EXACT TETRAHEDRON ARGUMENT FOR THE EXISTENCE OF STRESS TENSOR AND GENERAL EQUATION OF MOTION

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Abstract. The birth of modern continuum mechanics was the Cauchy's idea for traction vectors and his achievements of the existence of stress tensor and derivation of the general equation of motion. He gave a proof for the existence of stress tensor that is called Cauchy tetrahedron argument. But there are some challenges on the different versions of tetrahedron argument and the proofs for the existence of stress tensor. We give a new proof for the existence of stress tensor and derivation of the general equation of motion. The exact tetrahedron argument for the first time gives us a clear and deep insight into the origins and the nature of these fundamental concepts and equations in continuum mechanics. This new approach leads to the exact point-base definition and derivation of these fundamental parameters and relations in continuum mechanics. By the exact tetrahedron argument we derived the relation for the existence of stress tensor and the general equation of motion, simultaneously. In this new proof, there is no approximating or limiting process and all of the effective parameters are exact values not average values. Also, we show that in this proof, all the challenges on the previous tetrahedron arguments and the proofs for the existence of stress tensor are removed.

1. Introduction

The existence of stress tensor and the general equation of motion form the main part of the foundation of continuum mechanics. During 1822 to 1828, Cauchy for the first time, introduced the basic idea of *traction vector* and presented a proof for the existence of *stress tensor* that is called *Cauchy tetrahedron argument* and by another process obtained the general equation of motion that is called *Cauchy equation of motion*. He also gave some important properties for the state of stress, e.g. the *symmetry of stress tensor* [3], [4], [7], [8]. The basic idea of Cauchy was that the internal forces on the surface in continuum media in addition to the normal component can have the tangential components. From Truesdell in (1968, [8]), on pages 336 and 338:

> *Thus it might seem that CAUCHY's achievement in formulating and developing the general theory of stress was an easy one. It was not. CAUCHY's concept has the simplicity of genius. Its deep and thorough originality is fully outlined only against the background of the century of achievement by the brilliant geometers who preceded, treating special*

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kinds and cases of deformable bodies by complicated and sometimes incorrect ways without ever hitting upon this basic idea, which immediately became and has remained the foundation of the mechanics of gross bodies.

We already gave a comprehensive review of the different versions of tetrahedron argument and the proofs for the existence of stress tensor presented in the published books during nearly two centuries from the birth to this time, and considered the important challenges and the improvements of each one (2017, [1]).

In this article for the first time, we give the exact tetrahedron argument that removes all the stated challenges and opens a new and deep vision into the foundation of continuum mechanics and the nature of the traction vector, the stress tensor, and the general equation of motion.

For presenting the exact tetrahedron argument we first give the general forms of the conservation of linear momentum for a mass element and prove the important relation that is called *Cauchy lemma* for the traction vectors that act on the opposite sides of the same surface. Then, the exact tetrahedron argument will be presented. We also, discuss some aspects of this new proof and consider the challenges that hold for the previous tetrahedron arguments and the proofs for the existence of stress tensor, on this new proof.

The integral equation of conservation of linear momentum on a mass element in continuum media is:

$$\frac{d}{dt}\int_{\mathcal{M}} \rho\boldsymbol{v}\, dV = \int_{\partial\mathcal{M}} \boldsymbol{t}\, dS + \int_{\mathcal{M}} \rho\boldsymbol{b}\, dV \tag{1.1}$$

where $\rho = \rho(\boldsymbol{r}, t)$ is the density, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t)$ is the velocity vector, and $\rho\boldsymbol{v}$ is the linear momentum per unit volume of the mass element $\mathcal{M}$. On the right hand side, $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n})$ is the surface force per unit area that is called traction vector and acts on the surface of the mass element i.e. $\partial\mathcal{M}$, and $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{r}, t)$ is the body force per unit mass. Here $\boldsymbol{r}$ is the position vector, t is time, and $\boldsymbol{n}$ is the outward unit normal vector on the surface of mass element. By using the transport theorem and the conservation of mass [9], [5], the left hand side of the equation converts to:

$$\frac{d}{dt}\int_{\mathcal{M}} \rho\boldsymbol{v}\, dV = \int_{\mathcal{M}} \Big\{\frac{\partial(\rho\boldsymbol{v})}{\partial t} + \nabla.(\rho\boldsymbol{v})\Big\}\, dV = \int_{\mathcal{M}} \Big\{\rho\frac{\partial\boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}.\nabla)\boldsymbol{v}\Big\}\, dV = \int_{\mathcal{M}} \rho\boldsymbol{a}\, dV \tag{1.2}$$

where $\boldsymbol{a} = \partial\boldsymbol{v}/\partial t + (\boldsymbol{v}.\nabla)\boldsymbol{v}$ is the acceleration vector. By rearranging the equation (1.1):

$$\int_{\mathcal{M}} (\rho\boldsymbol{a} - \rho\boldsymbol{b})\, dV = \int_{\partial\mathcal{M}} \boldsymbol{t}\, dS \tag{1.3}$$

for simplicity, we use $\boldsymbol{B} = (\rho\boldsymbol{a} - \rho\boldsymbol{b})$ that is called *body term* within the proof. So, the equation (1.3) rewrites as:

$$\int_{\mathcal{M}} \boldsymbol{B}\, dV = \int_{\partial\mathcal{M}} \boldsymbol{t}\, dS \tag{1.4}$$

In general, $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{r}, t)$ and $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n})$ are continuous functions in their scope in continuum media.

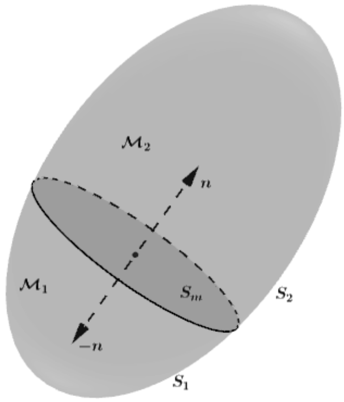

FIGURE 1. The mass elements $\mathcal{M}_1$ and $\mathcal{M}_2$, where $\partial\mathcal{M}_1 = S_1 \cup S_m$ and $\partial\mathcal{M}_2 = S_2 \cup S_m$, and the mass element $\mathcal{M}$ such that $V_\mathcal{M} = V_{\mathcal{M}_1} \cup V_{\mathcal{M}_2}$ and $\partial\mathcal{M} = S_1 \cup S_2$.

2. Cauchy Lemma

Cauchy lemma discusses about the traction vectors that act on the opposite sides of the same surface at a given point and time. There are some approaches to prove this lemma in the literature. Here we present a proof for the Cauchy lemma that is nearly similar to the proofs in [6] and [2]. Suppose the mass element $\mathcal{M}$ splits into $\mathcal{M}_1$ and $\mathcal{M}_2$ by the surface S_m in the way that $V_\mathcal{M} = V_{\mathcal{M}_1} \cup V_{\mathcal{M}_2}$, $\partial\mathcal{M}_1 = S_1 \cup S_m$, $\partial\mathcal{M}_2 = S_2 \cup S_m$, and $\partial\mathcal{M} = S_1 \cup S_2$, see Figure 1. If the equation (1.4) applies to $\mathcal{M}_1$ and $\mathcal{M}_2$, then the sum of these equations is:

$$\int_{\mathcal{M}_1} \boldsymbol{B}_1\, dV + \int_{\mathcal{M}_2} \boldsymbol{B}_2\, dV = \int_{\partial\mathcal{M}_1} \boldsymbol{t}_1\, dS + \int_{\partial\mathcal{M}_2} \boldsymbol{t}_2\, dS$$

By $V_\mathcal{M} = V_{\mathcal{M}_1} \cup V_{\mathcal{M}_2}$, the sum of the body term integrals is equal to the integral of the body term on $\mathcal{M}$. In addition, by $\partial\mathcal{M}_1 = S_1 \cup S_m$ and $\partial\mathcal{M}_2 = S_2 \cup S_m$, the surface integrals split as:

$$\int_{\mathcal{M}} \boldsymbol{B}\, dV = \int_{S_1} \boldsymbol{t}_1\, dS + \int_{S_m} \boldsymbol{t}_1\, dS + \int_{S_2} \boldsymbol{t}_2\, dS + \int_{S_m} \boldsymbol{t}_2\, dS$$

By $\partial\mathcal{M} = S_1 \cup S_2$, the sum of surface integrals on S_1 and S_2 is equal to the surface integral of $\boldsymbol{t}$ on $\partial\mathcal{M}$, so:

$$\int_{\mathcal{M}} \boldsymbol{B}\, dV = \int_{\partial\mathcal{M}} \boldsymbol{t}\, dS + \int_{S_m} \boldsymbol{t}_1\, dS + \int_{S_m} \boldsymbol{t}_2\, dS$$

Comparing this integral equation with the integral equation (1.4), implies that:

$$\int_{S_m} \boldsymbol{t}_1\, dS + \int_{S_m} \boldsymbol{t}_2\, dS = \boldsymbol{0}$$

But $\boldsymbol{t}_1$ on S_m is $\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n})$, and $\boldsymbol{t}_2$ on S_m is $\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n})$, so:

$$\int_{S_m} \big\{\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) + \boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n})\big\}\, dS = \boldsymbol{0}$$

therefore, we have

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n}) \tag{2.1}$$

This is the Cauchy lemma that is derived by using the integral equation of conservation of linear momentum (1.4). It states *"the traction vectors acting on opposite sides of the same surface at a given point and time are equal in magnitude but opposite in direction"*.

