*International Journal for Uncertainty Quantification*, 6 (4): 361–375 (2016)

# NOVEL SINGLE-VALUED NEUTROSOPHIC AGGREGATED OPERATORS UNDER FRANK NORM OPERATION AND ITS APPLICATION TO DECISION-MAKING PROCESS

***Nancy & Harish Garg****

*School of Mathematics, Thapar University Patiala 147004, Punjab, India*

*Address all correspondence to: Harish Garg, E-mail: harishg58iitr@gmail.com

*Original Manuscript Submitted: 8/18/2016; Final Draft Received: 9/28/2016*

*Uncertainties play a dominant role during the aggregation process and hence their corresponding decisions are made fuzzier. Single-value neutrosophic numbers (SVNNs) contain the three ranges: truth, indeterminacy, and falsity membership degrees, and are very useful for describing and handling the uncertainties in the day-to-day life situations. In this study, some operations of SVNNs such as sum, product, and scalar multiplication are defined under Frank norm operations and, based on it, some averaging and geometric aggregation operators have been developed. We further establish some of its properties. Moreover, a decision-making method based on the proposed operators is established and illustrated with a numerical example.*

**KEY WORDS:** *score function, neutrosophic set, expert system, decision making*

## 1. INTRODUCTION

Decision making (DM) is one of the most widely used phenomena in our day-to-day life. Almost all decisions take several steps to reach the final destination and some of them may be vague in nature. On the other hand, with the growing complexities of the systems day-by-day, it is difficult for the decision maker to make a decision within a reasonable time by using uncertain, imprecise, and vague information. For handling this, researchers pay more attention to the fuzzy set (FS) theory (Zadeh, 1965) and corresponding extensions such as an intuitionistic fuzzy set (IFS) theory (Attanassov, 1986), interval-valued IFS (IVIFS) (Atanassov and Gargov, 1989), neutrosophic set (NS) (Smarandache, 1999), etc. To date, IFSs and IVIFSs have been widely applied by the various researchers in different decision-making problems. For instance, various researchers (Xu and Yager, 2006; Garg, 2016a; Xu, 2007; Garg, 2016c,d; Yager, 1988; Xu and Hu, 2010; Garg, 2016b; Xu and Chen, 2007; Garg et al., 2015; Wang and Liu, 2012; Garg, 2015) proposed an aggregation operator for handling the different preferences of the decision makers towards the alternatives under IFS or IVIFS environments. Xu and Zhao (2016) presented a comprehensive analysis of the various methods under IFSs and/or IVIFSs and their corresponding applications in DM problems. Although the FSs or IFSs have been widely used by the researchers, but it cannot deal with indeterminate and inconsistent information. For example, if an expert takes an opinion from a certain person about a certain object, then the person may say that 0.5 is the possibility that the statement is true, 0.7 say that the statement is false, and 0.2 say that he or she is not sure of it. This issue is not handled by the FSs or IFSs. To resolve this, Smarandache (1999) introduced a new component called the "indeterminacy-membership function" and added into the "truth membership function" and "falsity membership function," all are independent components lying in $]0^+, 1^+[$, and hence the corresponding sets are known as neutrosophic sets (NSs), which is the generalization of IFS and FS. However, without specification, NSs are difficult to apply in real-life problems. Thus, an extension of the NS, called a single-valued NSs (SVNSs) and interval-valued NSs (IVNSs) were proposed by Wang et al. (2005, 2010), respectively. Majumdar and Samant (2014) and Ye (2014b) proposed an entropy and similarity measures of SVNSs and IVNSs, respectively. Ye (2013) and Broumi and Smarandache (2013) proposed a correlation coefficient of SVNS and IVNSs. Ye (2014a) and Zhang et al. (2014) proposed an aggregation operator for SVNSs and IVNSs. Later on, Peng et al. (2016) showed that some operations in Ye (2014a) may be unrealistic and hence define the novel operations and aggregation operators for MCDM problems. Rather than ranking of the sets, the various authors (Liu et al., 2014; Ye, 2015; Li et al., 2016; Liu and Shi, 2015; Tian et al., 2016; Broumi and Smarandache, 2014) have studied the aggregation operators in the NS environment by using algebraic, Einstein, Hamacher, etc., t-norm and t-conorm operations of SVNSs. Frank norms are one of the most important compatibility norm. These norms involve the parameter which provides the different choices to the decision maker during the information fusion process and hence make it more adequate to model the decision-making problems than others.

Therefore, in this paper, we present a new method to deal with fuzzy DM problems based on SVNSs under Frank norm operations. To do this, an operational law on different SVNSs and their corresponding averaging and geometric aggregation operators has been proposed. Further, a method within the multicriteria decision analysis based on these operators of SVNSs has been proposed for handling the uncertainties in the collective information. The remainder of the text has been summarized as follows. Section 2 describes the basic concepts of NSs. Section 3 introduces some averaging and geometric aggregation operators under Frank norm operations. In Section 4, an approach to DM, a practical example to validate and demonstrate the approach has been presented and compares it with the previous work. Section 5 concludes the paper.

## 2. PRELIMINARIES

An overview of NS and SVNS has been addressed here on the universal set $X$.

**Definition 2.1.** (Smarandache, 1999) A NS $A$ in $X$ is defined by its "truth membership function" $(T_A(x))$, an "indeterminacy-membership function" $(I_A(x))$, and a "falsity membership function" $(F_A(x))$ where all are the subset of $]0^-, 1^+[$ such that $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$ for all $x \in X$.

**Definition 2.2.** (Ye, 2014a) A NS $A$ is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$$

and is called SVNS where $T_A(x), I_A(x), F_A(x) \in [0,1]$. The pairs of these are called single-valued neutrosophic numbers (SVNNs) denoted by $\alpha = \{\langle T_A(x), I_A(x), F_A(x)\rangle\}$ or $\alpha = \langle a, b, c\rangle$.

To compare the different SVNNs, a comparison law has been defined as follows (Ye, 2014a):

**Definition 2.3.** For a SVNN $\alpha$, $sc(\alpha) = a - b - c$ is called the score function of $\alpha$. For two SVNNs $\alpha$ and $\beta$, if $sc(\alpha) > sc(\beta)$ then $\alpha \succ \beta$;

**Definition 2.4.** The t-norms $\zeta$ and t-conorms $\xi$ defined by $\zeta, \xi : [0,1]^2 \longrightarrow [0,1]$, related by $\xi(x,y) = 1 - \zeta(1-x, 1-y), \forall x, y \in [0,1]$. Based on these norms, a generalized union and intersection for SVNNs $\alpha_1 = \langle a_1, b_1, c_2\rangle$ and $\alpha_2 = \langle a_2, b_2, c_2\rangle$ are defined as $\alpha_1 \bigcap_{\zeta,\xi} \alpha_2 = \langle \zeta(a_1,a_2), \xi(b_1,b_2), \xi(c_1,c_2)\rangle$ and $\alpha_1 \bigcup_{\zeta,\xi} \alpha_2 = \langle \xi(a_1,a_2), \zeta(b_1,b_2), \zeta(c_1,c_2)\rangle$.

