

# The moment-generating function of normal variate functionals using the star probability measure

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**Abstract** The motivation of this study is to investigate new methods for the calculation of the moment-generating function of the lognormal distribution. Taylor expansion method on the moments of the lognormal suffers from divergence issues, saddle-point approximation is not exact, and integration methods can be complicated. In the present paper we introduce a new probability measure that we refer to as the star probability measure as an alternative approach to compute the moment-generating function of normal variate functionals such as the lognormal distribution.

**Keywords** moment-generating function · lognormal distribution · star probability measure

## 1 Introduction

In the present study, we investigate how we might use a stochastic approach to compute the moment-generating function of normal variate functionals. The moment-generating function of a random variable  $X$  is expressed as  $M(\theta) = \mathbb{E}(e^{\theta X})$  by definition. By Taylor expansion of  $e^{\theta X}$  centered in zero, we would get  $\mathbb{E}(e^{\theta X}) = \mathbb{E}\left(1 + \frac{\theta X}{1!} + \frac{\theta^2 X^2}{2!} + \dots + \frac{\theta^n X^n}{n!}\right)$ . The approach using Taylor expansion fails when applied to the moment-generating function of the lognormal distribution. Its expression  $M(\theta) = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} e^{n\mu + n^2\sigma^2/2}$  diverges for all  $\theta$  - mainly because the lognormal distribution is skewed to the right. The skew increases the likelihood of having occurrences that depart from the central point of the Taylor expansion. These occurrences produce moments of higher order, which are not offset by the factorial of  $n$  in the denominator; therefore, the Taylor series diverges.

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Saddle points and their representations are worth considering, although they are not always accurate in all domains. An expression for the characteristic function using the classical saddle-point approximation as described in [4] was obtained by [6]. Several other representations of the characteristic function have been obtained using asymptotic series [2, 7]. Integration methods to evaluate the characteristic function such as the Hermite-Gauss quadrature proposed by [5] appear to be a natural choice. It was noted in [3] that, due to the oscillatory integrand and the slow decay rate of the tail of the lognormal density function, numerical integration is difficult. Other numerical methods include the work of [8], which employs a contour integral passing through the saddle point at the steepest descent rate to overcome the oscillatory behavior of the integrand and the slow decay rate at the tail. A closed-form approximation of the Laplace transform of the lognormal distribution, which is fairly accurate over its domain of definition was derived by [1]. Although this equation is not a true benchmark, we can compare it with our results due to its simplicity and because the Laplace transform of the density function is the moment-generating function with a negative sign in front of the argument  $\theta$ .

In the present approach we introduce the stochastic process associated to the problem, which is obtained by applying Itô's lemma with the function  $f(x) = e^{\theta Y(x)}$  to the base process  $dx_t = \mu dt + \sigma dW_t$ , where  $\mu$  and  $\sigma$  are respectively the drift and volatility parameters,  $W_t$  is a Wiener process, and  $Y(x)$  is the random variable of the distribution expressed as a function of a normal random variable  $x$ . The purpose is to compute  $\mathbb{E}(f_t)$  to evaluate the moment-generating function of the distribution. While the method leads to the known closed form for the normal distribution, numerical methods cannot be avoided for the calculation of the moment-generating function of the lognormal distribution. Because of the correlation between the real and imaginary parts of the associated stochastic process, the method fails when computing the characteristic function of the lognormal distribution due to excess complexity. It appears that the stochastic approach provides some interesting information about the resolvability of the moment-generating function of normal variate functionals.

Section 2 introduces the theoretical background with the star probability measure and its use to compute the expected value of normal variate functionals. An illustration of the method is provided in section 3 for the calculation of the moment-generating function of the normal distribution. In section 4, we apply the method to the calculation of the moment-generating function of the lognormal distribution. This section covers the derivation of the stochastic differential equation, the differential equation of the first moment and of the variance. The various issues arising when extending the calculations to the characteristic function are also highlighted. Lastly, we perform a numerical calculation of the moment-generating function of the lognormal distribution, where we compare the present study's results against the closed-form approximation of the Laplace transform. In section 5, we offer our conclusion.