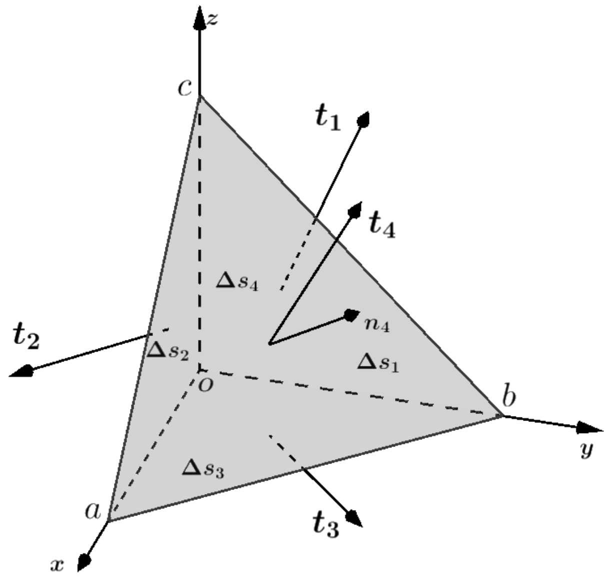

FIGURE 2. Tetrahedron geometry and the exact traction vectors on the faces.

3. Exact Tetrahedron Argument

There is a belief today that the foundation of mechanics is a dead subject, but this is not correct.

Here for the first time, we present the exact proof of tetrahedron argument. Imagine a tetrahedron element in the continuum media that its vortex is at point $\boldsymbol{o}$ and its three orthogonal faces are parallel to the three orthogonal planes of the Cartesian coordinate system. The fourth surface of the tetrahedron, i.e. its base, has the outward unit normal vector $\boldsymbol{n}_4$. For simplicity, the vortex point is at the origin of the coordinate system. The geometry parameters are shown in Figure 2. The vector $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$ is the position vector from the origin of the coordinate system. Now the integral equation of conservation of linear momentum (1.4) applies to this tetrahedron mass element:

$$\int_{\Delta s_4} \boldsymbol{t}_4 \, dS + \int_{\Delta s_1} \boldsymbol{t}_1 \, dS + \int_{\Delta s_2} \boldsymbol{t}_2 \, dS + \int_{\Delta s_3} \boldsymbol{t}_3 \, dS = \int_{\mathcal{M}} \boldsymbol{B} \, dV \tag{3.1}$$

The key idea of this proof is to write the variables of this equation in terms of the exact Taylor series about a point in the domain. Here, we derive these series about the vortex point of tetrahedron (point $\boldsymbol{o}$), where the three orthogonal planes pass through it. Note that time (t) is the same in all terms, so it does not exist in the Taylor series. For $\boldsymbol{B}(\boldsymbol{r}, t)$ at any point in the domain of the mass element, we have:

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}_o + \frac{\partial \boldsymbol{B}_o}{\partial x}x + \frac{\partial \boldsymbol{B}_o}{\partial y}y + \frac{\partial \boldsymbol{B}_o}{\partial z}z \\ &+ \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{B}_o}{\partial x^2}x^2 + \frac{\partial^2 \boldsymbol{B}_o}{\partial y^2}y^2 + \frac{\partial^2 \boldsymbol{B}_o}{\partial z^2}z^2 + 2\frac{\partial^2 \boldsymbol{B}_o}{\partial x \partial y}xy + 2\frac{\partial^2 \boldsymbol{B}_o}{\partial x \partial z}xz + 2\frac{\partial^2 \boldsymbol{B}_o}{\partial y \partial z}yz\Big) \\ &+ \ldots = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!n!k!} \frac{\partial^{(m+n+k)} \boldsymbol{B}}{\partial x^m \partial y^n \partial z^k}\Big|_o x^m y^n z^k \end{aligned} \tag{3.2}$$

Here $\boldsymbol{B}_o$ and $\partial \boldsymbol{B}_o/\partial x$ are the exact values of $\boldsymbol{B}$ and $\partial \boldsymbol{B}/\partial x$ at point $\boldsymbol{o}$, respectively. Similarly, the other derivatives are the exact values of the related derivatives of $\boldsymbol{B}$ at

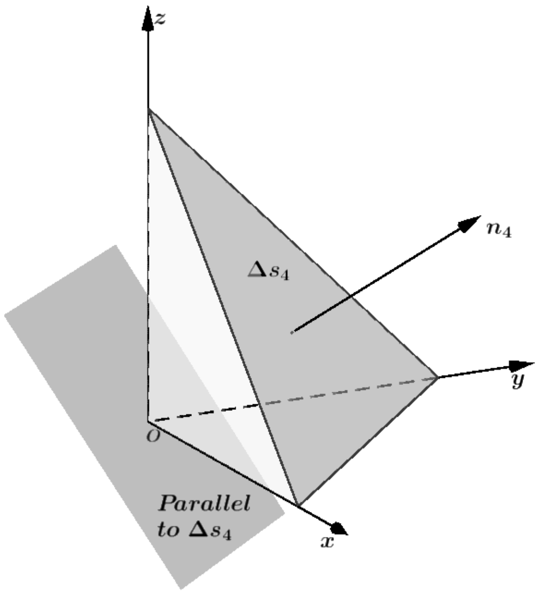

FIGURE 3. Inclined plane that is parallel to Δs_4 and passes through point $\boldsymbol{o}$.

point $\boldsymbol{o}$. On the surface Δs_1, $x = 0$ and $\boldsymbol{n}_1$ does not change, so:

$$\begin{aligned} \boldsymbol{t}_1 &= \boldsymbol{t}_{1_o} + \frac{\partial \boldsymbol{t}_{1_o}}{\partial y} y + \frac{\partial \boldsymbol{t}_{1_o}}{\partial z} z + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2} y^2 + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2} z^2 + 2\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y \partial z} yz\Big) \\ &+ \ldots = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!k!} \frac{\partial^{(m+k)} \boldsymbol{t}_1}{\partial y^m \partial z^k}\Big|_o y^m z^k \end{aligned} \tag{3.3}$$

where $\boldsymbol{t}_{1_o}$ is the exact value of the traction vector $\boldsymbol{t}_1$ on Δs_1 at point $\boldsymbol{o}$. On the surface Δs_2, $y = 0$ and $\boldsymbol{n}_2$ does not change, and on the surface Δs_3, $z = 0$ and $\boldsymbol{n}_3$ does not change, so:

$$\begin{aligned} \boldsymbol{t}_2 &= \boldsymbol{t}_{2_o} + \frac{\partial \boldsymbol{t}_{2_o}}{\partial x} x + \frac{\partial \boldsymbol{t}_{2_o}}{\partial z} z + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2} x^2 + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2} z^2 + 2\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x \partial z} xz\Big) \\ &+ \ldots = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!k!} \frac{\partial^{(m+k)} \boldsymbol{t}_2}{\partial x^m \partial z^k}\Big|_o x^m z^k \end{aligned} \tag{3.4}$$

$$\begin{aligned} \boldsymbol{t}_3 &= \boldsymbol{t}_{3_o} + \frac{\partial \boldsymbol{t}_{3_o}}{\partial x} x + \frac{\partial \boldsymbol{t}_{3_o}}{\partial y} y + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2} x^2 + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2} y^2 + 2\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x \partial y} xy\Big) \\ &+ \ldots = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!k!} \frac{\partial^{(m+k)} \boldsymbol{t}_3}{\partial x^m \partial y^k}\Big|_o x^m y^k \end{aligned} \tag{3.5}$$

Similarly, $\boldsymbol{t}_{2_o}$ and $\boldsymbol{t}_{3_o}$ are the exact values of $\boldsymbol{t}_2$ and $\boldsymbol{t}_3$ at point $\boldsymbol{o}$ on Δs_2 and Δs_3, respectively. For the traction vector on surface Δs_4 a more explanation is needed. The traction vector on Δs_4 expands based on the traction vector on the plane that is parallel to Δs_4 and passes through the vortex point of tetrahedron (point $\boldsymbol{o}$). Because the unit normal vectors of these two planes are the same, see Figure 3. Therefore:

$$
\begin{aligned}
\boldsymbol{t}_4 &= \boldsymbol{t}_{4_o} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial x}x + \frac{\partial \boldsymbol{t}_{4_o}}{\partial y}y + \frac{\partial \boldsymbol{t}_{4_o}}{\partial z}z \\
&+ \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}x^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}y^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}z^2 + 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial y}xy + 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial z}xz + 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y \partial z}yz\Big) \\
&+ \ldots = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{m!n!k!}\frac{\partial^{(m+n+k)}\boldsymbol{t}_4}{\partial x^m \partial y^n \partial z^k}\Big|_o x^m y^n z^k
\end{aligned}
\tag{3.6}
$$

Here $\boldsymbol{t}_{4_o}$ is the exact traction vector at point $\boldsymbol{o}$ on the plane with unit normal vector $\boldsymbol{n}_4$, that this plane passes exactly through point $\boldsymbol{o}$, the vertex point of tetrahedron element. x, y, and z are the components of the position vector $\boldsymbol{r}$ on the surface Δs_4.