**Definition 2.5.** (Frank triangular norm:) Frank t-norm ($\oplus_F$) and t-conorm ($\otimes_F$) are defined as (Frank,1979).

$$\begin{aligned} x \oplus_F y &= 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda - 1}\right), \quad \lambda > 1 \quad \forall (x,y) \in [0,1]^2, \\ x \otimes_F y &= \log_\lambda\left(1 + \frac{(\lambda^{x}-1)(\lambda^{y}-1)}{\lambda - 1}\right), \quad \lambda > 1 \quad \forall (x,y) \in [0,1]^2. \end{aligned}$$

It has been easily verified that the Frank sum and product have the following properties:

- $\left(x \oplus_F y\right) + \left(x \otimes_F y\right) = x + y$
- $(\partial(x \oplus_F y))/\partial x + (\partial(x \otimes_F y))/\partial x = 1.$

**Remark 2.1.** *For some special cases of* $\lambda$*, we see that Frank operations reduces to algebraic and Lukasiewicz sum and product operations.*

*(i) If* $\lambda \to 1$*, then* $x \oplus_F y \equiv x + y - xy$*,* $x \otimes_F y \equiv xy$*, and hence it reduces to an algebraic sum and product operations, respectively.*

*(ii) If* $\lambda \to \infty$*, then* $x \oplus_F y \equiv \min(x+y, 1)$*,* $x \otimes_F y \equiv \max(0, x+y-1)$*, which are the Lukasiewicz sum and product operations, respectively.*

## 3. AGGREGATION OPERATORS FOR SVNNs

Based on the Definition 2.5, we will establish the basic operation laws for SVNNs and their corresponding aggregation operators in this section.

**Definition 3.1.** Let $\alpha_1 = \langle a_1, b_1, c_1 \rangle$ and $\alpha_2 = \langle a_2, b_2, c_2 \rangle$ be two SVNNs, then the operational rules based on Frank norms are defined as follows:

$$\begin{aligned}
\alpha_1 \oplus_F \alpha_2 &= \Bigg\langle 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a_1} - 1)(\lambda^{1-a_2} - 1)}{\lambda - 1}\right), \log_\lambda \left(1 + \frac{(\lambda^{b_1} - 1)(\lambda^{b_2} - 1)}{\lambda - 1}\right), \\
&\quad \log_\lambda \left(1 + \frac{(\lambda^{c_1} - 1)(\lambda^{c_2} - 1)}{\lambda - 1}\right)\Bigg\rangle, \quad \lambda > 1; \\
\alpha_1 \otimes_F \alpha_2 &= \Bigg\langle \log_\lambda \left(1 + \frac{(\lambda^{a_1} - 1)(\lambda^{a_2} - 1)}{\lambda - 1}\right), 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-b_1} - 1)(\lambda^{1-b_2} - 1)}{\lambda - 1}\right), \\
&\quad 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-c_1} - 1)(\lambda^{1-c_2} - 1)}{\lambda - 1}\right)\Bigg\rangle, \quad \lambda > 1.
\end{aligned}$$

**Theorem 3.1.** *The operations defined in Definition 3.1 for two SVNNs* $\alpha_1$ *and* $\alpha_2$ *are also SVNNs.*

*Proof.* Since $\alpha_i$'s are SVNNs and hence $0 \le a_i, b_i, c_i \le 1$ for $i = 1, 2$ so

$$\log_\lambda \left(1 + \frac{(\lambda^{1-1} - 1)(\lambda^{1-1} - 1)}{\lambda - 1}\right) \le \log_\lambda \left(1 + \frac{(\lambda^{1-a_1} - 1)(\lambda^{1-a_2} - 1)}{\lambda - 1}\right) \le \log_\lambda \left(1 + \frac{(\lambda^{1-0} - 1)(\lambda^{1-0} - 1)}{\lambda - 1}\right);$$

i.e.,

$$0 \le \log_\lambda \left(1 + \frac{(\lambda^{1-a_1} - 1)(\lambda^{1-a_2} - 1)}{\lambda - 1}\right) \le 1.$$

Hence

$$0 \le 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a_1} - 1)(\lambda^{1-a_2} - 1)}{\lambda - 1}\right) \le 1.$$

Similarly,

$$0 \le \log_\lambda \left(1 + \frac{(\lambda^{b_1} - 1)(\lambda^{b_2} - 1)}{\lambda - 1}\right) \le 1$$

and

$$0 \le \log_\lambda \left(1 + \frac{(\lambda^{c_1} - 1)(\lambda^{c_2} - 1)}{\lambda - 1}\right) \le 1.$$

Further,

$$0 \le 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a_1} - 1)(\lambda^{1-a_2} - 1)}{\lambda - 1}\right) + \log_\lambda \left(1 + \frac{(\lambda^{b_1} - 1)(\lambda^{b_2} - 1)}{\lambda - 1}\right) + \log_\lambda \left(1 + \frac{(\lambda^{c_1} - 1)(\lambda^{c_2} - 1)}{\lambda - 1}\right) \le 3$$

which indicates $\alpha_1 \oplus_F \alpha_2$ is SVNN. Similarly, we can prove that $\alpha_1 \otimes_F \alpha_2$ is also SVNN. □

**Theorem 3.2.** *Let $n$ be any positive integer and $\alpha_1 = \langle a, b, c\rangle$ is a SVNN, then*

$$n \cdot_F \alpha = \left\langle 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a} - 1)^n}{(\lambda - 1)^{n-1}}\right), \log_\lambda \left(1 + \frac{(\lambda^{b} - 1)^n}{(\lambda - 1)^{n-1}}\right), \log_\lambda \left(1 + \frac{(\lambda^{c} - 1)^n}{(\lambda - 1)^{n-1}}\right) \right\rangle, \lambda > 0 \tag{1}$$

*is also SVNN, where $n \cdot_F \alpha = \alpha \bigoplus_F \alpha \bigoplus_F \cdots \bigoplus_F \alpha$.*

*Proof.* We prove the results by induction on $n$. For $n = 2$, we have by Definition 3.1

$$\begin{aligned} 2 \cdot_F \alpha &= \left\langle 1 - \log_\lambda \left(1 + \frac{((\lambda^{1-a} - 1)^1)((\lambda^{1-a} - 1)^1)}{(\lambda - 1)}\right), \log_\lambda \left(1 + \frac{((\lambda^{b} - 1)^1)((\lambda^{b} - 1)^1)}{(\lambda - 1)}\right),\right. \\ &\quad \left. \log_\lambda \left(1 + \frac{((\lambda^{c} - 1)^1)((\lambda^{c} - 1)^1)}{(\lambda - 1)}\right)\right\rangle \\ &= \left\langle 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a} - 1)^2}{(\lambda - 1)^{2-1}}\right), \log_\lambda \left(1 + \frac{(\lambda^{b} - 1)^2}{(\lambda - 1)^{2-1}}\right), \log_\lambda \left(1 + \frac{(\lambda^{c} - 1)^2}{(\lambda - 1)^{2-1}}\right)\right\rangle. \end{aligned}$$

Thus, result holds for $n = 2$. Assume it holds for $n = k$. Now, for $n = k + 1$, we have to prove

$$(k+1) \cdot_F \alpha = \left\langle 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a} - 1)^{k+1}}{(\lambda - 1)^{k}}\right), \log_\lambda \left(1 + \frac{(\lambda^{b} - 1)^{k+1}}{(\lambda - 1)^{k}}\right), \log_\lambda \left(1 + \frac{(\lambda^{c} - 1)^{k+1}}{(\lambda - 1)^{k}}\right)\right\rangle.$$