## 2 Theoretical background

### 2.1 The star probability measure

Let us assume that  $f(x)$  is a continuous function and  $x$  a random variable. Let us say  $F(x)$  is the antiderivative of  $f(x)$ . The star probability measure relative to  $F(x)$  is defined such that:

$$\mathbb{E}_*(F(x)) = F(\mathbb{E}(x)), \quad (1)$$

where  $\mathbb{E}_*$  is expectation in the star probability measure and  $\mathbb{E}$  expectation in the real-world probability measure.

Note that the star probability measure as defined in (1) does not exist in all cases. For example, if  $F(x) = x^2$  and  $x$  is normally distributed centered in zero, then the star probability measure does not exist. If  $F(x)$  is an increasing function on the domain of definition of  $x$ , then the star probability measure exists. Conversely, if  $F(x)$  is a decreasing function on the domain of definition of  $x$ , then the star probability measure exists.

Because of (1) we can write:

$$\mathbb{E}_* \left( \int_{x_0}^x f(x) dx \right) = \int_{x_0}^{\mathbb{E}(x)} f(x) dx. \quad (2)$$

Below are some properties of the star probability measure:

$$\mathbb{E}_*(a + bX) = a + b\mathbb{E}_*(X), \quad (3)$$

where  $a$  and  $b$  are constants, and  $X$  a random variable.

$$\mathbb{E}_*(W_t) = 0, \quad (4)$$

where  $W_t$  is a Wiener process.

Let us consider an Itô process in its integral form:

$$x_t = x_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s. \quad (5)$$

where the first integral on the right is a Riemann sum, and the second term an Itô integral.

We have  $\mathbb{E}_*(x_t) = x_0 + \int_0^t \mu_s ds$  and  $\mathbb{E}(x_t) = x_0 + \int_0^t \mu_s ds$ . Therefore, for the process  $x_t$ , we have

$$\mathbb{E}_*(x_t) = \mathbb{E}(x_t). \quad (6)$$

This property is valid for all processes of the form (5); yet it may not hold in general.

## 2.2 The stochastic differential equation

Assume  $x_t$  is a stochastic process that satisfies the below stochastic differential equation:

$$dx_t = \mu_1 dt + \sigma_1 dW_t, \quad (7)$$

where  $W_t$  is Wiener process,  $\mu_1$  a constant drift, and  $\sigma_1$  a constant volatility parameter.

Let us define a twice differentiable continuous function  $f(x)$  such that:

$$f(x) = \int_{x_0}^x \sigma_2(s) ds, \quad (8)$$

Let us apply Itô's lemma to the Itô process (7) with the function defined in (8). We have

$$\frac{\partial f(x)}{\partial x} = \sigma_2(x), \quad (9)$$

and

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{\partial \sigma_2(x)}{\partial x} = \sigma_2'(x), \quad (10)$$

By applying Itô's lemma we get:

$$df = \left( \frac{\partial f}{\partial t} + \mu_1 \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_1^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_1 \frac{\partial f}{\partial x} dW_t, \quad (11)$$

Therefore

$$df = \left( \mu_1 \sigma_2(f^{-1}(f_t)) + \frac{1}{2} \sigma_1^2 \sigma_2'(f^{-1}(f_t)) \right) dt + \sigma_1 \sigma_2(f^{-1}(f_t)) dW_t, \quad (12)$$

where  $f^{-1}$  is the inverse function of  $f(x)$ , and  $f_t$  the random variable of the stochastic process defined by  $f_t = f(x_t)$ .

