The quintic : $z^5+z^4-1=0$

Edgar Valdebenito

Abstract

In this note we briefly explore the equation : $z^5+z^4-1=0$

1. Introduction: Roots

$$p(z)=z^5+z^4-1=0 \Rightarrow \begin{cases} z_1=r\in\mathbb{R} \\ z_2=a+bi\in\mathbb{C} \\ z_3=\overline{z}_2=a-bi\in\mathbb{C} \\ z_4=c+di\in\mathbb{C} \\ z_5=\overline{z}_4=c-di\in\mathbb{C} \end{cases} \tag{1}$$

$$z_1=r=0.856674... \tag{2}$$

$$\begin{cases} z_2=0.150051...+i\,0.897460... \\ z_3=0.150051...-i\,0.897460... \end{cases} \tag{3}$$

$$\begin{cases} z_4=-1.078388...+i\,0.496939... \\ z_5=-1.078388...-i\,0.496939... \end{cases} \tag{4}$$

$$p(x+iy)=0 \Rightarrow \begin{cases} x^5-10x^3y^2+5xy^4+x^4-6x^2y^2+y^4-1=0 \\ 5x^4y-10x^2y^3+y^5+4x^3y-4xy^3=0 \end{cases} \tag{5}$$

$$q(z)=z^5 p\left(\frac{1}{z}\right)=-z^5+z+1 \tag{6}$$

$$q(x+iy)=0 \Rightarrow \begin{cases} -x^5+10x^3y^2-5xy^4+x+1=0 \\ -5x^4y+10x^2y^3-y^5+y=0 \end{cases} \tag{7}$$

$$q(z)=0 \Rightarrow z=\left\{\frac{1}{z_1},\frac{1}{z_2},\frac{1}{z_3},\frac{1}{z_4},\frac{1}{z_5}\right\} \tag{8}$$

$$\left\{\mathrm{Re}\left(p(x+iy)\right)\right\}\cap\left\{\mathrm{Im}\left(p(x+iy)\right)\right\}=\{z_1,z_2,z_3,z_4,z_5\} \tag{9}$$

$$\left\{\mathrm{Re}\left(q(x+iy)\right)\right\}\cap\left\{\mathrm{Im}\left(q(x+iy)\right)\right\}=\left\{\frac{1}{z_1},\frac{1}{z_2},\frac{1}{z_3},\frac{1}{z_4},\frac{1}{z_5}\right\} \tag{10}$$

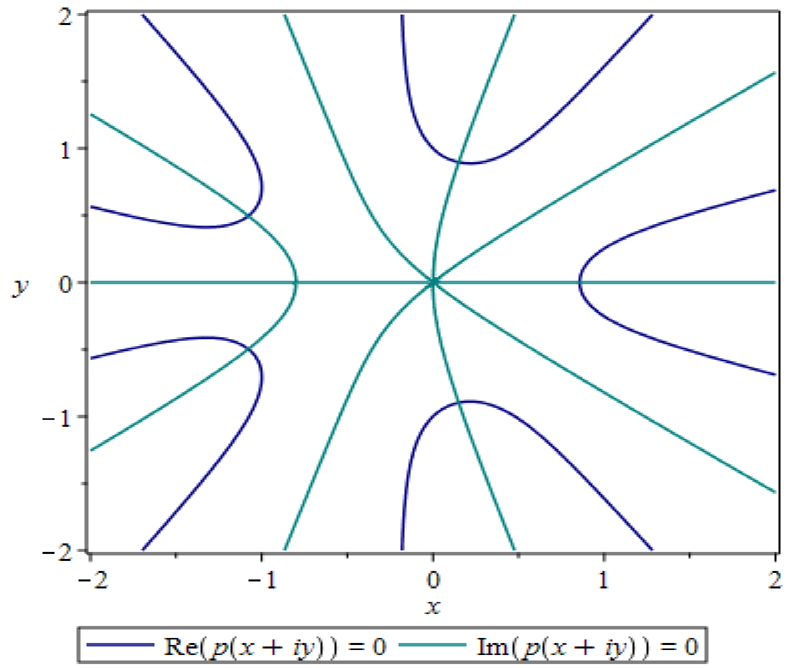

Figure 1.