Note that $\boldsymbol{t}_{1_o}$, $\boldsymbol{t}_{2_o}$, $\boldsymbol{t}_{3_o}$, and $\boldsymbol{t}_{4_o}$ are the exact traction vectors at point $\boldsymbol{o}$ but on the different planes with unit normal vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$, and $\boldsymbol{n}_4$, respectively. The body term $\boldsymbol{B}_o$ is exactly defined at point $\boldsymbol{o}$. So, all the traction vectors and the body term vector with subscript o and their all derivatives, such as $\partial^2 \boldsymbol{t}_{4_o}/\partial x \partial y$, are defined exactly at point $\boldsymbol{o}$ and are bounded. As a result, for the convergence of the above Taylor series it is enough that we have $|\boldsymbol{r}| \leqslant 1$ in the domain of the mass element $\mathcal{M}$. But the scale of the coordinate system is arbitrary and we can define this scale such that the greatest distance in the domain of the mass element from the origin, is equal to one, i.e. $|\boldsymbol{r}|_{max} = 1$. By this scale, in the entire of the tetrahedron mass element we have $|\boldsymbol{r}| \leqslant 1$, that leads to the convergence condition for the above Taylor series.

Now all of the variables are prepared for integration in the integral equation (3.1). The integration of $\boldsymbol{B}$ on the volume of $\mathcal{M}$:

$$
\begin{aligned}
\int_{\mathcal{M}} \boldsymbol{B}\, dV &= \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} \Big\{\boldsymbol{B}_o + \frac{\partial \boldsymbol{B}_o}{\partial x}x + \frac{\partial \boldsymbol{B}_o}{\partial y}y + \frac{\partial \boldsymbol{B}_o}{\partial z}z + \ldots\Big\}\, dx\, dy\, dz \\
&= \frac{1}{6}abc\Big\{\boldsymbol{B}_o + \frac{1}{4}\Big(\frac{\partial \boldsymbol{B}_o}{\partial x}a + \frac{\partial \boldsymbol{B}_o}{\partial y}b + \frac{\partial \boldsymbol{B}_o}{\partial z}c\Big) + \ldots\Big\}
\end{aligned}
\tag{3.7}
$$

The integration of $\boldsymbol{t}_4$ on Δs_4:

$$
\begin{aligned}
\int_{\Delta s_4} \boldsymbol{t}_4\, dS &= \int_0^b \int_0^{a(1-\frac{y}{b})} \Big\{\sqrt{\big(-\frac{c}{a}\big)^2 + \big(-\frac{c}{b}\big)^2 + 1}\Big(\boldsymbol{t}_{4_o} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial x}x + \frac{\partial \boldsymbol{t}_{4_o}}{\partial y}y \\
&+ \frac{\partial \boldsymbol{t}_{4_o}}{\partial z}\big(c(1-\frac{x}{a}-\frac{y}{b})\big) + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}x^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}y^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}\big(c(1-\frac{x}{a}-\frac{y}{b})\big)^2 \\
&+ 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial y}xy + 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial z}x\big(c(1-\frac{x}{a}-\frac{y}{b})\big) + 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y \partial z}y\big(c(1-\frac{x}{a}-\frac{y}{b})\big)\Big) \\
&+ \ldots\Big)\Big\}\, dx\, dy \\
&= \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}\Big\{\boldsymbol{t}_{4_o} + \frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{4_o}}{\partial x}a + \frac{\partial \boldsymbol{t}_{4_o}}{\partial y}b + \frac{\partial \boldsymbol{t}_{4_o}}{\partial z}c\Big) \\
&+ \frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}a^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}b^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}c^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial y}ab + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial z}ac + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y \partial z}bc\Big) + \ldots\Big\}
\end{aligned}
\tag{3.8}
$$

The integration of $\boldsymbol{t}_1$ on Δs_1:

$$\begin{aligned}\int_{\Delta s_1} \boldsymbol{t}_1\, dS &= \int_0^c \int_0^{b(1-\frac{z}{c})} \Big\{ \boldsymbol{t}_{1_o} + \frac{\partial \boldsymbol{t}_{1_o}}{\partial y} y + \frac{\partial \boldsymbol{t}_{1_o}}{\partial z} z \\ &\quad + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2} y^2 + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2} z^2 + 2\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y \partial z} yz \Big) + \dots \Big\}\, dy\, dz \\ &= \frac{1}{2} bc \Big\{ \boldsymbol{t}_{1_o} + \frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{1_o}}{\partial y} b + \frac{\partial \boldsymbol{t}_{1_o}}{\partial z} c \Big) + \frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2} b^2 + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2} c^2 + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y \partial z} bc \Big) + \dots \Big\}\end{aligned} \tag{3.9}$$

The integration of $\boldsymbol{t}_2$ on Δs_2:

$$\begin{aligned}\int_{\Delta s_2} \boldsymbol{t}_2\, dS &= \int_0^c \int_0^{a(1-\frac{z}{c})} \Big\{ \boldsymbol{t}_{2_o} + \frac{\partial \boldsymbol{t}_{2_o}}{\partial x} x + \frac{\partial \boldsymbol{t}_{2_o}}{\partial z} z \\ &\quad + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2} x^2 + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2} z^2 + 2\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x \partial z} xz \Big) + \dots \Big\}\, dx\, dz \\ &= \frac{1}{2} ac \Big\{ \boldsymbol{t}_{2_o} + \frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{2_o}}{\partial x} a + \frac{\partial \boldsymbol{t}_{2_o}}{\partial z} c \Big) + \frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2} a^2 + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2} c^2 + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x \partial z} ac \Big) + \dots \Big\}\end{aligned} \tag{3.10}$$

The integration of $\boldsymbol{t}_3$ on Δs_3:

$$\begin{aligned}\int_{\Delta s_3} \boldsymbol{t}_3\, dS &= \int_0^b \int_0^{a(1-\frac{y}{b})} \Big\{ \boldsymbol{t}_{3_o} + \frac{\partial \boldsymbol{t}_{3_o}}{\partial x} x + \frac{\partial \boldsymbol{t}_{3_o}}{\partial y} y \\ &\quad + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2} x^2 + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2} y^2 + 2\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x \partial y} xy \Big) + \dots \Big\}\, dx\, dy \\ &= \frac{1}{2} ab \Big\{ \boldsymbol{t}_{3_o} + \frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{3_o}}{\partial x} a + \frac{\partial \boldsymbol{t}_{3_o}}{\partial y} b \Big) + \frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2} a^2 + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2} b^2 + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x \partial y} ab \Big) + \dots \Big\}\end{aligned} \tag{3.11}$$

The geometrical relations for area of the faces and the volume of the tetrahedron are:

$$\begin{aligned}&\Delta s_1 = \frac{1}{2} bc, \qquad \Delta s_2 = \frac{1}{2} ac, \qquad \Delta s_3 = \frac{1}{2} ab \\ &\Delta s_4 = \frac{1}{2}\sqrt{a^2 b^2 + a^2 c^2 + b^2 c^2}, \qquad \Delta V = \frac{1}{6} abc\end{aligned} \tag{3.12}$$

By substituting the obtained relations for the integrals of the traction vectors and the body term into the equation (3.1) and using the above geometrical relations:

$$\begin{aligned}
&\Delta s_4\Big\{\boldsymbol{t}_{4_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{4_o}}{\partial x}a+\frac{\partial \boldsymbol{t}_{4_o}}{\partial y}b+\frac{\partial \boldsymbol{t}_{4_o}}{\partial z}c\Big)\\
&\quad+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}a^2+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}b^2+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}c^2+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial y}ab+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial z}ac+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y\partial z}bc\Big)+\dots\Big\}\\
&\quad+\Delta s_1\Big\{\boldsymbol{t}_{1_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{1_o}}{\partial y}b+\frac{\partial \boldsymbol{t}_{1_o}}{\partial z}c\Big)+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2}b^2+\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2}c^2+\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y\partial z}bc\Big)+\dots\Big\}\\
&\quad+\Delta s_2\Big\{\boldsymbol{t}_{2_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{2_o}}{\partial x}a+\frac{\partial \boldsymbol{t}_{2_o}}{\partial z}c\Big)+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2}a^2+\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2}c^2+\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x\partial z}ac\Big)+\dots\Big\}\\
&\quad+\Delta s_3\Big\{\boldsymbol{t}_{3_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{3_o}}{\partial x}a+\frac{\partial \boldsymbol{t}_{3_o}}{\partial y}b\Big)+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2}a^2+\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2}b^2+\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x\partial y}ab\Big)+\dots\Big\}\\
&\quad-\Delta V\Big\{\boldsymbol{B}_o+\frac{1}{4}\Big(\frac{\partial \boldsymbol{B}_o}{\partial x}a+\frac{\partial \boldsymbol{B}_o}{\partial y}b+\frac{\partial \boldsymbol{B}_o}{\partial z}c\Big)+\dots\Big\}=\boldsymbol{0}
\end{aligned}\tag{3.13}$$

In the geometry of tetrahedron, h is the height of the vertex $\boldsymbol{o}$ from the base face, i.e. Δs_4. So, we have the following geometrical relations for a tetrahedron with $\boldsymbol{n}_4 = n_x\boldsymbol{e}_x + n_y\boldsymbol{e}_y + n_z\boldsymbol{e}_z$, where a, b, and c are greater than zero, see Figure 2.

$$\begin{aligned}
&h=n_x a,\qquad h=n_y b,\qquad h=n_z c\\
&\frac{1}{h^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2},\qquad \Delta s_4=\frac{abc}{2h}\\
&\Delta s_1=n_x\Delta s_4,\qquad \Delta s_2=n_y\Delta s_4,\qquad \Delta s_3=n_z\Delta s_4\\
&\Delta V=\frac{1}{6}abc=\frac{1}{3}h\Delta s_4
\end{aligned}\tag{3.14}$$