The left-hand side can be rewritten as $(k + 1)\alpha = k\alpha \oplus_F \alpha$, and based on operations defined in Definition 3.1, we have

$$\begin{aligned} (k \cdot_F \alpha) \oplus_F \alpha &= \left\langle 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a} - 1)^{k}}{(\lambda - 1)^{k-1}}\right), \log_\lambda \left(1 + \frac{(\lambda^{b} - 1)^{k}}{(\lambda - 1)^{k-1}}\right),\right. \\ &\quad \left. \log_\lambda \left(1 + \frac{(\lambda^{c} - 1)^{k}}{(\lambda - 1)^{k-1}}\right)\right\rangle \oplus_F (a, b, c) \\ &= \left\langle 1 - \log_\lambda \left(1 + \frac{\left(\lambda^{1-\left(1-\log_\lambda(1+[(\lambda^{1-a}-1)^k/(\lambda-1)^{k-1}])\right)} - 1\right)(\lambda^{1-a} - 1)}{(\lambda - 1)}\right),\right. \\ &\quad \log_\lambda \left(1 + \frac{(\lambda^{\log_\lambda(1+[(\lambda^{b}-1)^k/(\lambda-1)^{k-1}])} - 1)(\lambda^{b} - 1)}{(\lambda - 1)}\right), \\ &\quad \left. \log_\lambda \left(1 + \frac{(\lambda^{\log_\lambda(1+[(\lambda^{c}-1)^k/(\lambda-1)^{k-1}])} - 1)(\lambda^{c} - 1)}{(\lambda - 1)}\right)\right\rangle \\ &= \left\langle 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a} - 1)^{k+1}}{(\lambda - 1)^{k}}\right), \log_\lambda \left(1 + \frac{(\lambda^{b} - 1)^{k+1}}{(\lambda - 1)^{k}}\right), \log_\lambda \left(1 + \frac{(\lambda^{c} - 1)^{k+1}}{(\lambda - 1)^{k}}\right)\right\rangle. \end{aligned}$$

Therefore, the result holds for $n = k + 1$. It can easily be verified that

$$\begin{aligned} 0 &= 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-0} - 1)^n}{(\lambda - 1)^{n-1}}\right) \leq 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a} - 1)^n}{(\lambda - 1)^{n-1}}\right) \leq 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-1} - 1)^n}{(\lambda - 1)^{n-1}}\right) = 1, \\ 0 &= \log_\lambda \left(1 + \frac{(\lambda^{0} - 1)^n}{(\lambda - 1)^{n-1}}\right) \leq \log_\lambda \left(1 + \frac{(\lambda^{b} - 1)^n}{(\lambda - 1)^{n-1}}\right) \leq \log_\lambda \left(1 + \frac{(\lambda^{1} - 1)^n}{(\lambda - 1)^{n-1}}\right) = 1, \\ 0 &= \log_\lambda \left(1 + \frac{(\lambda^{0} - 1)^n}{(\lambda - 1)^{n-1}}\right) \leq \log_\lambda \left(1 + \frac{(\lambda^{c} - 1)^n}{(\lambda - 1)^{n-1}}\right) \leq \log_\lambda \left(1 + \frac{(\lambda^{1} - 1)^n}{(\lambda - 1)^{n-1}}\right) = 1. \end{aligned}$$

Clearly,

$$0 \leq 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a} - 1)^n}{(\lambda - 1)^{n-1}}\right) + \log_\lambda \left(1 + \frac{(\lambda^{b} - 1)^n}{(\lambda - 1)^{n-1}}\right) + \log_\lambda \left(1 + \frac{(\lambda^{c} - 1)^n}{(\lambda - 1)^{n-1}}\right) \leq 3$$

and hence $n \cdot_F \alpha$ is SVNN. □

**Theorem 3.3.** *If $n \in Z^+$ and $\alpha = \langle a, b, c\rangle$ is SVNN, then operation $\alpha^n$ defined as*

$$\alpha^n = \left\langle \log_\lambda \left(1 + \frac{(\lambda^{a} - 1)^n}{(\lambda - 1)^{n-1}}\right), 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-b} - 1)^n}{(\lambda - 1)^{n-1}}\right), 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-c} - 1)^n}{(\lambda - 1)^{n-1}}\right) \right\rangle$$

*is SVNN, where $\alpha^n = \alpha \otimes_F \alpha \otimes_F \ldots \otimes_F \alpha$.*

*Proof.* Follow from Theorem 3.2. □

**Theorem 3.4.** *(Commutative law) Let $\alpha_i = \langle a_i, b_i, c_i\rangle (i = 1, 2)$ be two SVNNs, then*

*(i)* $\alpha_1 \oplus_F \alpha_2 = \alpha_2 \oplus_F \alpha_1$,

*(ii)* $\alpha_1 \otimes_F \alpha_2 = \alpha_2 \otimes_F \alpha_1$.

**Theorem 3.5.** *(Associative law) Let $\alpha_i = \langle a_i, b_i, c_i\rangle (i = 1, 2, 3)$ be two SVNNs, then*

*(i)* $(\alpha_1 \oplus_F \alpha_2) \oplus_F \alpha_3 = \alpha_1 \oplus_F (\alpha_2 \oplus_F \alpha_3)$,

*(ii)* $(\alpha_1 \otimes_F \alpha_2) \otimes_F \alpha_3 = \alpha_1 \otimes_F (\alpha_2 \otimes_F \alpha_3)$.

Theorems 3.4 and 3.5 are straightforward and we omit their proofs.

**Theorem 3.6.** *If $\alpha_i = \langle a_i, b_i, c_i\rangle (i = 1, 2)$ are two SVNNs, and $\eta > 0$ is a real number, then*

*(i)* $\eta(\alpha_1 \oplus_F \alpha_2) = \eta\alpha_1 \oplus_F \eta\alpha_2$,

*(ii)* $(\alpha_1 \otimes_F \alpha_2)^\eta = (\alpha_1)^\eta \otimes_F (\alpha_2)^\eta$,

*(iii)* $\eta_1\alpha_1 \oplus_F \eta_2\alpha_1 = (\eta_1 + \eta_2)\alpha_1$,

*(iv)* $(\alpha_1)^{\eta_1} \otimes_F (\alpha_1)^{\eta_2} = (\alpha_1)^{\eta_1+\eta_2}$.

*Proof.* We prove parts (i) and (iii) and hence similarly for other.

