + e(\dot{\mathbf{r}}/c)\cdot\mathbf{A} + (e\hbar/(2mc))(\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E}) - \\ (e\hbar/(2mc))\left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})\times \\ \left[2 + \left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}|\dot{\mathbf{r}}/c|^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}\right]\end{gathered} \tag{13j}$$

The complicated structure of the last term of Eq. (13j) can be simplified markedly, with the result,

$$\begin{gathered}\dot{\mathbf{r}}\cdot\mathbf{P} = m|\dot{\mathbf{r}}|^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + e(\dot{\mathbf{r}}/c)\cdot\mathbf{A} + (e\hbar/(2mc))(\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E}) - \\ (e\hbar/(2mc))(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}\end{gathered} \tag{13k}$$

Putting Eqs. (13i) and (13k) together produces,

$$\begin{gathered}E_{\mathrm{rel}} = \dot{\mathbf{r}}\cdot\mathbf{P} - L_{\mathrm{rel}} = mc^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + e\phi - \\ (e\hbar/(2mc))\left[(\vec{\sigma}\cdot\mathbf{B}) + (\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})\left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}\right].\end{gathered} \tag{13l}$$

We now use the relations given by Eq. (13f) to express the $E_{\mathrm{rel}}$ of Eq. (13l) entirely in terms of free-particle momentum $\mathbf{p}$ instead of in terms of the particle velocity $\dot{\mathbf{r}}$,

$$\begin{gathered}E_{\mathrm{rel}} = (m^2c^4 + |c\mathbf{p}|^2)^{\frac{1}{2}} + e\phi - \\ (e\hbar/(2mc))\left[(\vec{\sigma}\cdot\mathbf{B}) + (\vec{\sigma}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B})\left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}(mc)^{-1}\right].\end{gathered} \tag{13m}$$

Eq. (13m) is to be used in conjunction with the Eq. (13h) successive approximation scheme for obtaining the free-particle momentum $\mathbf{p}$ in terms of the canonical momentum $\mathbf{P}$, in order to generate successive approximations to the relativistic Pauli Hamiltonian $H_{\mathrm{rel}}$.

*In those cases where* $\mathbf{B} = \mathbf{0}$, Eq. (13h) immediately yields the *exact* relationship of the canonical momentum $\mathbf{P}$ to the free-particle momentum $\mathbf{p}$, namely,

$$\mathbf{p} = \mathbf{P} - (e/c)\mathbf{A} - (e\hbar/(2mc^2))(\vec{\sigma} \times \mathbf{E}), \tag{13n}$$

and for *those* $\mathbf{B} = \mathbf{0}$ cases Eq. (13m) yields the following the *exact* relativistic Pauli Hamiltonian, namely,

$$H_{\mathrm{rel}} = \left(m^2c^4 + |c\mathbf{P} - e\mathbf{A} - (e\hbar/(2mc))(\vec{\sigma} \times \mathbf{E})|^2\right)^{\frac{1}{2}} + e\phi. \tag{13o}$$

The relativistically extended Pauli Hamiltonian of Eq. (13o) clearly bears *a very close resemblance* to the *relativistic Lorentz Hamiltonian*, which describes a *spinless* relativistic charged particle interacting with an electromagnetic field. That notwithstanding, the *relativistically extended* Pauli Hamiltonian of Eq. (13o) *also* very clearly *incorporates* the interaction of a *moving* particle's *spin* with *an electric field*, a phenomenon *that is utterly and completely foreign to the the nonrelativistic Pauli Hamiltonian* of Eq. (1), which Eq. (13o) *exactly relativistically extends in those special cases where* $\mathbf{B} = \mathbf{0}$. The *purely relativistic* interaction of a *moving* particle's *spin* with *an electric field* is, of course *the essence* of the *hydrogen atom's spin-orbit interaction*. Thus the Eq. (13o) $\mathbf{B} = \mathbf{0}$ special case of the relativistically extended Pauli Hamiltonian is obviously useful for the hydrogen atom.

The very close resemblance to the physically irreproachable Lorentz Hamiltonian which the Eq. (13o) $\mathbf{B} = \mathbf{0}$ special case of the relativistically extended Pauli Hamiltonian manifests shows that the latter has *none* of the pathologies which are so typical of the Dirac Hamiltonian.

The most *compelling* feature of the relativistic extension of the nonrelativistic Pauli Hamiltonian that has been carried out in this article is that *it follows uniquely* from its nonrelativistic counterpart *because it is obtained entirely from the required instantaneous Lorentz boosting of that nonrelativistic physical behavior*.

# References


[1] J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964), Eq. (1.34), p. 12.

[2] J. D. Bjorken and S. D. Drell, op. cit., Eq. (1.13), p. 6.

[3] P. A. M. Dirac, Proc. Roy. Soc. (London) **A117**, 610 (1928); Proc. Roy. Soc. (London) **A118**, 351 (1928).

[4] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972), Eq. (2.9.6), p. 47.

[5] L. I. Schiff Quantum Mechanics (McGraw-Hill, New York, 1955), Eqs. (43.26), (43.27). pp. 329–330.

[6] J. D. Bjorken and S. D. Drell, op. cit., Eqs. (1.29)–(1.31), p. 12.

[7] J. D. Bjorken and S. D. Drell, op. cit., p. 13.

[8] J. D. Bjorken and S. D. Drell, op. cit., Chapter 2, pp. 16–24.