We normalize (12) by  $\sigma_2(f^{-1}(f_t))$  to obtain:

$$\frac{df}{\sigma_2(f^{-1}(f_t))} = \left( \mu_1 + \frac{1}{2} \sigma_1^2 \frac{\sigma_2'(f^{-1}(f_t))}{\sigma_2(f^{-1}(f_t))} \right) dt + \sigma_1 dW_t, \quad (13)$$

Let us set the drift of the stochastic differential equation (12) to  $\mu_2(f^{-1}(f_t)) = (\mu_1 \sigma_2(f^{-1}(f_t)) + \frac{1}{2} \sigma_1^2 \sigma_2'(f^{-1}(f_t)))$ ; therefore we get:

$$\frac{df}{\sigma_2(f^{-1}(f))} = \frac{\mu_2(f^{-1}(f_t))}{\sigma_2(f^{-1}(f_t))} dt + \sigma_1 dW_t. \quad (14)$$

We integrate (14) between the boundary conditions to obtain:

$$\int_{f_0}^{f_t} \frac{df}{\sigma_2(f^{-1}(f))} = \int_0^t \frac{\mu_2(f^{-1}(f_s))}{\sigma_2(f^{-1}(f_s))} ds + \int_0^t \sigma_1 dW_t. \quad (15)$$

The derivative of the inverse of a function  $h(x)$  is expressed as follows:

$$[h^{-1}]'(x) = \frac{1}{h'(h^{-1}(x))}. \quad (16)$$

Hence:

$$f^{-1}(\chi) = \int_{f_0}^{\chi} \frac{ds}{\sigma_2(f^{-1}(s))}, \quad (17)$$

Therefore, we get:

$$f^{-1}(f_t) = \int_0^t \frac{\mu_2(f^{-1}(f_s))}{\sigma_2(f^{-1}(f_s))} ds + \sigma_1 W_t, \quad (18)$$

Because of the structure of (18), we can say that  $f^{-1}(f_t)$  is Gaussian with distribution  $\mathcal{N}(m_t, v_t)$  where  $m_t$  is the mean and  $v_t$  the variance of the process. It can be shown that, for a Gaussian process the below identity holds:

$$\mathbb{E}_*(f^{-1}(f_t)) = \mathbb{E}_*(m_t + \sqrt{v_t}Z), \quad (19)$$

where  $Z$  is a standard normal random variable.

If (14) is an Itô process, we can use property (6) and directly apply the star probability measure with respect to  $f^{-1}(f_t)$ . Hence, we get:

$$\mathbb{E}_*(f^{-1}(f_t)) = \mathbb{E}(f^{-1}(f_t)) = m_t. \quad (20)$$

Therefore

$$f^{-1}(\mathbb{E}(f_t)) = m_t. \quad (21)$$

Finally

$$\mathbb{E}(f_t) = f(m_t). \quad (22)$$

This is generally true for Itô processes. We could easily imagine a counter example where the mean and the variance of the process (18) are different under the real world and under the star probability measure. However, there is a similitude among these processes. We can say that they belong to a broad category characterized by the Gaussianity of their distributions. Let us show that for a process  $X_t$  defined in the real world by  $X_t = \mu_t + \sigma_t Z$ , where  $\mu_t$  and  $\sigma_t$  are the mean and the standard deviation, and  $Z$  is a standard normal random variable; the identity  $\mathbb{E}(X_t) = \mathbb{E}_*(X_t)$  holds for any arbitrary star probability measure.

Let us denote by  $\Lambda$  the class of random variable functionals that have a mean equal to  $\mu_t$  in the real world. Let us say  $\Sigma$  is the standard deviation of the process  $X_t$ , which can be different under the real world and under the star probability measure.

Suppose there exists a star probability measure such that  $\mathbb{E}(X_t) \neq \mathbb{E}_*(X_t)$ . The latter is true if there is a ‘‘convexity adjustment’’ effect. Then,  $\Lambda$  must be a convex or concave function of  $Z$ .

Hence we can write the distribution in its general form  $X_t = \Lambda + \Sigma Z$ , which should hold in either the real world or star probability measure.

Because  $X_t$  is Gaussian, we must have  $\Lambda = a_t + b_t Z$ , where  $a_t$  and  $b_t$  are deterministic functions. Note that  $\Lambda$  is no longer a convex or concave function. As a result  $X_t = a_t + (b_t + \Sigma) Z$  and we get  $\mathbb{E}(X_t) = \mathbb{E}_*(X_t)$ . Therefore, the hypothesis  $\mathbb{E}(X_t) \neq \mathbb{E}_*(X_t)$  is false, and we can conclude that (22) is valid for any Gaussian distribution.