(i) For SVNNs $\alpha_1, \alpha_2$ and real number $\eta > 0$, we have

$$
\begin{aligned}
&\eta(\alpha_1 \oplus_F \alpha_2) \\
&= \left\langle 1 - \log_\lambda \left( 1 + \frac{\left(\lambda^{\log_\lambda\left(1+\left[\left(\lambda^{1-a_1}-1\right)\left(\lambda^{1-a_2}-1\right)/(\lambda-1)\right]\right)} - 1\right)^\eta}{(\lambda-1)^{\eta-1}} \right), \right. \\
&\quad \log_\lambda \left( 1 + \frac{\left(\lambda^{\log_\lambda\left(1+\left[\left(\lambda^{b_1}-1\right)\left(\lambda^{b_2}-1\right)/(\lambda-1)\right]\right)} - 1\right)^\eta}{(\lambda-1)^{(\eta-1)}} \right), \\
&\quad \left. \log_\lambda \left( 1 + \frac{\left(\lambda^{\log_\lambda(1+[(\lambda^{c_1}-1)(\lambda^{c_2}-1)/(\lambda-1)])} - 1\right)^\eta}{(\lambda-1)^{(\eta-1)}} \right) \right\rangle \\
&= \left\langle 1 - \log_\lambda \left( 1 + \frac{\left[\left(\lambda^{(1-a_1)} - 1\right)^\eta \left(\lambda^{(1-a_2)} - 1\right)^\eta / (\lambda-1)^\eta\right]}{(\lambda-1)^{(\eta-1)}} \right), \right. \\
&\log_\lambda \left( 1 + \frac{\left[\left(\lambda^{b_1} - 1\right)^\eta \left(\lambda^{b_2} - 1\right)^\eta / (\lambda-1)^{(\eta)}\right]}{(\lambda-1)^{(\eta-1)}} \right), \\
&\left. \log_\lambda \left( 1 + \frac{\left[(\lambda^{c_1} - 1)^\eta (\lambda^{c_2} - 1)^\eta / (\lambda-1)^{(\eta)}\right]}{(\lambda-1)^{(\eta-1)}} \right) \right\rangle \\
&= \left\langle 1 - \log_\lambda \left( 1 + \frac{\left(\lambda^{(1-a_1)} - 1\right)^\eta \left(\lambda^{(1-a_2)} - 1\right)^\eta}{(\lambda-1)^{(2\eta-1)}} \right), \right. \\
&\left. \log_\lambda \left( 1 + \frac{\left(\lambda^{b_1} - 1\right)^\eta \left(\lambda^{b_2} - 1\right)^\eta}{(\lambda-1)^{(2\eta-1)}} \right), \log_\lambda \left( 1 + \frac{(\lambda^{c_1} - 1)^\eta (\lambda^{c_2} - 1)^\eta}{(\lambda-1)^{(2\eta-1)}} \right) \right\rangle \\
&= \left\langle 1 - \log_\lambda \left( 1 + \frac{\left[\left(\lambda^{(1-a_1)} - 1\right)^\eta / (\lambda-1)^{(\eta-1)}\right] \left[\left(\lambda^{(1-a_2)} - 1\right)^\eta / (\lambda-1)^{(\eta-1)}\right]}{(\lambda-1)} \right), \right. \\
&\quad \log_\lambda \left( 1 + \frac{\left[\left(\lambda^{b_1} - 1\right)^\eta / (\lambda-1)^{(\eta-1)}\right] \left[\left(\lambda^{b_2} - 1\right)^\eta / (\lambda-1)^{(\eta-1)}\right]}{\lambda-1} \right), \\
&\quad \left. \log_\lambda \left( 1 + \frac{\left[(\lambda^{c_1} - 1)^\eta / (\lambda-1)^{(\eta-1)}\right] \left[(\lambda^{c_2} - 1)^\eta / (\lambda-1)^{(\eta-1)}\right]}{\lambda-1} \right) \right\rangle \\
&= \left\langle 1 - \log_\lambda \left( \frac{\left(\lambda^{\log_\lambda\left(1+\left[\left(\lambda^{(1-a_1)^\eta}-1\right)/(\lambda-1)^{(\eta-1)}\right]\right)} - 1\right) \left(\lambda^{\log_\lambda\left(1+\left[\left(\lambda^{(1-a_2)^\eta}-1\right)/(\lambda-1)^{(\eta-1)}\right]\right)} - 1\right)}{(\lambda-1)} \right), \right. \\
&\quad \log_\lambda \left( 1 + \frac{\left(\lambda^{\log_\lambda\left(1+\left[\left(\lambda^{b_1}-1\right)^\eta/(\lambda-1)^{(\eta-1)}\right]\right)} - 1\right) \left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{a_2}-1)^\eta/(\lambda-1)^{(\eta-1)}\right]\right)} - 1\right)}{(\lambda-1)} \right), \\
&\quad \left. \log_\lambda \left( 1 + \frac{\left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{c_1}-1)^\eta/(\lambda-1)^{(\eta-1)}\right]\right)} - 1\right) \left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{c_2}-1)^\eta/(\lambda-1)^{(\eta-1)}\right]\right)} - 1\right)}{(\lambda-1)} \right) \right\rangle \\
&= \eta\alpha_1 \oplus_F \eta\alpha_2,
\end{aligned}
$$

(iii) For real $\eta_1, \eta_2 > 0$, we have

$$\eta_1\alpha_1 \oplus_F \eta_2\alpha_1$$
$$= \Bigg\langle 1 - \log_\lambda \Bigg(1 + \frac{\Big(\lambda^{\log_\lambda\left(1+\left[(\lambda^{(1-a_1)}-1)^{\eta_1}/(\lambda-1)^{(\eta_1-1)}\right]\right)} - 1\Big)}{\lambda - 1}$$
$$\times \Big(\lambda^{\log_\lambda\left(1+\left[(\lambda^{(1-a_1)}-1)^{\eta_2}/(\lambda-1)^{(\eta_2-1)}\right]\right)} - 1\Big)\Bigg),$$
$$\log_\lambda\Bigg(1 + \frac{\Big(\lambda^{\log_\lambda\left(1+\left[(\lambda^{(b_1)}-1)^{\eta_1}/(\lambda-1)^{(\eta_1-1)}\right]\right)} - 1\Big)\Big(\lambda^{\log_\lambda\left(1+\left[(\lambda^{(b_1)}-1)^{\eta_2}/(\lambda-1)^{(\eta_2-1)}\right]\right)} - 1\Big)}{\lambda - 1}\Bigg),$$
$$\log_\lambda\Bigg(1 + \frac{\Big(\lambda^{\log_\lambda\left(1+\left[(\lambda^{(c_1)}-1)^{\eta_1}/(\lambda-1)^{(\eta_1-1)}\right]\right)} - 1\Big)\Big(\lambda^{\log_\lambda\left(1+\left[(\lambda^{(c_1)}-1)^{\eta_2}/(\lambda-1)^{(\eta_2-1)}\right]\right)} - 1\Big)}{\lambda - 1}\Bigg)\Bigg\rangle$$
$$= \Bigg\langle 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-a_1}-1)^{(\eta_1+\eta_2)}}{(\lambda-1)^{(\eta_1+\eta_2-1)}}\right), \log_\lambda\left(1 + \frac{(\lambda^{b_1}-1)^{(\eta_1+\eta_2)}}{(\lambda-1)^{(\eta_1+\eta_2-1)}}\right),$$
$$\log_\lambda\left(1 + \frac{(\lambda^{c_1}-1)^{(\eta_1+\eta_2)}}{(\lambda-1)^{(\eta_1+\eta_2-1)}}\right)\Bigg\rangle$$
$$= (\eta_1 + \eta_2)\alpha_1.$$

□

Based on the Definition 3.1, we will discuss some averaging and geometric aggregation operators for the set of all SVNNs denoted by $\Omega$.

### 3.1 Weighted Averaging Operator

**Definition 3.2.** Let $\alpha_i = \langle a_i, b_i, c_i\rangle$ be $n$ collections of SVNNs, then *SVNFWA* operator is a mapping, *SVNFWA*: $\Omega^n \to \Omega$, defined by

$$\textit{SVNFWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = (w_1 \cdot_F \alpha_1) \oplus_F (w_2 \cdot_F \alpha_2) \oplus_F \ldots \oplus_F (w_n \cdot_F \alpha_n), \tag{2}$$

where $w = (w_1, w_2, ..., w_n)^T$ is the normalized weight factor of $\alpha_i$'s.

**Theorem 3.7.** *The aggregated value by using the SVNFWA operator is also SVNN and is expressed as*

$$\textit{SVNFWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \Bigg\langle 1 - \log_\lambda\big(1 + \prod_{i=1}^{n}(\lambda^{1-a_i}-1)^{w_i}\big), \log_\lambda\big(1 + \prod_{i=1}^{n}(\lambda^{b_i}-1)^{w_i}\big),$$
$$\log_\lambda\big(1 + \prod_{i=1}^{n}(\lambda^{c_i}-1)^{w_i}\big)\Bigg\rangle. \tag{3}$$

*Proof.* In order to prove the above result, it is sufficient to prove that Eq. (4) holds for any vector $w$.

$$\textit{SVNFWA}(\alpha_1, \alpha_2, ..., \alpha_n) = \Bigg\langle 1 - \log_\lambda\left(1 + \frac{\prod_{i=1}^{n}(\lambda^{1-a_i}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{n} w_i - 1}}\right), \log_\lambda\left(1 + \frac{\prod_{i=1}^{n}(\lambda^{b_i}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{n} w_i - 1}}\right),$$
$$\log_\lambda\left(1 + \frac{\prod_{i=1}^{n}(\lambda^{c_i}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{n} w_i - 1}}\right)\Bigg\rangle. \tag{4}$$

