On Rational vs. Adelic Homotopy Theory
(following some ideas of David Edwards)

Robert Varley

Informal Notes of lectures given in the University of Georgia Algebraic Topology Seminar in December, 2010

Starting with homotopy type over the rationals, I am going to emphasize using a DGA over Q (for instance the Sullivan-Thom-Whitney deRham complex, or a minimal model) and simply considering its tensorization with Q_p (for every finite prime p) and R.

With the "limited, practical definition" that we're giving for a homotopy type (simply connected and with finite dimensional cohomology algebra, say) over a field of characteristic 0, the following 2 points seem to hold:

a) There exists a rational homotopy type X such that a rational homotopy type Y is equal to X if and only if it is equal p-adically (for every prime p) and also over the reals.

b) There can be X and Y with distinct cohomology algebras over Q (hence distinct RHT's) but having the same adelic homotopy type in the sense of having the same HT over every Q_p (p-adic numbers) and over R (real numbers).

As to (a), one can take a formal space whose rational cohomology is completely determined by a single (scalar-valued) quadratic form over Q; then the local-to-global principle kicks in for quadratic forms over Q (without any coherence required as p ranges through the primes and infinity).

Assume for now that the cohomology algebra is actually finite dimensional (e.g. for the RHT of a finite CW complex).

It seems that there is going to be a finiteness result for each set of rational homotopy types which agree over B (= the "big adeles", as we defined them). In other words, the meaning is that given one RHT, all the RHT's for which there are p-adic isomorphisms for all p and a real isomorphism - in the sense that we defined - with the given one, must form a finite set (of equivalence classes in the rational homotopy category). I have not checked all the details yet, but I will record the main ingredients here.

First, for each finite dimensional 1-connected graded algebra H* over the rationals, consider DGA's R together with an isomorphism i from H* to the cohomology of R. Then the work of Halperin, Schlessinger, and Stasheff shows that there is a conical affine algebraic variety V defined over Q, a unipotent affine algebraic group U defined over Q, and an action of U on V defined over Q such that V(Q)/U(Q) is the set of RHT's. And by the same token, the set V(B)/U(B) will be the set of homotopy types (in our sense) over B. In fact the map V(Q)/U(Q) -> V(B)/U(B) is one-to-one by a general finiteness result of Borel, applied to this case of the (connected) unipotent affine algebraic group U. The finiteness result about rational to adelic equivalence is related to Galois cohomology of affine algebraic groups over Q, and becomes particularly strong in the unipotent case.

Second, to get rid of the isomorphism i, consider the action of the automorphism group G of H*, another affine algebraic group defined over Q. Now the same finiteness result gives that only finitely many G(Q)-orbits can go to a G(B)-orbit. (Technically, since the quotient V/U might not exist as an algebraic variety we should probably act by a single affine algebraic group combining the successive actions of U and G into one action on V, (or else stratify V and V/U - with finitely many strata - and apply Borel's finiteness result stratum-wise).)

Notice that we already discussed examples of non-singleton fibers for RHT's (= HT's over Q) -> HT's over B (in the sense we defined), coming from distinct graded algebras over Q which became isomorphic over the p-adic numbers for every prime p and over the reals. E.g. when there was a 3-variable cubic form involved in the multiplication $H^{\wedge 2} \times H^{\wedge 2} \times H^{\wedge 2} \to H^{\wedge 6}$ on the diagonal copy of a 3-dimensional $H^{\wedge 2}$. I started to realize that the openness of the Tate-Shafarevich finiteness conjecture was not a problem since the groups involved are affine, not abelian varieties. Also it was obvious that V(Q) -> V(B) was one-to-one, and so (assuming the Halperin-Schlessinger-Stasheff formulation above is exactly right) it basically became the following question: if a linear algebraic group L acts on affine n-space, then are the fibers of $Q^n / L(Q) \to B^n / L(B)$ finite. And this is essentially the result of Borel, with one-to-one in the (connected) unipotent case as a striking special case. In fact, this kind of rational -> adelic finiteness result is almost the same as what Sullivan was invoking (for integral -> rational and real invariants). There is a fantastic 1993 Bulletin article by B. Mazur on the local-to-global principle that was very helpful in sorting the stuff out (still on a preliminary basis, but so far everything seems to fit consistently together).