### 3 An illustration of the star probability measure with the moment-generating function of the normal distribution

In the present section we show how to derive the moment-generating function of the normal distribution with parameter  $\mu$  and  $\sigma$  using the star probability measure.

Let us consider the stochastic process  $x_t$  which satisfies the following stochastic differential equation:

$$dx_t = \mu dt + \sigma dW_t, \quad (23)$$

where  $W_t$  is a Wiener process,  $\mu$  the drift,  $\sigma$  the volatility, and  $x_0 = 0$ . This process has distribution  $x_t \sim \mathcal{N}(\mu t, \sigma^2 t)$ .

Let us apply Itô's lemma to (23) using the function  $f(x) = e^{\theta x}$ ; we get:

$$\frac{df}{f} = \left( \mu\theta + \frac{1}{2}\sigma^2\theta^2 \right) dt + \sigma dW_t, \quad (24)$$

where  $\frac{df}{f} = d(\ln f)$ .

Let us integrate (24) between the boundary conditions; we get:

$$\int_{f_0}^{f_t} \frac{df}{f} = \int_0^t \left( \mu\theta + \frac{1}{2}\sigma^2\theta^2 \right) dt + \sigma W_t. \quad (25)$$

Hence

$$\ln(f_t) = \left( \mu\theta + \frac{1}{2}\sigma^2\theta^2 \right) t + \sigma W_t. \quad (26)$$

The stochastic process (24) is an Itô process; therefore, we can apply property (6). When we apply the star probability measure with respect to  $\ln(f_t)$  to (26), we get:

$$\mathbb{E}_*(\ln(f_t)) = \mathbb{E}_* \left( \left( \mu\theta + \frac{1}{2}\sigma^2\theta^2 \right) t \right) + \mathbb{E}_*(\sigma W_t). \quad (27)$$

Using (1) and property (3) and (4), we get:

$$\ln(\mathbb{E}(f_t)) = \left( \mu\theta + \frac{1}{2}\sigma^2\theta^2 \right) t. \quad (28)$$

Therefore

$$\mathbb{E}(f_t) = e^{(\mu\theta + \frac{1}{2}\sigma^2\theta^2)t}. \quad (29)$$

The moment-generating function of the normal distribution is  $M(\theta) = \mathbb{E}(f_1)$ , which is:

$$M(\theta) = e^{(\mu\theta + \frac{1}{2}\sigma^2\theta^2)}. \quad (30)$$

where  $\theta \in \mathbb{R}$ .

The characteristic function  $\varphi(\theta)$  of the normal distribution is identical to the expression (30) with  $\theta \in \mathbb{C}$  and  $\Re(\theta)$  invariant, and in its specific form  $\theta$  is a pure imaginary number.

For a pair of random variables defined by their respective stochastic processes, the realization of both processes is pairwise correlated. In this context, the pair of random variables is said to be monotonically correlated provided both processes are independent, have finite variance, and their respective distributions form a white noise. The real and imaginary parts of (24) are monotonically correlated when  $\theta$  is in the complex plane.

#### 4 Application to the moment-generating function of the lognormal distribution

In this section we apply the previous method to the moment generating function of the lognormal distribution. The moment generating function of the lognormal distribution of parameter  $\mu$  and  $\sigma$  is expressed as  $M(\theta) = \mathbb{E}(e^{\theta e^x})$  where  $x \sim \mathcal{N}(\mu, \sigma^2)$  and  $\theta \in \mathbb{R}$ .

##### 4.1 The stochastic differential equation

Let us consider the stochastic process  $x_t$  which satisfies the following equation:

$$dx_t = \mu dt + \sigma dW_t, \quad (31)$$

where  $W_t$  is a Wiener process,  $\mu$  the drift,  $\sigma$  the volatility, and  $x_0 = 0$ . This process has distribution  $x_t \sim \mathcal{N}(\mu t, \sigma^2 t)$ .