We prove this by induction on $n$. Now, for $n = 2$, we have

$$w_1 \cdot_F a_1 = \left\langle 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-a_1}-1)^{w_1}}{(\lambda-1)^{w_1-1}}\right), \log_\lambda\left(1 + \frac{(\lambda^{b_1}-1)^{w_1}}{(\lambda-1)^{w_1-1}}\right), \log_\lambda\left(1 + \frac{(\lambda^{c_1}-1)^{w_1}}{(\lambda-1)^{w_1-1}}\right)\right\rangle,$$

$$w_2 \cdot_F a_2 = \left\langle 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-a_2}-1)^{w_2}}{(\lambda-1)^{w_2-1}}\right), \log_\lambda\left(1 + \frac{(\lambda^{b_2}-1)^{w_1}}{(\lambda-1)^{w_2-1}}\right), \log_\lambda\left(1 + \frac{(\lambda^{c_2}-1)^{w_2}}{(\lambda-1)^{w_2-1}}\right)\right\rangle,$$

and hence

$$\begin{aligned}
&\mathit{SVNFWA}(\alpha_1,\alpha_2) = (w_1 \cdot_F \alpha_1) \oplus_F (w_2 \cdot_F \alpha_2)\\
&= \left\langle 1 - \log_\lambda\left(1 + \frac{\left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{1-a_1}-1)^{w_1}/(\lambda-1)^{w_1-1}\right]\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{1-a_2}-1)^{w_2}/(\lambda-1)^{w_2-1}\right]\right)}-1\right)}{\lambda-1}\right),\right.\\
&\quad \log_\lambda\left(1 + \frac{\left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{b_1}-1)^{w_1}/(\lambda-1)^{w_1-1}\right]\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{b_2}-1)^{w_2}/(\lambda-1)^{w_2-1}\right]\right)}-1\right)}{\lambda-1}\right),\\
&\quad \left.\log_\lambda\left(1 + \frac{\left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{c_1}-1)^{w_1}/(\lambda-1)^{w_1-1}\right]\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{c_2}-1)^{w_2}/(\lambda-1)^{w_2-1}\right]\right)}-1\right)}{\lambda-1}\right)\right\rangle\\
&= \left\langle 1 - \log_\lambda\left(1 + \frac{\prod_{i=1}^{2}(\lambda^{1-a_i}-1)^{w_i}}{(\lambda-1)^{w_1+w_2-1}}\right), \log_\lambda\left(1 + \frac{\prod_{i=1}^{2}(\lambda^{b_i}-1)^{w_i}}{(\lambda-1)^{w_1+w_2-1}}\right), \log_\lambda\left(1 + \frac{\prod_{i=1}^{2}(\lambda^{c_i}-1)^{w_i}}{(\lambda-1)^{w_1+w_2-1}}\right)\right\rangle.
\end{aligned}$$

Thus the result is true for $n = 2$. Assume the result holds for $n = k$, then for $n = k+1$, we have

$$\begin{aligned}
&\mathit{SVNFWA}(\alpha_1,\alpha_2,\ldots,\alpha_k,\alpha_{k+1}) = \mathit{SVNFWA}(\alpha_1,\alpha_2,\ldots,\alpha_k) \oplus_F (w_{k+1} \cdot_F \alpha_{k+1})\\
&= \left\langle 1 - \log_\lambda\left(1 + \frac{\left(\lambda^{\log_\lambda\left(1+\left[\prod_{i=1}^{k}(\lambda^{1-a_i}-1)^{w_i}/(\lambda-1)^{\sum_{i=1}^{k}w_i-1}\right]-1\right)}\left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{1-a_{k+1}}-1)^{w_{k+1}}/(\lambda-1)^{w_{k+1}-1}\right]\right)}\right)-1\right)}{\lambda-1}\right),\right.\\
&\quad \log_\lambda\left(1 + \frac{\left(\lambda^{\log_\lambda\left(1+\left[\prod_{i=1}^{k}(\lambda^{b_i}-1)^{w_i}/(\lambda-1)^{\sum_{i=1}^{k}w_i-1}\right]-1\right)}\left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{I_{k+1}}-1)^{w_{k+1}}/(\lambda-1)^{w_{k+1}-1}\right]\right)}\right)-1\right)}{\lambda-1}\right),\\
&\quad \left.\log_\lambda\left(1 + \frac{\left(\lambda^{\log_\lambda\left(1+\left[\prod_{i=1}^{k}\left(\lambda^{c_i}-1\right)^{w_i}/(\lambda-1)^{\sum_{i=1}^{k}w_i-1}\right]-1\right)}\left(\lambda^{\log_\lambda\left(1+\left[(\lambda^{F_{k+1}}-1)^{w_{k+1}}/(\lambda-1)^{w_{k+1}-1}\right]\right)}\right)-1\right)}{\lambda-1}\right)\right\rangle\\
&= \left\langle 1 - \log_\lambda\left(1 + \frac{\prod_{i=1}^{k+1}(\lambda^{1-a_i}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k+1}w_i-1}}\right), \log_\lambda\left(1 + \frac{\prod_{i=1}^{k+1}(\lambda^{b_i}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k+1}w_i-1}}\right), \log_\lambda\left(1 + \frac{\prod_{i=1}^{k+1}(\lambda^{b_i}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k+1}w_i-1}}\right)\right\rangle.
\end{aligned}$$

Therefore, the result holds for $n = k+1$. □

**Property 3.1.** If all SVNNs $\alpha_i$'s are equal to $\alpha$ then we have

$$\mathit{SVNFWA}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \alpha.$$

*Proof.*

$$\begin{aligned}
&SVNFWA(\alpha, \alpha, ..., \alpha)\\
&= \left\langle 1 - \log_\lambda\left(1 + \prod_{i=1}^{n}\left(\lambda^{1-a_i} - 1\right)^{w_i}\right), \log_\lambda\left(1 + \prod_{i=1}^{n}\left(\lambda^{b_i} - 1\right)^{w_i}\right), \log_\lambda\left(1 + \prod_{i=1}^{n}\left(\lambda^{c_i} - 1\right)^{w_i}\right)\right\rangle\\
&= \left\langle 1 - \log_\lambda\left(1 + \left(\lambda^{1-a} - 1\right)^{\sum_{i=1}^{n} w_i}\right), \log_\lambda\left(1 + \left(\lambda^{b} - 1\right)^{\sum_{i=1}^{n} w_i}\right), \log_\lambda\left(1 + \left(\lambda^{c} - 1\right)^{\sum_{i=1}^{n} w_i}\right)\right\rangle\\
&= \left\langle 1 - \log_\lambda\left(1 + \left(\lambda^{1-a} - 1\right)\right), \log_\lambda\left(1 + \left(\lambda^{b} - 1\right)\right), \log_\lambda\left(1 + \left(\lambda^{c} - 1\right)\right)\right\rangle\\
&= \langle a, b, c\rangle\\
&= \alpha.
\end{aligned}$$

□

**Property 3.2.** (Monotonicity) Let $\alpha_i = \langle a_i, b_i, c_i\rangle$ and $\alpha_i' = \langle a_i', b_i', c_i'\rangle, (i = 1, 2, ..., n)$ be two collections of SVNNs such that $\alpha_i \leq \alpha_i'$, i.e., $a_i \leq a_i'$, $b_i \geq b_i'$ and $c_i \geq c_i'$, for all $i$, then *SVNFWA*$(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq$ *SVNFWA*$(\alpha_1', \alpha_2', \ldots, \alpha_n')$.