If we first divide the equation (3.13) by Δs_4 and use the relations in (3.14) for the areas and volume of the tetrahedron, then substitute the relations $a = h/n_x$, $b = h/n_y$, and $c = h/n_z$ into the equation:

$$\begin{aligned}
&\Big\{\boldsymbol{t}_{4_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{4_o}}{\partial x}\frac{1}{n_x}+\frac{\partial \boldsymbol{t}_{4_o}}{\partial y}\frac{1}{n_y}+\frac{\partial \boldsymbol{t}_{4_o}}{\partial z}\frac{1}{n_z}\Big)h\\
&+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}\frac{1}{n_x^2}+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}\frac{1}{n_y^2}+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}\frac{1}{n_z^2}+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial y}\frac{1}{n_xn_y}+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial z}\frac{1}{n_xn_z}+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y\partial z}\frac{1}{n_yn_z}\Big)h^2+\dots\Big\}\\
&+n_x\Big\{\boldsymbol{t}_{1_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{1_o}}{\partial y}\frac{1}{n_y}+\frac{\partial \boldsymbol{t}_{1_o}}{\partial z}\frac{1}{n_z}\Big)h+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2}\frac{1}{n_y^2}+\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2}\frac{1}{n_z^2}+\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y\partial z}\frac{1}{n_yn_z}\Big)h^2+\dots\Big\}\\
&+n_y\Big\{\boldsymbol{t}_{2_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{2_o}}{\partial x}\frac{1}{n_x}+\frac{\partial \boldsymbol{t}_{2_o}}{\partial z}\frac{1}{n_z}\Big)h+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2}\frac{1}{n_x^2}+\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2}\frac{1}{n_z^2}+\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x\partial z}\frac{1}{n_xn_z}\Big)h^2+\dots\Big\}\\
&+n_z\Big\{\boldsymbol{t}_{3_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{3_o}}{\partial x}\frac{1}{n_x}+\frac{\partial \boldsymbol{t}_{3_o}}{\partial y}\frac{1}{n_y}\Big)h+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2}\frac{1}{n_x^2}+\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2}\frac{1}{n_y^2}+\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x\partial y}\frac{1}{n_xn_y}\Big)h^2+\dots\Big\}\\
&-\frac{1}{3}h\Big\{\boldsymbol{B}_o+\frac{1}{4}\Big(\frac{\partial \boldsymbol{B}_o}{\partial x}\frac{1}{n_x}+\frac{\partial \boldsymbol{B}_o}{\partial y}\frac{1}{n_y}+\frac{\partial \boldsymbol{B}_o}{\partial z}\frac{1}{n_z}\Big)h+\dots\Big\}=\boldsymbol{0}
\end{aligned}\tag{3.15}$$

Now by rearranging the equation based on the powers of h, we have:

$$
\begin{aligned}
&\Big\{\boldsymbol{t}_{4_o} + n_x\boldsymbol{t}_{1_o} + n_y\boldsymbol{t}_{2_o} + n_z\boldsymbol{t}_{3_o}\Big\} \\
&\quad + \Big\{\Big(\frac{\partial \boldsymbol{t}_{4_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial y}\frac{1}{n_y} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial z}\frac{1}{n_z}\Big) + n_x\Big(\frac{\partial \boldsymbol{t}_{1_o}}{\partial y}\frac{1}{n_y} + \frac{\partial \boldsymbol{t}_{1_o}}{\partial z}\frac{1}{n_z}\Big) \\
&\quad + n_y\Big(\frac{\partial \boldsymbol{t}_{2_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{2_o}}{\partial z}\frac{1}{n_z}\Big) + n_z\Big(\frac{\partial \boldsymbol{t}_{3_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{3_o}}{\partial y}\frac{1}{n_y}\Big) - \boldsymbol{B}_o\Big\}\frac{1}{3}h \\
&\quad + \Big\{\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial y}\frac{1}{n_x n_y} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial z}\frac{1}{n_x n_z} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y\partial z}\frac{1}{n_y n_z}\Big) \\
&\quad + n_x\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y\partial z}\frac{1}{n_y n_z}\Big) + n_y\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x\partial z}\frac{1}{n_x n_z}\Big) \\
&\quad + n_z\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x\partial y}\frac{1}{n_x n_y}\Big) - \Big(\frac{\partial \boldsymbol{B}_o}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{B}_o}{\partial y}\frac{1}{n_y} + \frac{\partial \boldsymbol{B}_o}{\partial z}\frac{1}{n_z}\Big)\Big\}\frac{1}{12}h^2 \\
&\quad + \ldots = \boldsymbol{0}
\end{aligned}
\tag{3.16}
$$

Note that by the coordinate system here and by $\Delta V \neq 0$, no one of n_x, n_y, and n_z is zero exactly. So, all of the expressions in the braces $\{\}$ of the equation (3.16) exist. We can rename the expressions in the braces and rewrite the equation as:

$$\boldsymbol{E}_0 + \boldsymbol{E}_1\frac{1}{3}h + \boldsymbol{E}_2\frac{1}{12}h^2 + \ldots = \boldsymbol{0} \tag{3.17}$$

If we continue to integrate the higher order derivatives of Taylor series of all terms, that is a long time and complicated process and we do not present it here, we have:

$$\boldsymbol{E}_0 + \boldsymbol{E}_1\frac{1}{3}h + \boldsymbol{E}_2\frac{1}{12}h^2 + \boldsymbol{E}_3\frac{1}{60}h^3 + \ldots + \boldsymbol{E}_m\frac{2}{(m+2)!}h^m + \ldots = \boldsymbol{0} \tag{3.18}$$

or

$$\sum_{m=0}^{\infty} \boldsymbol{E}_m\frac{2}{(m+2)!}h^m = \boldsymbol{0} \tag{3.19}$$

This is a great equation in the foundation of continuum mechanics. $\boldsymbol{E}_0$, $\boldsymbol{E}_1$, and $\boldsymbol{E}_2$ are shown in the braces of the equation (3.16) and $\boldsymbol{E}_3$ and other $\boldsymbol{E}_m$'s will be presented. We now discuss some aspects of the equation (3.18):

- $\boldsymbol{E}_m$'s are formed by the expressions of traction vectors, body term and their derivatives, and the components of unit normal vector of the oriented plane.
- Each one of $\boldsymbol{E}_m$'s exists, because the surface terms, body term and their derivatives are defined as continuous functions in continuum media and by the coordinate system here and by $\Delta V \neq 0$, no one of n_x, n_y, and n_z is zero exactly.
- Each one of $\boldsymbol{E}_m$'s depends on the variables at point $\boldsymbol{o}$ and the components of unit normal vector of the oriented surface that is parallel to Δs_4 and passes through point $\boldsymbol{o}$. Because the surface terms, body term and their derivatives are defined at point $\boldsymbol{o}$.

- $\boldsymbol{E}_m$'s do not depend on the volume of tetrahedron.
- h is a geometrical variable and by the scale of the coordinate system on the tetrahedron mass element such that $|\boldsymbol{r}|_{max} \leqslant 1$, the altitude of the tetrahedron, h is not greater than one.
- Note that $h = 0$ is not valid, because the integral equation of conservation of linear momentum (1.4) is defined for the mass elements with nonzero volume.

By these properties, we return to the equation (3.18).

$$\boldsymbol{E}_0 + \boldsymbol{E}_1 \frac{1}{3} h + \boldsymbol{E}_2 \frac{1}{12} h^2 + \boldsymbol{E}_3 \frac{1}{60} h^3 + \ldots + \boldsymbol{E}_m \frac{2}{(m+2)!} h^m + \ldots = \boldsymbol{0}$$

We must find $\boldsymbol{E}_m$'s. We know $\boldsymbol{E}_m$'s are independent of h, so the only solution is that $\boldsymbol{E}_m$'s must be exactly equal to zero, i.e.:

$$\boldsymbol{E}_m = \boldsymbol{0}, \qquad m = 0, 1, 2, \ldots, \infty \tag{3.20}$$

Proof:
If we rewrite the equation (3.18) as:

$$\boldsymbol{E}_0 = -\boldsymbol{E}_1 \frac{1}{3} h - \boldsymbol{E}_2 \frac{1}{12} h^2 - \boldsymbol{E}_3 \frac{1}{60} h^3 - \ldots - \boldsymbol{E}_m \frac{2}{(m+2)!} h^m - \ldots$$

We know that $\boldsymbol{E}_m$'s are independent of h. So, the left hand side of the equation i.e. $\boldsymbol{E}_0$ is independent of h. This implies, the right hand side of the equation must be independent of h. Thus, the coefficients of the powers of h must be exactly equal to zero. So:

$$\boldsymbol{E}_m = \boldsymbol{0}, \qquad m = 1, 2, \ldots, \infty$$

As a result, the right hand side of the equation is zero. Therefore, the left hand side of the equation, i.e. $\boldsymbol{E}_0$, is equal to zero, as well. So:

$$\boldsymbol{E}_m = \boldsymbol{0}, \qquad m = 0, 1, 2, \ldots, \infty$$

and the proof is completed.

Note that this proof is valid not only for $h \to 0$, but also for all values of h in the domain. This means that the results (3.20) are valid not only for an infinitesimal tetrahedron, but also for any tetrahedron in the scaled coordinate system in continuum media. In addition, we have not done any approximation process during derivation of the equation (3.18) and within this proof. So, the results (3.20) hold exactly not approximately.