Let us apply Itô's lemma to (31) using the function  $f(x) = e^{\theta e^x}$  and apply the normalisation in (13); we get:

$$\frac{df}{f \ln f} = \left( \mu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 \ln f \right) dt + \sigma dW_t, \quad (32)$$

where  $\frac{df}{f \ln f} = d(\ln \ln f)$ .

Let us integrate (32) between the boundary conditions; we get:

$$\int_{f_0}^{f_t} \frac{df}{f \ln f} = \int_0^t \left( \mu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 \ln f \right) dt + \sigma W_t. \quad (33)$$

Hence

$$\ln \ln(f_t) = \ln \theta + \left( \mu + \frac{1}{2} \sigma^2 \right) t + \frac{1}{2} \sigma^2 \int_0^t \ln(f_s) ds + \sigma W_t. \quad (34)$$

Whenever  $\theta \in \mathbb{R}$ ,  $\ln \ln(f_t)$  is Gaussian with distribution  $\mathcal{N}(m_t, v_t)$ , where  $m_t$  is the first moment and  $v_t$  the variance.

#### 4.2 The dependence structure between the real and imaginary parts

If we set  $y_t = \ln \ln f_t$ , we can rewrite (34) as follows:

$$y_t = \ln \theta + \left( \mu + \frac{1}{2} \sigma^2 \right) t + \frac{1}{2} \sigma^2 \int_0^t e^{y_s} ds + \sigma W_t. \quad (35)$$

As we move into the complex plane we need to introduce  $y_t = y_{R,t} + i y_{I,t}$ , where  $y_{R,t}$  is the real part and  $y_{I,t}$  the imaginary part. In a number of settings, the bivariate normal assumption relies on the law of large numbers, a prior holistic to multivariate distributions. Given (35), the dependence structure between the marginals in our problem is as follows:

$$y_{R,t} = \Re(\ln \theta) + \left( \mu + \frac{1}{2} \sigma^2 \right) t + \frac{1}{2} \sigma^2 \int_0^t \cos(y_{I,s}) e^{y_{R,s}} ds + \sigma W_t, \quad (36)$$

and

$$y_{I,t} = \Im(\ln \theta) + \frac{1}{2} \sigma^2 \int_0^t \sin(y_{I,s}) e^{y_{R,s}} ds. \quad (37)$$

We note that the real part  $y_{R,t}$  diffuses according to  $y_{R,t} \sim \mathcal{N}(m_t, v_t)$ , whereas  $y_{I,t}$  is correlated to its real counterpart. The joint distribution of the variables  $y_{R,t}$  and  $y_{I,t}$  deviates from the multivariate normal distribution due to the correlation between the variables. In the present case, we can say that the real part is normally distributed given (36) and that  $y_{I,t}$  is a function of  $y_{R,t}$ . Nonetheless, the distribution of the imaginary part is not well defined in the proper way of a parametric distribution. We can say that  $y_{I,t}$  is not Gaussian because  $y_{I,t}$  is not a linear function of  $y_{R,t}$  which is Gaussian.

The parametrization of a bivariate normal distribution yields :

$$Z = \rho_t Z_1 + \sqrt{1 - \rho_t^2} Z_2, \quad (38)$$

where  $\rho_t$  is the correlation between  $Z$  and  $Z_1$  at time  $t$ , and  $Z_1$  and  $Z_2$  are two independent standard normal random variables.

Assuming the real and imaginary parts of  $y_t$  have bivariate normal distribution, we would get:



$$y_t = a + bt + \sigma_1 Z_1 + i\sigma_2 \left( \rho_t Z_1 + \sqrt{1 - \rho_t^2} Z_2 \right), \quad (39)$$

where  $a$  and  $b$  are complex numbers,  $\sigma_1$  and  $\sigma_2$  are respectively the standard deviations of the real and imaginary parts, and  $Z_1$  and  $Z_2$  are two independent standard normal random variables.