*Proof.* Let $\alpha_i$ and $\alpha_i'$ are two SVNNs such that for all $i$, $a_i \leq a_i'$, $b_i \geq b_i'$, and $c_i \geq c_i'$ and let $\lambda \geq 1$ be a real number. Therefore,

$$\begin{aligned}
\lambda^{1-a_i} \geq \lambda^{1-a_i'} &\Leftrightarrow 1 + \prod_{i=1}^{n}(\lambda^{1-a_i} - 1)^{w_i} \geq 1 + \prod_{i=1}^{n}(\lambda^{1-a_i'} - 1)^{w_i}\\
&\Leftrightarrow 0 \leq \frac{1 + \prod_{i=1}^{n}(\lambda^{1-ai'} - 1)^{w_i}}{1 + \prod_{i=1}^{n}(\lambda^{1-a_i} - 1)^{w_i}} \leq 1 \Leftrightarrow \log_\lambda\left(\frac{1 + \prod_{i=1}^{n}(\lambda^{1-a_i'} - 1)^{w_i}}{1 + \prod_{i=1}^{n}(\lambda^{1-a_i} - 1)^{w_i}}\right) \leq 0\\
&\Leftrightarrow 1 - \log_\lambda\left(1 + \prod_{i=1}^{n}(\lambda^{1-a_i} - 1)^{w_i}\right) \leq 1 - \log_\lambda\left(1 + \prod_{i=1}^{n}(\lambda^{1-a_i'} - 1)^{w_i}\right).
\end{aligned}$$

Further,

$$\begin{aligned}
b_i \geq b_i' &\Leftrightarrow 1 + \prod_{i=1}^{n}(\lambda^{b_i} - 1)^{w_i} \geq 1 + \prod_{i=1}^{n}(\lambda^{b_i'} - 1)^{w_i} \Leftrightarrow \log_\lambda\left(\frac{1 + \prod_{i=1}^{n}(\lambda^{b_i} - 1)^{w_i}}{1 + \prod_{i=1}^{n}(\lambda^{b_i'} - 1)^{w_i}}\right) \geq 0\\
&\Leftrightarrow \log_\lambda\left(1 + \prod_{i=1}^{n}(\lambda^{b_i} - 1)^{w_i}\right) \geq \log_\lambda\left(1 + \prod_{i=1}^{n}(\lambda^{b_i'} - 1)^{w_i}\right).
\end{aligned}$$

Similarly,

$$\log_\lambda\left(1 + \prod_{i=1}^{n}(\lambda^{c_i} - 1)^{w_i}\right) \geq \log_\lambda\left(1 + \prod_{i=1}^{n}(\lambda^{c_i'} - 1)^{w_i}\right).$$

Therefore, by the score function of SVNN, we get *SVNFWA*$(\alpha_1, \alpha_2, ..., \alpha_n) \leq$ *SVNFWA*$(\alpha_1', \alpha_2', ..., \alpha_n)$. □

**Property 3.3.** For a collection of SVNNs $\alpha_i$'s, take $\alpha^- = \langle \min_i a_i, \max_i b_i, \max_i c_i\rangle$ and $\alpha^+ = \langle \max_i a_i, \min_i b_i, \min_i c_i\rangle$, then $\alpha^- \leq$ *SVNFWA*$(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+$

*Proof.* Proof follows from the above property. □

### 3.2 Weighted Geometric Operator

**Definition 3.3.** Let $\alpha_i = \langle a_i, b_i, c_i \rangle$ be $n$ collections of SVNNs, then *SVNFWG* operator is a mapping, *SVNFWG*: $\Omega^n \to \Omega$, defined by

$$\mathit{SVNFWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_1^{w_1} \otimes_F \alpha_2^{w_2} \otimes_F \ldots \otimes_F \alpha_n^{w_n}, \tag{5}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the normalized weight factor of $\alpha_i$'s.

**Theorem 3.8.** *The aggregated value by using Definition 3.3 is SVNN and is given by*

$$\begin{aligned}\mathit{SVNFWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \quad = \quad & \Bigg\langle \log_\lambda \left(1 + \prod_{i=1}^{n} (\lambda^{a_i} - 1)^{w_i}\right), 1 - \log_\lambda \left(1 + \prod_{i=1}^{n} (\lambda^{1-b_i} - 1)^{w_i}\right), \\ & 1 - \log_\lambda \left(1 + \prod_{i=1}^{n} (\lambda^{1-c_i} - 1)^{w_i}\right) \Bigg\rangle. \end{aligned} \tag{6}$$

*Proof.* Follows from Theorem 3.7. □

Based on this theorem, some desirable properties of it have been pointed out for a collection of SVNNs $\alpha_i$'s as

(P1) (Idempotency:) If $\alpha_i = \alpha$ for each $i$ then $\mathit{SVNFWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha$.

(P2) (Monotonicity:) If $\alpha_i \leq \alpha_i'$ for each $i$ then $\mathit{SVNFWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \mathit{SVNFWG}(\alpha_1', \alpha_2', \ldots, \alpha_n')$.

(P3) (Monotonicity:) Let $\alpha^-$ and $\alpha^+$ be lower and upper limits of $\alpha_i$'s, then $\alpha^- \leq \mathit{SVNFWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+$.

## 4. DECISION-MAKING METHOD BASED ON PROPOSED OPERATORS

This section describes the decision-making method based on proposed operators followed by an illustrative example for demonstrating and effectiveness of it. A sensitivity analysis of the decision parameter has also been given.

### 4.1 Proposed Approach

Consider a problem of DM in which a decision maker wants to select the best alternative out of $A_1, A_2, \ldots, A_m$ which are to be evaluated under the set of criteria $C_1, C_2, \ldots, C_n$ whose normalized weight vector is $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$. Assume that they are evaluated and give their preferences in terms of SVNNs $\alpha_{ij} = \langle a_{ij}, b_{ij}, c_{ij} \rangle$ where $a_{ij}, b_{ij}$ and $c_{ij}$ represent the degrees of "truth membership function," "indeterminacy-membership function," and a "falsity membership function" such that $0 \leq a_{ij}, b_{ij}, c_{ij} \leq 1$ and $a_{ij} + b_{ij} + c_{ij} \leq 3$. Therefore, the overall collective neutrosophic matrix is $D = (\alpha_{ij})_{m \times n}$. Since the different criteria may be of different types, namely benefit or cost, then there is a need to normalize it. For this, the value of the benefit type is converted into the cost type by using the following equation (Xu and Hu, 2010):

$$r_{ij} = \begin{cases} \alpha_{ij}^c, & \text{for benefit criteria} \\ \alpha_{ij}, & \text{for cost criteria} \end{cases}, \tag{7}$$

where $\alpha_{ij}^c$ is the complement of SVNNs $\alpha_{ij}$ and hence the matrix $D$ is converted into matrix $R = (r_{ij})_{m \times n}$. Then we have the following methods for MCDM based on the proposed function.

Step 1: Transform the matrix $D$ into matrix $R$ by using Eq. (7).

Step 2: Aggregate the SVNNs into the collective SVNN either by using

(i) *SVNFWA* operator:

$$r_i = \textit{SVNFWA}(r_{i1}, r_{i2}, \ldots, r_{in})$$

or

(ii) *SVNFWG* operator:

$$r_i = \textit{SVNFWG}(r_{i1}, r_{i2}, \ldots, r_{in}).$$

Step 3: Compare $r_i (i = 1, 2, \ldots, m)$ by Definition 2.3 and hence select the best alternative(s).

Step 4: End.

## 4.2 Illustrative Example

A computer center in a certain university wants to improve the work productivity. To do this they want to select a new information system from the set of four different alternatives $A_i, i = 1, 2, 3, 4$ which are evaluated by the decision maker under the different criteria, namely, the "cost of hardware/software" $(C_1)$, "contribution to organizational performance" $(C_2)$, and "effort to transform from current system" $(C_3)$ whose weight vector is $\omega = (0.4, 0.2, 0.4)^T$. After evaluation, the rating values of these alternatives are summarized in the form of SVNNs as below.