Furthermore, the subscript o in the expressions of $\boldsymbol{E}_m$'s in the equation (3.16) indicates the vortex point of the tetrahedron. But any point in the domain in continuum media can be regarded as the vertex point of a tetrahedron and we could consider that tetrahedron. So, the point $\boldsymbol{o}$ can be any point in the continuum domain. Thus, we conclude that $\boldsymbol{E}_m$'s are equal to zero at any point in continuum media. This implies that all of their derivatives are equal to zero, as well. For example, we have for $\boldsymbol{E}_0$:

$$\frac{\partial \boldsymbol{E}_0}{\partial x} = \frac{\partial \boldsymbol{E}_0}{\partial y} = \frac{\partial \boldsymbol{E}_0}{\partial z} = \boldsymbol{0} \tag{3.21}$$

and the other higher derivatives of $\boldsymbol{E}_0$ are equal to zero. This trend holds for other $\boldsymbol{E}_m$'s.

But what are $\boldsymbol{E}_m$'s? In the following we will consider them and see that they lead to the important results.

For $\boldsymbol{E}_0 = \mathbf{0}$, from the equation (3.16):

$$\boldsymbol{E}_0 = \boldsymbol{t}_{4_o} + n_x \boldsymbol{t}_{1_o} + n_y \boldsymbol{t}_{2_o} + n_z \boldsymbol{t}_{3_o} = \mathbf{0} \tag{3.22}$$

This relation is similar to the relation of Cauchy tetrahedron argument. The Cauchy relation was:

$$\boldsymbol{t}_4 + n_x \boldsymbol{t}_1 + n_y \boldsymbol{t}_2 + n_z \boldsymbol{t}_3 = \mathbf{0} \tag{3.23}$$

But there are some important conceptual differences between them:

- In the Cauchy relation (3.23), the traction vectors are not exactly defined at the point $\boldsymbol{o}$ and they are the sequence of the limit $h \to 0$ on the tetrahedron volume. But here in the equation (3.22) the traction vectors are exactly defined at point $\boldsymbol{o}$.
- In the Cauchy relation (3.23), the traction vectors are average values on the tetrahedron faces. But here in (3.22) the traction vectors are defined at point $\boldsymbol{o}$ on the surfaces that pass exactly through point $\boldsymbol{o}$.
- In the Cauchy relation (3.23), the traction vector $\boldsymbol{t}_4$ is defined on the surface Δs_4 of the tetrahedron. This surface does not pass through point $\boldsymbol{o}$ even in the limit $h \to 0$ for an infinitesimal tetrahedron. But here in (3.22), $\boldsymbol{t}_{4_o}$ is defined on the surface that passes through point $\boldsymbol{o}$ and is parallel to Δs_4, see Figure 3.

These differences are very important because by them the relation (3.22) is exactly point base but the relation (3.23) is average value base.

Let us return to the relation (3.22) for $\boldsymbol{E}_0 = \mathbf{0}$, we have:

$$\boldsymbol{t}_{4_o} + n_x \boldsymbol{t}_{1_o} + n_y \boldsymbol{t}_{2_o} + n_z \boldsymbol{t}_{3_o} = \mathbf{0}$$

The traction vector $\boldsymbol{t}_{1_o}$ is defied on the negative side of the coordinate plane yz i.e., $\boldsymbol{n}_1 = -1\boldsymbol{e}_x$ at point $\boldsymbol{o}$. If $\boldsymbol{t}_{x_o}$ is the traction vector on the positive side of the coordinate plane yz at point $\boldsymbol{o}$, then by the equation (2.1) i.e., $\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n})$ we have:

$$\boldsymbol{t}_{1_o} = -\boldsymbol{t}_{x_o} \tag{3.24}$$

Similarly for $\boldsymbol{t}_{2_o}$ and $\boldsymbol{t}_{3_o}$:

$$\boldsymbol{t}_{2_o} = -\boldsymbol{t}_{y_o}, \qquad \boldsymbol{t}_{3_o} = -\boldsymbol{t}_{z_o} \tag{3.25}$$

By substituting these relations into (3.22)

$$\boldsymbol{t}_{4_o} + n_x(-\boldsymbol{t}_{x_o}) + n_y(-\boldsymbol{t}_{y_o}) + n_z(-\boldsymbol{t}_{z_o}) = \mathbf{0}$$

so

$$\boldsymbol{t}_{4_o} = n_{x4} \boldsymbol{t}_{x_o} + n_{y4} \boldsymbol{t}_{y_o} + n_{z4} \boldsymbol{t}_{z_o} \tag{3.26}$$

where $n_{x4} = n_x$, $n_{y4} = n_y$, and $n_{z4} = n_z$. So, the traction vector $\boldsymbol{t}_{4_o}$ can be obtained by a linear relation between the traction vectors on the three orthogonal planes and the components of its unit normal vector. But can we use the equation (3.26) for any unit normal vector rather than $\boldsymbol{n}_{4_o}$?

By considering the equations (3.13) and (3.16) we find that the equation (3.26) is really below equation:

$$\boldsymbol{t}_{4_o} = \frac{\Delta s_1}{\Delta s_4}\boldsymbol{t}_{x_o} + \frac{\Delta s_2}{\Delta s_4}\boldsymbol{t}_{y_o} + \frac{\Delta s_3}{\Delta s_4}\boldsymbol{t}_{z_o} \tag{3.27}$$

and this equation is

$$\boldsymbol{t}_{4_o} = |n_{x4}|\boldsymbol{t}_{x_o} + |n_{y4}|\boldsymbol{t}_{y_o} + |n_{z4}|\boldsymbol{t}_{z_o} \tag{3.28}$$

In Figure 2, by $a > 0$, $b > 0$, and $c > 0$, the components of unit normal vector on the oriented surface are greater than zero. So, the equation (3.26) is valid for these cases.

For the surfaces with negative components of the unit normal vector but not equal to zero, imagine a tetrahedron mass element by the unit normal vector of its oriented surface (base face), $\boldsymbol{n}_{-4}$, that all the components are negative. So, we have $\boldsymbol{n}_{-4_o} = n_{x-4}\boldsymbol{e}_x + n_{y-4}\boldsymbol{e}_y + n_{z-4}\boldsymbol{e}_z = -n_x\boldsymbol{e}_x - n_y\boldsymbol{e}_y - n_z\boldsymbol{e}_z$, where $\boldsymbol{n}_{-4_o}$ is the outward unit normal vector of the parallel plane to the oriented surface that passes through the vortex point of the tetrahedron (point $\boldsymbol{o}$), and n_x, n_y, and n_z are positive values. Applying the process of exact tetrahedron argument to this new tetrahedron, leads to the following relation similar to the equation (3.22):

$$\boldsymbol{E}_0 = \boldsymbol{t}_{-4_o} + |n_{x-4}|\boldsymbol{t}_{x_o} + |n_{y-4}|\boldsymbol{t}_{y_o} + |n_{z-4}|\boldsymbol{t}_{z_o} = \boldsymbol{0} \tag{3.29}$$

As compared with the equation (3.22), in this equation we have $\boldsymbol{t}_{x_o}$, $\boldsymbol{t}_{y_o}$, and $\boldsymbol{t}_{z_o}$ rather than $\boldsymbol{t}_{1_o}$, $\boldsymbol{t}_{2_o}$, and $\boldsymbol{t}_{3_o}$, respectively. Because the outward sides of orthogonal faces of this new tetrahedron are at positive directions of the coordinate system. By (3.29) and the components of $\boldsymbol{n}_{-4_o}$, we have:

$$\begin{aligned}
\boldsymbol{t}_{-4_o} &= -|n_{x-4}|\boldsymbol{t}_{x_o} - |n_{y-4}|\boldsymbol{t}_{y_o} - |n_{z-4}|\boldsymbol{t}_{z_o} \\
&= -|-n_x|\boldsymbol{t}_{x_o} - |-n_y|\boldsymbol{t}_{y_o} - |-n_z|\boldsymbol{t}_{z_o} \\
&= -n_x\boldsymbol{t}_{x_o} - n_y\boldsymbol{t}_{y_o} - n_z\boldsymbol{t}_{z_o} \\
&= n_{x-4}\boldsymbol{t}_{x_o} + n_{y-4}\boldsymbol{t}_{y_o} + n_{z-4}\boldsymbol{t}_{z_o}
\end{aligned} \tag{3.30}$$

So, the traction vector $\boldsymbol{t}_{-4_o}$ can be obtained from a linear relation between the traction vectors on the three orthogonal planes and the components of its unit normal vector. For the surfaces that one or two components of their unit normal vectors are negative, the same process can be done.

For the other surfaces that one or two components of their unit normal vectors are equal to zero, the tetrahedron does not form, but due to the continuous property of the traction vectors on $\boldsymbol{n}$ and the arbitrary choosing for any orthogonal basis for the coordinate system, the traction vectors on these surfaces can be described by the equation (3.26), as well. So, in general, the normal unit vector $\boldsymbol{n}_4$ can be related to any surface that passes through point $\boldsymbol{o}$ in three-dimensional continuum media. Thus, the subscript 4 removes from the equation (3.26) and we have for every $\boldsymbol{n} = n_x\boldsymbol{e}_x + n_y\boldsymbol{e}_y + n_z\boldsymbol{e}_z$:

$$\boldsymbol{t}_o = n_x\boldsymbol{t}_{x_o} + n_y\boldsymbol{t}_{y_o} + n_z\boldsymbol{t}_{z_o} \tag{3.31}$$

The subscript o in this equation indicates the vortex point of the tetrahedron. But any point in the domain in continuum media can be the vertex point of a tetrahedron and we could consider this tetrahedron. So, the point $\boldsymbol{o}$ can be any point in continuum media and the subscript o removes from the equation:

$$\boldsymbol{t} = n_x \boldsymbol{t}_x + n_y \boldsymbol{t}_y + n_z \boldsymbol{t}_z \tag{3.32}$$

or

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = n_x \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x) + n_y \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y) + n_z \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z) \tag{3.33}$$

It means that if we have the traction vectors on the three orthogonal surfaces at a given point and time, then we can get the traction vector on any surface that passes through that point at that time by using the unit normal vector of the surface and the linear relation (3.33).