Note the following example: let GL1 act on the affine line by (t,a) -> (t^{2}) times a, (for t in GL1 and a in the affine line). Then if we just look at the real component of the (big) adeles B, and nonzero points on the affine line, then we have {nonzero rational numbers}/{squares of nonzero rationals} -> {+/- 1} with infinite fibres; this is an analogue of Brown-Szczarba, just using the reals. But in this case (even though GL1 is not unipotent), the map {nonzero rational numbers}/{squares of nonzero rationals} -> {invertible elements of B}/{squares of invertible elements of B} is one-to-one, as a special case of the Hasse principle for quadratic forms, but nontrivial even for 1-variable quadratic forms.

The working statement is that for 1-connected RHT's with finite dimensional rational cohomology, there are only finitely many (over Q) that become isomorphic over the (big) adeles, where isomorphism of HT's over a field means equivalence under the relation of quasi-isomorphism of the DGA models (and the big adeles is a product of fields). There are basic counterexamples if the hypothesis is relaxed to (1) the rational cohomology is of "finite type" (meaning finite dimensional in each degree), or to (2) the rational cohomology vanishes in large degree. The counterexamples are built immediately from the single Selmer example of the 2 rational cubic forms in 3 variables that fail the local-to-global equivalence principle, using enough room (allowed by infinite dimensionality of rational cohomology) to spread out the problem degree wise and have (2) hold or within low degree and have (1) hold.

One still gets the finiteness result after removing any finite number of completions of Q. Taking care of that case uses the Borel-Serre generalization of Borel's earlier finiteness result about Q -> adeles. (For the "unipotent part of the problem", any single field Q_p or R is enough; but for the other part of the problem - related to isomorphisms of graded algebras - for a finiteness result, it's OK to remove any finite # of the completions. As I was saying, if you do remove more than 1 completion, this finite number of Q-things is almost always going to be more than just a singleton, even if the cohomology algebra is just based on a single quadratic form over Q.)

But given that Bousfield & Gugenheim provide (i) infrastructure towards the general relationship between simplicial sets and

DGA's over Q, (ii) prove their "equivalence theorem" for minimal algebras of finite Q-type, and (iii) sometimes work over an arbitrary field of characteristic 0; it would seem that everything's ready for addressing the question of topologizing the vector spaces in the appropriate cochain complexes. However, note that one little cloud appears right now. It can't be the case that the natural categories of DGA's over every field of characteristic 0 are all isomorphic to some single fixed category of simplicial sets, as nice as that category might be. So, maybe question (1) would be to identify good ways to place topological conditions on appropriate categories of chain and cochain complexes over a single topological field, consider the functoriality properties under change of field (e.g. from Q to a completion), and then see what happens to the RHT's under passage to the big adeles. In other words, we get question (2) why did nothing change in Bousfield-Kan in going from a given "solid ring" of theirs, such as Q, to a larger ring such as a single completion of Q or the whole adeles? In other words, why can the field change so much in a completion process and yet some co-simplicial structure over the the field not really change? Are there other formulations in which the structure can change?

From the moduli point of view, whenever there are actual moduli for the isomorphism classes (i.e. parameters varying in the field F, and not just jumping around among finitely many points in a non-Hausdorff quotient space), the map HT(Q) -> HT(F) can't be onto, even for a single field F (as in one example of Brown-Szczarba with F = the reals). In other words, the Platonic model (which is very nearly the precise situation) is that there is an algebraic variety V defined over Q and an equivalence relation E on V also defined over Q, so that HT(F) = V(F)/E(F), so as F gets big you're typically getting more and more points, e.g. if F is uncountable such as Q_p or R. When F is algebraically closed, you expect to have V(F)/E(F) = (V/E)(F), to the extent that a quotient variety V/E exists. As the simplest example, if W = V/E is (more or less) a 1-dimensional variety defined over Q, then as you enlarge the field F beyond Q you will typically get more points of W over F. (What is cute is that as long as your field is countable it might be hard to quantify "more points", but once you get to an uncountable set of points of W over some field F, the distinction with the Q-points is clear.) Thus, the product B over all the completions F of Q can allow an astronomical difference between the rational homotopy types with fixed cohomology dimensions over Q and the adelic homotopy types with the corresponding fixed cohomology dimensions over B.