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccc} \quad C_1 \quad & \quad C_2 \quad & \quad C_3 \quad \end{array} \\ \begin{bmatrix} \langle 0.265, 0.350, 0.385 \rangle & \langle 0.280, 0.610, 0.330 \rangle & \langle 0.245, 0.275, 0.480 \rangle \\ \langle 0.345, 0.245, 0.410 \rangle & \langle 0.280, 0.710, 0.430 \rangle & \langle 0.245, 0.375, 0.380 \rangle \\ \langle 0.365, 0.300, 0.335 \rangle & \langle 0.205, 0.685, 0.480 \rangle & \langle 0.340, 0.370, 0.290 \rangle \\ \langle 0.430, 0.300, 0.270 \rangle & \langle 0.295, 0.755, 0.460 \rangle & \langle 0.310, 0.520, 0.170 \rangle \end{bmatrix} \end{array}.$$

Then by utilizing the proposed *SVNFWA* operator, we obtain the most desirable alternative(s) as follows.

Step 1: Since $C_1$ & $C_3$ are the cost criteria and $C_2$ is the benefit criterion, hence the transform matrix by using Eq. (7) becomes

$$R = \begin{bmatrix} \langle 0.265, 0.350, 0.385 \rangle & \langle 0.330, 0.390, 0.280 \rangle & \langle 0.245, 0.275, 0.480 \rangle \\ \langle 0.345, 0.245, 0.410 \rangle & \langle 0.430, 0.290, 0.280 \rangle & \langle 0.245, 0.375, 0.380 \rangle \\ \langle 0.365, 0.300, 0.335 \rangle & \langle 0.480, 0.315, 0.205 \rangle & \langle 0.340, 0.370, 0.290 \rangle \\ \langle 0.430, 0.300, 0.270 \rangle & \langle 0.460, 0.245, 0.295 \rangle & \langle 0.310, 0.520, 0.170 \rangle \end{bmatrix}.$$

Step 2: Aggregate these preferences $r_{ij}$ into collective $r_i$ by Eq. (3) (here, without loss of generality, we use $\lambda = 2$).

$r_1 = \langle 0.2705, 0.3955, 0.3251 \rangle$, $\quad r_2 = \langle 0.3249, 0.3689, 0.3010 \rangle$,
$r_3 = \langle 0.3799, 0.2871, 0.3296 \rangle$, $\quad r_4 = \langle 0.3907, 0.2289, 0.3612 \rangle$.

Step 3: By Definition 2.3, score values of $r_i$'s are $sc(r_1) = -0.4501$, $sc(r_2) = -0.3450$, $sc(r_3) = -0.2368$, and $sc(r_4) = -0.1994$ and hence ranking order is $A_4 \succ A_3 \succ A_2 \succ A_1$. Thus, the best one is $A_4$.

Further, if we utilize the *SVNFWG* operator for aggregating these SVNNs, then the results are as follows.

Step 1: Similar to that of above.

Step 2: Aggregate these values by Eq. (6) into the collective $r_i$.

$r_1 = (0.2685, 0.3292, 0.4056)$, $\quad r_2 = (0.3152, 0.3080, 0.3734)$,
$r_3 = (0.3752, 0.3316, 0.2922)$, $\quad r_4 = (0.3831, 0.3865, 0.2364)$.

Step 3: Score values of $r_i$'s are $sc(r_1) = -0.4664$, $sc(r_2) = -0.3661$, $sc(r_3) = -0.2486$, and $sc(r_4) = -0.2399$ and hence the best one is $A_4$.

On the other hand, if we apply the various existing approaches (Liu et al., 2014; Ye, 2013, 2014c; Majumdar and Samant, 2014; Broumi and Samarandache, 2013; Sahin, 2014; Ye, 2014b) from the field of decision making to the considered problem, then their corresponding rating values as well as ranking of the alternatives are summarized in Table 1. These results, have been analyzed and it was found that the best alternatives coincide with the proposed ones and hence the proposed methods have a suitable tool for solving the decision-making problems under the uncertain environment.

### 4.3 Sensitivity Analysis

In order to see the influence of the parameter λ on the decision making, an analysis has been conducted in which different values of $\lambda (= 1, 1.5, 2, 2.5, 3, 5, 10, 15)$ have been taken for the considered problem. Based on these parameters, the proposed approach has been applied and their corresponding score values as well as ranking of the alternatives are summarized in Table 2. From this, it has been observed that with the increase of λ, score values by *SVNFWA* operators are decreasing while they are increasing for *SVNFWG* operators. Further, it has been concluded that the ranking of the given alternative is symmetric and it was found that the most suitable alternative is $A_4$, and $A_1$ is the least suitable.

## 5. CONCLUSIONS

Aggregation operators play a crucial role during the decision-making process as most of the data related to system identification are uncertain in nature. For this, the neutrosophic set theory has been utilized in the present manuscript and hence the performance of each object has been measured in terms of SVNNs. In order to aggregate all these preferences, a Frank operator based an aggregation operator such as *SVNFWA* and *SVNFWG* has been proposed in

**TABLE 1:** Comparative analysis

| | Method | Calculated values of | | | | Ranking |
|---|---|---|---|---|---|---|
| | | $A_1$ | $A_2$ | $A_3$ | $A_4$ | |
| Liu et al. (2014) | Hamacher operator | | | | | |
| | $\gamma = 1$ | 0.2707 | 0.3257 | 0.3804 | 0.3913 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | $\gamma = 2$ | –0.0445 | 0.0555 | 0.1544 | 0.1628 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | $\gamma = 2.5$ | –0.0804 | 0.0332 | 0.1429 | 0.1435 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | $\gamma = 3$ | –0.0922 | 0.0303 | 0.1462 | 0.1398 | $A_3 \succ A_4 \succ A_2 \succ A_1$ |
| | $\gamma = 5$ | –0.0661 | 0.0694 | 0.1900 | 0.1656 | $A_3 \succ A_4 \succ A_2 \succ A_1$ |
| Ye (2014c) | Cross entropy | 1.9099 | 1.7331 | 1.5431 | 1.5296 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| Ye (2013) | Correlation coefficient | 0.4559 | 0.5471 | 0.6453 | 0.6387 | $A_3 \succ A_4 \succ A_2 \succ A_1$ |
| Majumdar and Samant (2014) | Similarity measure | 0.5200 | 0.5600 | 0.5967 | 0.6000 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| Broumi and Smarandache (2013) | Distance measure | 0.7300 | 0.6780 | 0.6220 | 0.6120 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| Sahin (2014) | Score function | 0.1133 | 0.1782 | 0.2174 | 0.2225 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| Ye (2014b) | Hamming distance | 0.4867 | 0.4520 | 0.4147 | 0.4080 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | Euclidean distance | 0.5196 | 0.4839 | 0.4424 | 0.4446 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| Ye (2015) | Cosine similarity measure | 0.4110 | 0.4815 | 0.5575 | 0.5695 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | Cosine similarity measure | 0.7214 | 0.7563 | 0.7941 | 0.7997 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |

**TABLE 2:** Effect of the parameter λ on ranking of the alternatives

| λ | Operator | Score value of alternative | | | | Ranking |
|---|---|---|---|---|---|---|
| | | $A_1$ | $A_2$ | $A_3$ | $A_4$ | |
| $\to 1$ | *SVNFWA* | –0.4486 | –0.3433 | –0.2357 | –0.1960 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | *SVNFWG* | –0.4677 | –0.3678 | –0.2496 | –0.2434 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 1.5 | *SVNFWA* | –0.4495 | –0.3443 | –0.2363 | –0.1980 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | *SVNFWG* | –0.4669 | –0.3668 | –0.2490 | –0.2413 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 2 | *SVNFWA* | –0.4501 | –0.3450 | –0.2368 | –0.1994 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | *SVNFWG* | –0.4664 | –0.3661 | –0.2486 | –0.2399 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 2.5 | *SVNFWA* | –0.4506 | –0.3456 | –0.2371 | –0.2004 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | *SVNFWG* | –0.4660 | –0.3656 | –0.2484 | –0.2389 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 3 | *SVNFWA* | –0.4509 | –0.3460 | –0.2373 | –0.2012 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | *SVNFWG* | –0.4657 | –0.3652 | –0.2482 | –0.2381 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 5 | *SVNFWA* | –0.4518 | –0.3471 | –0.2380 | –0.2033 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | *SVNFWG* | –0.4649 | –0.3642 | –0.2476 | –0.2360 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 10 | *SVNFWA* | –0.4530 | –0.3484 | –0.2388 | –0.2074 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | *SVNFWG* | –0.4639 | –0.3629 | –0.2470 | –0.2336 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 15 | *SVNFWA* | –0.4536 | –0.3484 | –0.2388 | –0.2060 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| | *SVNFWG* | –0.4635 | –0.3623 | –0.2466 | –0.2324 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |

the manuscript. Some of its desirable properties have also been investigated. Further, a decision-making approach has been presented based on these operators and illustrated with a numerical example in which each alternative is assessed in terms of SVNNs. By comparison with the existing approaches, it has been concluded that the proposed operators show a more stable, practical, and optimistic nature to the decision makers during the aggregation process. Measuring values corresponding to different values of λ will offer the various choices to the decision makers in assessing the alternatives. Therefore, the present approach becomes more consistent and reliable to present the degree of fuzziness. In the future, we will extend it to different fields.