So, we must define the traction vectors on the three orthogonal surfaces at any point and any time. The traction vector on the surface with unit normal vector $\boldsymbol{e}_x$ by its components:

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x) = T_{xx}(\boldsymbol{r}, t)\, \boldsymbol{e}_x + T_{xy}(\boldsymbol{r}, t)\, \boldsymbol{e}_y + T_{xz}(\boldsymbol{r}, t)\, \boldsymbol{e}_z \tag{3.34}$$

here $T_{xx}(\boldsymbol{r}, t)$, $T_{xy}(\boldsymbol{r}, t)$, and $T_{xz}(\boldsymbol{r}, t)$ are scalars that depend only on $\boldsymbol{r}$ and t. In each one the first subscript indicates the direction of normal unit vector of the surface that it acts on it and the second subscript indicate the direction of this component of traction vector. And similarly, we define the traction vectors on the surfaces with unit normal vectors $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$, respectively, as:

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y) = T_{yx}(\boldsymbol{r}, t)\, \boldsymbol{e}_x + T_{yy}(\boldsymbol{r}, t)\, \boldsymbol{e}_y + T_{yz}(\boldsymbol{r}, t)\, \boldsymbol{e}_z \tag{3.35}$$

and

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z) = T_{zx}(\boldsymbol{r}, t)\, \boldsymbol{e}_x + T_{zy}(\boldsymbol{r}, t)\, \boldsymbol{e}_y + T_{zz}(\boldsymbol{r}, t)\, \boldsymbol{e}_z \tag{3.36}$$

By substituting these equations in (3.33)

$$\begin{aligned}\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = {} & n_x \big\{T_{xx}(\boldsymbol{r}, t)\, \boldsymbol{e}_x + T_{xy}(\boldsymbol{r}, t)\, \boldsymbol{e}_y + T_{xz}(\boldsymbol{r}, t)\, \boldsymbol{e}_z\big\} \\ & + n_y \big\{T_{yx}(\boldsymbol{r}, t)\, \boldsymbol{e}_x + T_{yy}(\boldsymbol{r}, t)\, \boldsymbol{e}_y + T_{yz}(\boldsymbol{r}, t)\, \boldsymbol{e}_z\big\} \\ & + n_z \big\{T_{zx}(\boldsymbol{r}, t)\, \boldsymbol{e}_x + T_{zy}(\boldsymbol{r}, t)\, \boldsymbol{e}_y + T_{zz}(\boldsymbol{r}, t)\, \boldsymbol{e}_z\big\}\end{aligned}$$

by rearranging the equation

$$\begin{aligned}\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = {} & \big\{n_x T_{xx}(\boldsymbol{r}, t) + n_y T_{yx}(\boldsymbol{r}, t) + n_z T_{zx}(\boldsymbol{r}, t)\big\}\boldsymbol{e}_x \\ & + \big\{n_x T_{xy}(\boldsymbol{r}, t) + n_y T_{yy}(\boldsymbol{r}, t) + n_z T_{zy}(\boldsymbol{r}, t)\big\}\boldsymbol{e}_y \\ & + \big\{n_x T_{xz}(\boldsymbol{r}, t) + n_y T_{yz}(\boldsymbol{r}, t) + n_z T_{zz}(\boldsymbol{r}, t)\big\}\boldsymbol{e}_z\end{aligned}$$

this can be shown as

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = \begin{bmatrix} t_x(\boldsymbol{r}, t, \boldsymbol{n}) \\ t_y(\boldsymbol{r}, t, \boldsymbol{n}) \\ t_z(\boldsymbol{r}, t, \boldsymbol{n}) \end{bmatrix} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}^T \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \tag{3.37}$$

using the vector relations, we have

$$\boldsymbol{t} = \boldsymbol{T}^T . \boldsymbol{n} \tag{3.38}$$

where $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{r}, t)$ is a second order tensor that is called stress tensor. This tensor depends only on the position vector and time. This relation means that "for describing the state of stress on any surface at a given point and time we need the 9 components of the stress tensor at that point and time". So, $\boldsymbol{E}_0 = \mathbf{0}$ leads to the existence of stress tensor.

Note that here the stress tensor $\boldsymbol{T}$ is exactly defined as point-base but in the former tetrahedron arguments it is not. Because they used the average values of traction vectors on the surfaces that did not pass through the same point and by an approximating process the stress tensor is derived.

Let us see what $\boldsymbol{E}_1 = \mathbf{0}$ tells.
From the equation (3.16):

$$\begin{aligned}\boldsymbol{E}_1 = &\Big(\frac{\partial \boldsymbol{t}_{4_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial y}\frac{1}{n_y} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial z}\frac{1}{n_z}\Big) + n_x\Big(\frac{\partial \boldsymbol{t}_{1_o}}{\partial y}\frac{1}{n_y} + \frac{\partial \boldsymbol{t}_{1_o}}{\partial z}\frac{1}{n_z}\Big) \\ &+ n_y\Big(\frac{\partial \boldsymbol{t}_{2_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{2_o}}{\partial z}\frac{1}{n_z}\Big) + n_z\Big(\frac{\partial \boldsymbol{t}_{3_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{3_o}}{\partial y}\frac{1}{n_y}\Big) - \boldsymbol{B}_o\end{aligned} \tag{3.39}$$

As stated before, for the tetrahedron element with $\Delta V \neq 0$, no one of n_x, n_y, and n_z is zero exactly. So, $\boldsymbol{E}_1$ exists. Furthermore, the unit normal vector $\boldsymbol{n}_4$ is an arbitrary geometrical parameter and we have:

$$\frac{\partial \boldsymbol{n}_4}{\partial x} = \frac{\partial \boldsymbol{n}_4}{\partial y} = \frac{\partial \boldsymbol{n}_4}{\partial z} = \mathbf{0} \tag{3.40}$$

By using these relations and the equation (3.22), i.e. $\boldsymbol{t}_{4_o} = \boldsymbol{E}_0 - n_x\boldsymbol{t}_{1_o} - n_y\boldsymbol{t}_{2_o} - n_z\boldsymbol{t}_{3_o}$, we have for (3.39):

$$\boldsymbol{E}_1 = \frac{1}{n_x}\frac{\partial \boldsymbol{E}_0}{\partial x} + \frac{1}{n_y}\frac{\partial \boldsymbol{E}_0}{\partial y} + \frac{1}{n_z}\frac{\partial \boldsymbol{E}_0}{\partial z} - \frac{\partial \boldsymbol{t}_{1_o}}{\partial x} - \frac{\partial \boldsymbol{t}_{2_o}}{\partial y} - \frac{\partial \boldsymbol{t}_{3_o}}{\partial z} - \boldsymbol{B}_o$$

If we define $\boldsymbol{E}$ as:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{t}_{1_o}}{\partial x} - \frac{\partial \boldsymbol{t}_{2_o}}{\partial y} - \frac{\partial \boldsymbol{t}_{3_o}}{\partial z} - \boldsymbol{B}_o \tag{3.41}$$

so, we have

$$\boldsymbol{E}_1 = \frac{1}{n_x}\frac{\partial \boldsymbol{E}_0}{\partial x} + \frac{1}{n_y}\frac{\partial \boldsymbol{E}_0}{\partial y} + \frac{1}{n_z}\frac{\partial \boldsymbol{E}_0}{\partial z} + \boldsymbol{E} \tag{3.42}$$

But we saw in (3.21) that the derivatives of $\boldsymbol{E}_0$ are equal to zero. So, from (3.42) and $\boldsymbol{E}_1 = \mathbf{0}$, we have:

$$\boldsymbol{E}_1 = \boldsymbol{E} = \mathbf{0} \tag{3.43}$$

By (3.41), $\boldsymbol{E}$ is defined at the vertex point of tetrahedron. But we stated before that the vertex point of the tetrahedron can be at any point in continuum media. Therefore, by (3.43), $\boldsymbol{E} = \mathbf{0}$ at any point in continuum media. This leads to that all derivatives of $\boldsymbol{E}$ are equal to zero at any point in continuum media. So:

$$\frac{\partial \boldsymbol{E}}{\partial x} = \frac{\partial \boldsymbol{E}}{\partial y} = \frac{\partial \boldsymbol{E}}{\partial z} = \mathbf{0} \tag{3.44}$$