In the following sections, to avoid complexity arising from the correlation between the real and imaginary parts of the process, we restrict  $\theta$  to be real. This approach allows the calculation of the moment-generating function of the lognormal distribution, but its characteristic function remains intractable. This is a specific example which shows a limitation of the stochastic approach for the calculation of the characteristic function of normal variate functionals

### 4.3 Differential equation of the first moment

Let us rewrite (35):

$$y_t = \ln \theta + \left( \mu + \frac{1}{2}\sigma^2 \right) t + \frac{1}{2}\sigma^2 \int_0^t e^{y_s} ds + \sigma W_t, \quad (40)$$

where  $y_t = \ln \ln f_t$ .

Whenever  $\theta \in \mathbb{R}$ , we have  $y_t \sim \mathcal{N}(m_t, v_t)$ .

Let us take the expectation of the stochastic differential equation (40). In addition, we use the fact that  $\mathbb{E}(\int_0^t h(t) dt) = \int_0^t \mathbb{E}(h(t)) dt$ . Hence, we get:

$$\mathbb{E}(y_t) = \ln \theta + \left( \mu + \frac{1}{2}\sigma^2 \right) t + \frac{1}{2}\sigma^2 \int_0^t \mathbb{E}(e^{y_t}) ds. \quad (41)$$

Because the expected value of a lognormal variable  $z = e^x$ , where  $x \sim \mathcal{N}(\mu, \sigma)$  is  $\mathbb{E}(z) = e^{\mu + \frac{1}{2}\sigma^2}$ , we get:

$$\mathbb{E}(e^{y_t}) = e^{(m_t + \frac{1}{2}v_t)}. \quad (42)$$

Note that (42) holds when  $\theta \in \mathbb{R}$ , which is the case for the calculation of the moment-generating function. However, this equation is no longer valid when  $\theta$  is a complex number because  $y_t = y_{R,t} + i y_{I,t}$ , and only the real part  $y_{R,t}$  is Gaussian, whereas the imaginary part  $y_{I,t}$  as shown in the previous section, is not normally distributed. The linear combination of a Gaussian random variable with a non-Gaussian random variable is not Gaussian, hence we can no longer use the Gaussianity of  $y_t$  to compute the expected value  $\mathbb{E}(e^{y_t})$ .

We can rewrite (41) as follows:

$$m_t = \ln \theta + \left( \mu + \frac{1}{2}\sigma^2 \right) t + \frac{1}{2}\sigma^2 \int_0^t e^{(m_s + \frac{1}{2}v_s)} ds. \quad (43)$$

By deriving (43) with respect to time, we get:

$$\frac{\partial m_t}{\partial t} = \mu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 e^{(m_t + \frac{1}{2}v_t)}, \quad (44)$$

with initial condition  $m_0 = \ln(\theta)$ , which is the differential equation of the first moment. Note this equation is only valid when  $\theta$  is real but not in the complex plane.

#### 4.4 Differential equation of the variance

Whenever  $\theta \in \mathbb{R}$ , the process (40) is Gaussian with  $y_t \sim \mathcal{N}(m_t, v_t)$ . Hence, we can write:

$$\tilde{y}_t = m_t + \sqrt{v_t}Z, \quad (45)$$

where  $Z \sim \mathcal{N}(0, 1)$ . Before we can extract the differential equation of the variance, we first need to follow a few steps.

Let us derive (45) with respect to time. Hence, we get:

$$\frac{\partial \tilde{y}_t}{\partial t} = m'_t + \frac{1}{2} \frac{v'_t}{\sqrt{v_t}} Z, \quad (46)$$

The variance of (46) is as follows:

$$\text{Var} \left( \frac{\partial \tilde{y}_t}{\partial t} \right) = \frac{1}{4} \frac{v_t'^2}{v_t}. \quad (47)$$

Let us set  $y_t = \ln \ln f_t$ . Hence, (32) can be expressed as follows:

$$dy = \left( \mu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 e^{y_t} \right) dt + \sigma dW_t, \quad (48)$$

Let us integrate (48), therefore we get:

$$y_t - y_0 = \int_0^t \left( \mu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 e^{y_t} \right) dt + \sigma W_t, \quad (49)$$

Now let us derive (49) with respect to time introducing  $W_t = \sqrt{t}Z$ , where  $Z \sim \mathcal{N}(0, 1)$ . We get:

$$\frac{\partial y_t}{\partial t} = \mu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 e^{y_t} + \frac{1}{2}\sigma t^{-1/2}Z, \quad (50)$$

with  $y_t = m_t + \sqrt{v_t}Z$ .