## REFERENCES


Atanassov, K. and Gargov, G., Interval-valued intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, vol. **31**, pp. 343–349, 1989.

Atanassov, K. T., Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, vol. **20**, pp. 87–96, 1986.

Broumi, S. and Smarandache, F., Correlation coefficient of interval neutrosophic set, *Appl. Mech. Mater.*, vol. **436**, pp. 511–517, 2013.

Broumi, S. and Smarandache, F., Single valued neutrosophic trapezoid linguistic aggregation operators based multi-attribute decision making, *Bull. Pure Appl. Sci.-Math. Stat.*, vol. **33**, no. 2, pp. 135–155, 2014.

Frank, M. J., On the similtaneous associativity of $f(x, y)$ and $x + y - f(x, y)$, *Aequationes Mathematicae*, vol. **19**, no. 1, pp. 194–226, 1979.

Garg, H., A new generalized improved score function of interval-valued intuitionistic fuzzy sets and applications in expert systems, *Appl. Soft Comput.*, vol. **38**, pp. 988–999, 2016a.

Garg, H., A new generalized pythagorean fuzzy information aggregation using Einstein operations and its application to decision making, *Int. J. Intell. Syst.*, vol. **31**, no. 9, pp. 886–920, 2016b.

Garg, H., Generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision making, *Comp. Ind. Eng.*, vol. **101**, pp. 53–69, 2016c.

Garg, H., Generalized intuitionistic fuzzy multiplicative interactive geometric operators and their application to multiple criteria decision making, *Int. J. Machine Learn. Cybern.*, doi: 10.1007/s13042-015-0432-8, 2015.

Garg, H., Some series of intuitionistic fuzzy interactive averaging aggregation operators, *SpringerPlus*, vol. **5**, no. 1, pp. 1– 27, 2016d.

Garg, H., Agarwal, N., and Choubey, A., Entropy based multi-criteria decision making method under fuzzy environment and unknown attribute weights, *Global J. Technol. Optim.*, vol. **6**, no. 3, pp. 13–20, 2015.

Li, Y., Liu, P., and Chen, Y., Some single valued neutrosophic number Heronian mean operators and their application in multiple attribute group decision making, *Informatica*, vol. **27**, no. 1, pp. 85–110, 2016.

Liu, P. and Liu, X., The neutrosophic number generalized weighted power averaging operator and its application in multiple attribute group decision making, *Int. J. Machine Learn. Cybern.*, doi: 10.1007/s13042-016-0508-0, 2016.

Liu, P. and Shi, L., The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multiple attribute decision making, *Neural Comput. Appl.*, vol. **26**, no. 2, pp. 457–471, 2015.

Liu, P., Chu, Y., Li, Y., and Chen, Y., Some generalized neutrosophic number Hamacher aggregation operators and their application to group decision making, *Int. J. Fuzzy Syst.*, vol. **16**, no. 2, pp. 242–255, 2014.

Majumdar, P. and Samant, S. K., On similarity and entropy of neutrosophic sets, *J. Intell. Fuzzy Syst.*, vol. **26**, no. 3, pp. 1245–1252, 2014.

Peng, J. J., Wang, J. Q., Wang, J., Zhang, H. Y., and Chen, Z. H., Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems, *Int. J. Syst. Sci.*, vol. **47**, no. 10, pp. 2342–2358, 2016.

Şahin, R., Multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment, http://arxiv.org/abs/1412.5202, Dec. 17, 2014.

Smarandache, F., *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic*, Rehoboth, NM: American Research Press, 1999.

Tian, Z. P., Wang, J., Zhang, H. Y., and Wang, J. Q., Multi-criteria decision-making based on generalized prioritized aggregation operators under simplified neutrosophic uncertain linguistic environment, *Int. J. Machine Learn. Cybern.*, doi: 10.1007/s13042-016-0552-9, 2016.

Wang, W. and Liu, X., Intuitionistic fuzzy information aggregation using Einstein operations, *IEEE Trans. Fuzzy Syst.*, vol. **20**, no. 5, pp. 923–938, 2012.

Wang, H., Smarandache, F., Zhang, Y. Q., and Smarandache, R., *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*, Phoenix, AZ: Hexis, 2005.

Wang, H., Smarandache, F., Zhang, Y. Q., and Sunderraman, R., Single valued neutrosophic sets, *Multispace Multistruc.*, vol. **4**, pp. 410–413, 2010.

Xu, Z. S., Intuitionistic fuzzy aggregation operators, *IEEE Trans. Fuzzy Syst.*, vol. **15**, no. 6, pp. 1179–1187, 2007.

Xu, Z. and Chen, J., Approach to group decision making based on interval valued intuitionistic judgment matrices, *Syst. Eng.—Theory Pract.*, vol. **27**, no. 4, pp. 126–133, 2007.

Xu, Z. S. and Hu, H., Projection models for intuitionistic fuzzy multiple attribute decision making, *Int. J. Inf. Technol. Decis. Making*, vol. **9**, no. 2, pp. 267–280, 2010.

Xu, Z. S. and Yager, R. R., Some geometric aggregation operators based on intuitionistic fuzzy sets, *Int. J. General Syst.*, vol. **35**, pp. 417–433, 2006.

Xu, Z. and Zhao, N., Information fusion for intuitionistic fuzzy decision making: an overview, *Inf. Fusion*, vol. **28**, pp. 10–23, 2016.

Yager, R. R., On ordered weighted averaging aggregation operators in multi-criteria decision making, *IEEE Trans. Syst., Man Cybern.*, vol. **18**, no. 1, pp. 183–190, 1988.

Ye, J., A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets, *Int. J. Gen. Syst.*, vol. **26**, no. 5, pp. 2459–2466, 2014a.

Ye, J., Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses, *Artif. Intell. Med.*, vol. **63**, pp. 171–179, 2015.

Ye, J., Multicriteria decision making method using the correlation coefficient under single-value neutrosophic environment, *Int. J. Gen. Syst.*, vol. **42**, no. 4, pp. 386–394, 2013.

Ye, J., Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making, *Int. J. Intell. Fuzzy Syst.*, vol. **26**, no. 1, pp. 165–172, 2014b.

Ye, J., Single valued neutrosophic cross-entropy for multicriteria decision making problems, *Appl. Math. Modell.*, vol. **38**, no. 3, pp. 1170–1175, 2014c.

Zadeh, L. A., Fuzzy sets, *Inf. Control*, vol. **8**, pp. 338–353, 1965.

Zhang, H. Y., Wang, J. Q., and Chen, X. H., Interval neutrosophic sets and their application in multicriteria decision making problems, *Sci. World J.*, vol. **2014**, doi: 10.1155/2014/645953, 2014.