By using the relations (3.24) and (3.25) i.e., $\boldsymbol{t}_{1_o} = -\boldsymbol{t}_{x_o}$, $\boldsymbol{t}_{2_o} = -\boldsymbol{t}_{y_o}$, and $\boldsymbol{t}_{3_o} = -\boldsymbol{t}_{z_o}$, we have for (3.41):

$$\boldsymbol{E} = \frac{\partial \boldsymbol{t}_{x_o}}{\partial x} + \frac{\partial \boldsymbol{t}_{y_o}}{\partial y} + \frac{\partial \boldsymbol{t}_{z_o}}{\partial z} - \boldsymbol{B}_o \tag{3.45}$$

but $\boldsymbol{E} = \mathbf{0}$, so

$$\boldsymbol{B}_o = \frac{\partial \boldsymbol{t}_{x_o}}{\partial x} + \frac{\partial \boldsymbol{t}_{y_o}}{\partial y} + \frac{\partial \boldsymbol{t}_{z_o}}{\partial z} \tag{3.46}$$

As stated before, we can remove the subscript o from the equation and tell that this equation is valid at any point and any time in the continuum domain. Therefore:

$$\boldsymbol{B} = \frac{\partial \boldsymbol{t}_x}{\partial x} + \frac{\partial \boldsymbol{t}_y}{\partial y} + \frac{\partial \boldsymbol{t}_z}{\partial z} \tag{3.47}$$

or

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\partial \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x)}{\partial x} + \frac{\partial \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y)}{\partial y} + \frac{\partial \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z)}{\partial z} \tag{3.48}$$

This differential equation means that if we have the first derivatives of the traction vectors on the three orthogonal surfaces at a given point and time, then we can get the body term at that point and time by using the equation (3.48). By substituting the definitions of $\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x)$, $\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y)$, and $\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z)$ from the relations (3.34), (3.35), and (3.36) into the equation (3.48):

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{r}, t) = & \frac{\partial}{\partial x}\{T_{xx}(\boldsymbol{r}, t)\,\boldsymbol{e}_x + T_{xy}(\boldsymbol{r}, t)\,\boldsymbol{e}_y + T_{xz}(\boldsymbol{r}, t)\,\boldsymbol{e}_z\} \\ & + \frac{\partial}{\partial y}\{T_{xx}(\boldsymbol{r}, t)\,\boldsymbol{e}_x + T_{xy}(\boldsymbol{r}, t)\,\boldsymbol{e}_y + T_{xz}(\boldsymbol{r}, t)\,\boldsymbol{e}_z\} \\ & + \frac{\partial}{\partial z}\{T_{xx}(\boldsymbol{r}, t)\,\boldsymbol{e}_x + T_{xy}(\boldsymbol{r}, t)\,\boldsymbol{e}_y + T_{xz}(\boldsymbol{r}, t)\,\boldsymbol{e}_z\} \end{aligned} \tag{3.49}$$

by rearranging the equation and using $\boldsymbol{B} = \rho\boldsymbol{a} - \rho\boldsymbol{b}$ from the equation (1.3) we have for any $\boldsymbol{r}$ and t:

$$\begin{aligned} \rho\boldsymbol{a} - \rho\boldsymbol{b} = & \left\{\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z}\right\}\boldsymbol{e}_x + \left\{\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z}\right\}\boldsymbol{e}_y \\ & + \left\{\frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z}\right\}\boldsymbol{e}_z \end{aligned} \tag{3.50}$$

this can be shown as

$$\rho\boldsymbol{a} - \rho\boldsymbol{b} = \begin{bmatrix} \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \\ \\ \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} \\ \\ \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} = \nabla.\boldsymbol{T}$$

so, we have

$$\rho\boldsymbol{a} = \nabla.\boldsymbol{T} + \rho\boldsymbol{b} \tag{3.51}$$

or

$$\rho(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v}) = \nabla.\boldsymbol{T} + \rho\boldsymbol{b} \tag{3.52}$$

So, $\boldsymbol{E}_1 = \mathbf{0}$ leads to the general equation of motion that is called Cauchy equation of motion. Cauchy obtained this important equation by applying the conservation of linear momentum to a "cubic element" and did not obtain it from the tetrahedron argument. The tetrahedron argument that is represented by most of the scientists and authors in continuum mechanics leads only to the relation (3.23) i.e., $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \mathbf{0}$ for the existence of stress tensor. But here in addition to the exactly derivation of the stress tensor, the other fundamental equation in continuum mechanics i.e., the Cauchy equation of motion is exactly derived from the tetrahedron argument, simultaneously.

Let us see what $\boldsymbol{E}_2 = \mathbf{0}$ tells.
From the equation (3.16):

$$\begin{aligned}\boldsymbol{E}_2 = &\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial y}\frac{1}{n_x n_y} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial z}\frac{1}{n_x n_z} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y\partial z}\frac{1}{n_y n_z}\Big)\\ &+ n_x\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y\partial z}\frac{1}{n_y n_z}\Big) + n_y\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x\partial z}\frac{1}{n_x n_z}\Big)\\ &+ n_z\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x\partial y}\frac{1}{n_x n_y}\Big) - \Big(\frac{\partial \boldsymbol{B}_o}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{B}_o}{\partial y}\frac{1}{n_y} + \frac{\partial \boldsymbol{B}_o}{\partial z}\frac{1}{n_z}\Big)\end{aligned} \tag{3.53}$$

For $\boldsymbol{E}_2$, Similar to the process for $\boldsymbol{E}_1 = \mathbf{0}$, we have:

$$\begin{aligned}\boldsymbol{E}_2 = &\frac{1}{n_x^2}\frac{\partial^2 \boldsymbol{E}_0}{\partial x^2} + \frac{1}{n_y^2}\frac{\partial^2 \boldsymbol{E}_0}{\partial y^2} + \frac{1}{n_z^2}\frac{\partial^2 \boldsymbol{E}_0}{\partial z^2} + \frac{1}{n_x n_y}\frac{\partial^2 \boldsymbol{E}_0}{\partial x\partial y} + \frac{1}{n_x n_z}\frac{\partial^2 \boldsymbol{E}_0}{\partial x\partial z} + \frac{1}{n_y n_z}\frac{\partial^2 \boldsymbol{E}_0}{\partial y\partial z}\\ &+ \frac{1}{n_x}\frac{\partial \boldsymbol{E}}{\partial x} + \frac{1}{n_y}\frac{\partial \boldsymbol{E}}{\partial y} + \frac{1}{n_z}\frac{\partial \boldsymbol{E}}{\partial z}\end{aligned} \tag{3.54}$$

By the previous explanations, all derivatives of $\boldsymbol{E}_0$ and $\boldsymbol{E}$ are equal to zero so, the equation (3.54) is a correct result of $\boldsymbol{E}_2 = \mathbf{0}$.

For $\boldsymbol{E}_3 = \mathbf{0}$ we have:

$$\begin{aligned}\boldsymbol{E}_3 = &\Big(\frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x^3}\frac{1}{n_x^3} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial y^3}\frac{1}{n_y^3} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial z^3}\frac{1}{n_z^3} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x^2\partial y}\frac{1}{n_x^2 n_y} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x^2\partial z}\frac{1}{n_x^2 n_z} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial y^2\partial z}\frac{1}{n_y^2 n_z}\\ &+ \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x\partial y^2}\frac{1}{n_x n_y^2} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x\partial z^2}\frac{1}{n_x n_z^2} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial y\partial z^2}\frac{1}{n_y n_z^2} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x\partial y\partial z}\frac{1}{n_x n_y n_z}\Big)\\ &+ n_x\Big(\frac{\partial^3 \boldsymbol{t}_{1_o}}{\partial y^3}\frac{1}{n_y^3} + \frac{\partial^3 \boldsymbol{t}_{1_o}}{\partial z^3}\frac{1}{n_z^3} + \frac{\partial^3 \boldsymbol{t}_{1_o}}{\partial y^2\partial z}\frac{1}{n_y^2 n_z} + \frac{\partial^3 \boldsymbol{t}_{1_o}}{\partial y\partial z^2}\frac{1}{n_y n_z^2}\Big)\\ &+ n_y\Big(\frac{\partial^3 \boldsymbol{t}_{2_o}}{\partial x^3}\frac{1}{n_x^3} + \frac{\partial^3 \boldsymbol{t}_{2_o}}{\partial z^3}\frac{1}{n_z^3} + \frac{\partial^3 \boldsymbol{t}_{2_o}}{\partial x^2\partial z}\frac{1}{n_x^2 n_z} + \frac{\partial^3 \boldsymbol{t}_{2_o}}{\partial x\partial z^2}\frac{1}{n_x n_z^2}\Big)\\ &+ n_z\Big(\frac{\partial^3 \boldsymbol{t}_{3_o}}{\partial x^3}\frac{1}{n_x^3} + \frac{\partial^3 \boldsymbol{t}_{3_o}}{\partial y^3}\frac{1}{n_y^3} + \frac{\partial^3 \boldsymbol{t}_{3_o}}{\partial x^2\partial y}\frac{1}{n_x^2 n_y} + \frac{\partial^3 \boldsymbol{t}_{3_o}}{\partial x\partial y^2}\frac{1}{n_x n_y^2}\Big)\\ &- \Big(\frac{\partial^2 \boldsymbol{B}_o}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{B}_o}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{B}_o}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{B}_o}{\partial x\partial y}\frac{1}{n_x n_y} + \frac{\partial^2 \boldsymbol{B}_o}{\partial x\partial z}\frac{1}{n_x n_z} + \frac{\partial^2 \boldsymbol{B}_o}{\partial y\partial z}\frac{1}{n_y n_z}\Big)\end{aligned} \tag{3.55}$$