The variance of (50) is as follows:

$$\text{Var} \left( \frac{\partial y_t}{\partial t} \right) = \frac{1}{4}\sigma^4 \text{Var} \left( e^{m_t + \sqrt{v_t}Z} \right) + \frac{1}{4} \frac{\sigma^2}{t} + \frac{1}{2} \frac{\sigma^3}{t^{1/2}} \text{Cov} \left( e^{m_t + \sqrt{v_t}Z}, Z \right). \quad (51)$$

Note that to obtain (51) we used  $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$ .

For a lognormal random variable  $u = e^x$  where  $x \sim \mathcal{N}(\mu, \sigma^2)$ , we have  $\text{Var}(u) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$ . Therefore:

$$\text{Var}\left(e^{m_t + \sqrt{v_t}Z}\right) = (e^{v_t} - 1)e^{2m_t + v_t}. \quad (52)$$

To evaluate the covariance term we need to use the formula  $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ . Because  $\mathbb{E}(Z) = 0$ , the covariance term is equal to  $\mathcal{I} = \mathbb{E}\left(Ze^{m_t + \sqrt{v_t}Z}\right)$ .

We have

$$\mathcal{I} = \int_{-\infty}^{\infty} xe^{m_t + \sqrt{v_t}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx. \quad (53)$$

We can write

$$\mathcal{I} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{m_t + \sqrt{v_t}x - \frac{1}{2}x^2} dx, \quad (54)$$

which is equivalent to

$$\mathcal{I} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{v_t} - x)e^{m_t + \sqrt{v_t}x - \frac{1}{2}x^2} dx + \sqrt{\frac{v_t}{2\pi}} \int_{-\infty}^{\infty} e^{m_t + \sqrt{v_t}x - \frac{1}{2}x^2} dx. \quad (55)$$

The left hand side integral is equal to zero as both branches converge asymptotically to zero, hence:

$$\mathcal{I} = \sqrt{\frac{v_t}{2\pi}} \int_{-\infty}^{\infty} e^{m_t + \sqrt{v_t}x - \frac{1}{2}x^2} dx. \quad (56)$$

Because we have  $m_t + \sqrt{v_t}x - \frac{1}{2}x^2 = -\frac{1}{2}(x - \sqrt{v_t})^2 + m_t + \frac{1}{2}v_t$ , we can write:

$$\mathcal{I} = \sqrt{v_t} e^{(m_t + \frac{1}{2}v_t)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x - \sqrt{v_t})^2} dx. \quad (57)$$

The term inside the integral is the density function of a normal random variable with mean  $\sqrt{v_t}$  and unit variance. Its integral over the domain is equal to one. Therefore, we get:

$$\mathcal{I} = \sqrt{v_t} e^{(m_t + \frac{1}{2}v_t)}. \quad (58)$$

Therefore, we have

$$\text{Var}\left(\frac{\partial y_t}{\partial t}\right) = \frac{1}{4}\sigma^4 (e^{v_t} - 1)e^{2m_t + v_t} + \frac{1}{4}\frac{\sigma^2}{t} + \frac{1}{2}\frac{\sigma^3}{t^{1/2}}\sqrt{v_t} e^{(m_t + \frac{1}{2}v_t)}. \quad (59)$$

By coupling (47) with (59) under  $y_t = \tilde{y}_t$ , we get:

$$\frac{\partial v_t}{\partial t} = \sqrt{\frac{v_t \sigma^2}{t} + v_t \sigma^4 (e^{v_t} - 1)e^{2m_t + v_t} + 2\frac{\sigma^3}{t^{1/2}}v_t^{3/2} e^{(m_t + \frac{1}{2}v_t)}}, \quad (60)$$

with initial conditions  $v_0 = 0$  and  $v'_t|_{t=0} = \sigma^2$ .

Eq. (60) is the differential equation for the variance that we need to compute the moment-generating function of the lognormal distribution. As for the equation of the first moment, the differential equation of the variance is only valid when  $\theta$  is real but not in the complex plane.

#### 4.5 Numerical application

The moment-generating function of the lognormal distribution with parameter  $\mu$  and  $\sigma$  is  $M(\theta) = \mathbb{E}(f_1)$  where  $f_t$  is the random variable of the stochastic differential equation (34). The parameters  $m_1$  and  $v_1$  are evaluated by integration of the differential equation of the first moment (44) and the variance (60) over a unit time interval  $\Delta T = [0, 1]$ .

We solve these integrals by discretisation over the domain of integration  $\Delta T$ , introducing small time increments  $\delta t$ . We start the calculation from time  $t_0 = 0$  and iteratively compute  $\frac{\partial m_t}{\partial t}$  and  $\frac{\partial v_t}{\partial t}$  at each time step. Using piecewise linear segments, we compute  $m_t$  and  $v_t$  at the next time step until we reach  $t_1 = 1$ . The numerical scheme consists of:

$$m_{i+1} = m_i + \left. \frac{\partial m_t}{\partial t} \right|_i \delta t, \quad (61)$$

and

$$v_{i+1} = v_i + \left. \frac{\partial v_t}{\partial t} \right|_i \delta t, \quad (62)$$

for each iteration.

The initial conditions are given by  $m_0 = \ln \theta$ ,  $v_0 = 0$  and  $v'_t|_{t=0} = \sigma^2$ . Once we get the endpoint  $m_1$  we can compute the moment-generating function of the lognormal distribution  $M(\theta)$  using (22). This is the algorithm for the calculation of the moment-generating function of the lognormal distribution using the stochastic approach.

As a reference for this study let us use the approximation of the moment-generating function of the lognormal distribution given by the Laplace transform of the lognormal distribution [1]. This equation, derived via an asymptotic method, is as follows:

$$M(\theta) \approx \frac{\exp\left(-\frac{W^2(-\theta\sigma^2 e^\mu) + 2W(-\theta\sigma^2 e^\mu)}{2\sigma^2}\right)}{\sqrt{1 + W(-\theta\sigma^2 e^\mu)}}, \quad (63)$$

where  $W$  is the Lambert-W function.

The calculations of the moment-generating function of the lognormal distribution are summarized in the below table for a set of  $\theta$  values. We use the parameters  $\sigma = 0.1$  and  $\mu = 0$ . For the discretization algorithm of the stochastic approach we used 2,000 equidistant time steps, and an accuracy of  $1.0 \times 10^{-6}$  for the Lambert function.

**Table 1** Calculations of the moment-generating function of the lognormal distribution

$\theta$	0.1	0.3	0.5	1
Algorithm of the present study	1.105780	1.352504	1.654947	2.745844
Asmussen, Jensen and Rojas-Nandayapa approximation	1.105780	1.352504	1.654957	2.745950

## 5 Conclusion

In the present study we introduced the star probability measure and illustrated some of its application to the calculation of the moment function of the normal and lognormal distributions. While for the normal distribution there exists a closed-form solution; the lognormal distribution does not have a known closed form. At this stage, the star probability method has its own limitations. It can be used to compute the moment-generating function of the lognormal distribution; however, the method presents excess complexity when extended to the calculation of the characteristic function. This is due to the correlation between the real and imaginary parts and the non-Gaussianity of the latter. A sufficient condition for an analytical solution to exist is that the real and imaginary parts be monotonically correlated. Although, in theory, this method is applicable to any functional of a normal random variable, the characteristic function of the lognormal distribution remains a topic of research today.

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