Similar to the previous processes for $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, we have for $\boldsymbol{E}_3$:

$$\begin{aligned}\boldsymbol{E}_3 = &\frac{1}{n_x^3}\frac{\partial^3 \boldsymbol{E}_0}{\partial x^3} + \frac{1}{n_y^3}\frac{\partial^3 \boldsymbol{E}_0}{\partial y^3} + \frac{1}{n_z^3}\frac{\partial^3 \boldsymbol{E}_0}{\partial z^3} + \frac{1}{n_x^2 n_y}\frac{\partial^3 \boldsymbol{E}_0}{\partial x^2 \partial y} + \frac{1}{n_x^2 n_z}\frac{\partial^3 \boldsymbol{E}_0}{\partial x^2 \partial z} + \frac{1}{n_y^2 n_z}\frac{\partial^3 \boldsymbol{E}_0}{\partial y^2 \partial z} \\ &+ \frac{1}{n_x n_y^2}\frac{\partial^3 \boldsymbol{E}_0}{\partial x \partial y^2} + \frac{1}{n_x n_z^2}\frac{\partial^3 \boldsymbol{E}_0}{\partial x \partial z^2} + \frac{1}{n_y n_z^2}\frac{\partial^3 \boldsymbol{E}_0}{\partial y \partial z^2} + \frac{1}{n_x n_y n_z}\frac{\partial^3 \boldsymbol{E}_0}{\partial x \partial y \partial z} \\ &+ \frac{1}{n_x^2}\frac{\partial^2 \boldsymbol{E}}{\partial x^2} + \frac{1}{n_y^2}\frac{\partial^2 \boldsymbol{E}}{\partial y^2} + \frac{1}{n_z^2}\frac{\partial^2 \boldsymbol{E}}{\partial z^2} + \frac{1}{n_x n_y}\frac{\partial^2 \boldsymbol{E}}{\partial x \partial y} + \frac{1}{n_x n_z}\frac{\partial^2 \boldsymbol{E}}{\partial x \partial z} + \frac{1}{n_y n_z}\frac{\partial^2 \boldsymbol{E}}{\partial y \partial z}\end{aligned} \tag{3.56}$$

We saw that all derivatives of $\boldsymbol{E}_0$ and $\boldsymbol{E}$ are equal to zero. So, the equation (3.56) is a correct result of $\boldsymbol{E}_3 = \mathbf{0}$. This process for other $\boldsymbol{E}_m$'s, leads to the expressions that contain the higher derivatives of $\boldsymbol{E}_0$ and $\boldsymbol{E}$ and the higher powers of the components of the unit normal vector.

4. Discussion

In this section, we discuss some aspects of this new proof and compare it with the previous proofs for the existence of stress tensor and derivation of the Cauchy equation of motion. We gave a comprehensive review on the Cauchy tetrahedron argument and the proofs for the existence of stress tensor (2017, [1]). In that article, we stated some important and fundamental challenges on those proofs. For considering the stated challenges on this new proof, we start with the first challenge in [1].

The challenge 1 told us that applying the conservation of linear momentum to any volumes and shapes of a mass element must lead to the equation of motion. But in the previous proofs this process on an infinitesimal tetrahedron mass element leads to the equation $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \mathbf{0}$ that differs from the equation of motion. In that proofs, the equation of motion is obtained by using the stress tensor relation and applying the conservation of linear momentum to a cubic element or by using the divergence theorem in the integral equation of conservation of linear momentum. But in this proof, both the relation for the existence of stress tensor and the equation of motion are obtained, simultaneously. Therefore, the challenge 1 is removed in this proof.

The challenge 2 told us that the previous proofs for the existence of stress tensor are based on infinitesimal volumes by the expressions like "$\Delta V \to 0$", "$h \to 0$", *"when the tetrahedron shrinks to a point"* or *"when the tetrahedron shrinks to zero volume"*, while it must be proved that the existence of stress tensor at a point does not depend on the size of the mass element. In other words, the stress tensor exists for any size of mass element in continuum media, where the volume of mass element increases, decreases or does not change. Therefore, in that proofs the result is only valid for the infinitesimal volumes and does not show that the result can be applied to the mass elements with any volume in continuum media. But here we proved that the existence of stress tensor is independent of the volume of mass element and does not have any expression that indicate the using of infinitesimal volume or a limit to zero volume in this proof. So, this challenge is removed in this proof.

The challenge 3 is related to the average values of the traction vectors, body forces, and inertia terms on the surfaces and the volume of the mass element in the previous proofs. The average values lead to the approximate process even for the infinitesimal mass element. But in this proof the exact values are used and no approximate process within the proof. So, this proof is exact and the challenge 3 is removed in it.

The challenge 4 is related to the order of the surface forces in the limit $\Delta V \to 0$ or $h \to 0$. In the previous proofs, it was told that in the limit the order of surface forces is h^2 and the order of body forces and inertia is h^3. So, they told that in the limit the body and inertia terms go to zero and the surface forces remain in the equation of conservation of linear momentum. In that challenge, we showed that this is not correct. And in this proof, since we did not any limiting or approximating process, this challenge is removed.

In the challenges 5 and 6, it was told that to prove the existence of stress tensor as a point-base function from the relation $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \boldsymbol{0}$, the four surfaces that the traction vectors are defined on them must pass through the same point. But in this relation $\boldsymbol{t}_4$ is defined on Δs_4 and this surface, even for infinitesimal tetrahedron, does not pass through the vertex point of the tetrahedron that other three faces pass through it. But in this proof in the relation $\boldsymbol{t}_{4_o} + n_x\boldsymbol{t}_{1_o} + n_y\boldsymbol{t}_{2_o} + n_z\boldsymbol{t}_{3_o} = \boldsymbol{0}$, we define all of the traction vectors at the same point $\boldsymbol{o}$, where the four surfaces pass through it exactly. So, the stress tensor is obtained as a point-base function exactly. Thus, these challenges are removed in this proof.

The challenges 7 and 8 are related to the equation $\boldsymbol{t}_4+n_x\boldsymbol{t}_1+n_y\boldsymbol{t}_2+n_z\boldsymbol{t}_3 = \boldsymbol{0}$, where the traction vectors are the average values on the surfaces of an infinitesimal tetrahedron. It was told that by multiplying this equation by Δs_4, we have $\boldsymbol{t}_4\Delta s_4 + \boldsymbol{t}_1\Delta s_1 + \boldsymbol{t}_2\Delta s_2 + \boldsymbol{t}_3\Delta s_3 = \boldsymbol{0}$, this means that the sum of the surface forces on the infinitesimal tetrahedron is zero. This is not correct, because from the conservation of linear momentum (1.4), the surface forces on any mass element are equal to the body terms on that element. But in this proof, we used the exact traction vectors, so the equation $\boldsymbol{t}_{4_o}+n_x\boldsymbol{t}_{1_o}+n_y\boldsymbol{t}_{2_o}+n_z\boldsymbol{t}_{3_o} = \boldsymbol{0}$ is derived. In this equation, since all of the traction vectors are defined at point $\boldsymbol{o}$, so the equation $\boldsymbol{t}_{4_o}\Delta s_4 + \boldsymbol{t}_{1_o}\Delta s_1 + \boldsymbol{t}_{2_o}\Delta s_2 + \boldsymbol{t}_{3_o}\Delta s_3 = \boldsymbol{0}$ does not mean the sum of the traction vectors on the surface of the mass element is equal to zero.

5. Conclusion

We considered the general integral equation of conservation of linear momentum as:

$$\int_{\mathcal{M}} \rho\boldsymbol{a}\, dV = \int_{\partial\mathcal{M}} \boldsymbol{t}\, dS + \int_{\mathcal{M}} \rho\boldsymbol{b}\, dV$$

where $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n})$ is the traction vector (surface force per unit area). We first derived the Cauchy lemma for traction vectors from the above integral equation:

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n})$$

Then we showed by a new exact tetrahedron argument that applying the general integral equation of conservation of linear momentum to the tetrahedron mass element leads to the following fundamental equation:

$$\boldsymbol{E}_0 + \boldsymbol{E}_1\frac{1}{3}h + \boldsymbol{E}_2\frac{1}{12}h^2 + \boldsymbol{E}_3\frac{1}{60}h^3 + \ldots + \boldsymbol{E}_m\frac{2}{(m+2)!}h^m + \ldots = \mathbf{0}$$

where h is the altitude of the tetrahedron. $\boldsymbol{E}_m$'s are expressions that contain the traction vectors, inertia, body force, and their derivatives and the powers of the components of unit normal vector of the tetrahedron's base face. Then we showed that the only solution of this equation is:

$$\boldsymbol{E}_m = \mathbf{0}, \qquad m = 0, 1, 2, \ldots, \infty$$

i.e. $\boldsymbol{E}_m$'s must be equal to zero. Then, we proved that $\boldsymbol{E}_0 = \mathbf{0}$ leads to the existence of stress tensor:

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}^T \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \boldsymbol{T}^T.\boldsymbol{n}$$

and $\boldsymbol{E}_1 = \mathbf{0}$ leads to the derivation of the general equation of motion:

$$\rho(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v}) = \nabla.\boldsymbol{T} + \rho\boldsymbol{b}$$

for other $\boldsymbol{E}_m = \mathbf{0}$ these results are repeated. During this proof, there is no limiting or approximating process and the parameters are exact point-base functions not average values. This proof is not limited to $h \to 0$ for an infinitesimal tetrahedron mass element. Also, we showed that in this proof, all of the challenges on the previous tetrahedron arguments and the proofs for existence of stress tensor are removed.

Historical note: *The manuscript of the exact tetrahedron argument is prepared before writing the review article [1].*

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