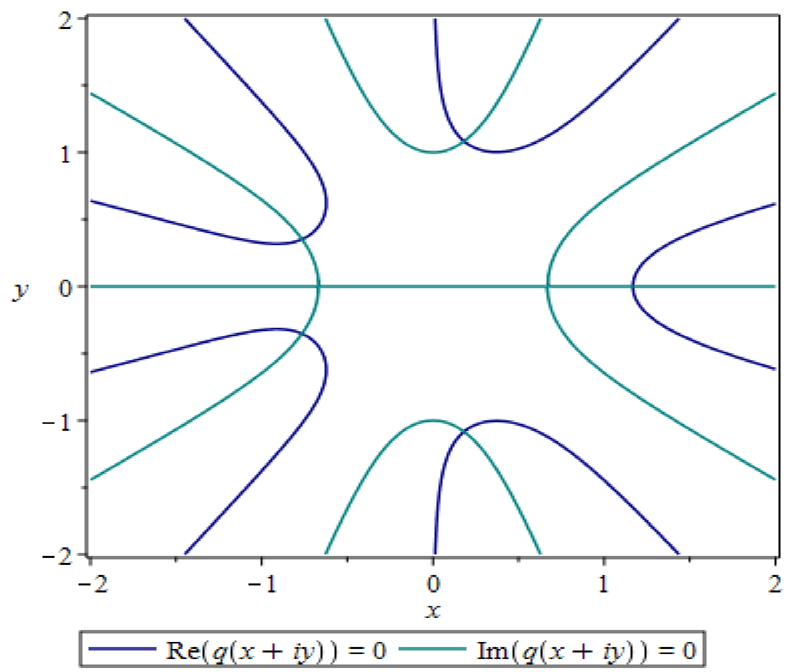

Figure 2.

2. Some Representations

$$z_1 = r = \cfrac{1}{\sqrt[4]{1+\cfrac{1}{\sqrt[4]{1+\cfrac{1}{\sqrt[4]{1+\dots}}}}}} \tag{11}$$

$$\frac{1}{z_1} = \frac{1}{r} = \sqrt[5]{1+\sqrt[5]{1+\sqrt[5]{1+\dots}}} \tag{12}$$

$$z_2 = a+bi = \cfrac{i}{\sqrt[4]{1+\cfrac{i}{\sqrt[4]{1+\cfrac{i}{\sqrt[4]{1+\dots}}}}}} \tag{13}$$

$$z_3 = a-bi = \cfrac{-i}{\sqrt[4]{1-\cfrac{i}{\sqrt[4]{1-\cfrac{i}{\sqrt[4]{1-\dots}}}}}} \tag{14}$$

$$z_4 = c+di = \cfrac{-1+i}{\sqrt[4]{-4+\cfrac{4-4i}{\sqrt[4]{-4+\cfrac{4-4i}{\sqrt[4]{-4+\dots}}}}}} \tag{15}$$

$$z_5 = c-di = \cfrac{-\sqrt[4]{2}\left(2+i\right)}{\sqrt[4]{-14+48i+\cfrac{\sqrt[4]{2}\left(76-82i\right)}{\sqrt[4]{-14+48i+\cfrac{\sqrt[4]{2}\left(76-82i\right)}{\sqrt[4]{-14+48i+\dots}}}}}} \tag{16}$$

3. Some Relations

$$\pi = 16\tan^{-1}\left(\frac{a}{b}\right) + 4\tan^{-1}\left(\frac{1+a-b}{1+a+b}\right) \tag{17}$$

$$\pi = 8\tan^{-1}\left(\frac{d}{-c}\right) + 2\tan^{-1}\left(\frac{1+c}{d}\right) \tag{18}$$

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty} r^n \sum_{k=[n/5]}^{[n/4]} \binom{k}{n-4k} \frac{(-3)^{-k}}{2k+1} \tag{19}$$

$$\binom{k}{n-4k} = \begin{cases} \dfrac{k!}{(n-4k)!(5k-n)!} & ,5k \geq n \\ 0 & ,5k \leq n \end{cases} \tag{20}$$

$$\pi = 4\tan^{-1}\left(\frac{c+d}{c-d}\right) - 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left((c+di)^{-5n}\right) \tag{21}$$

$$\pi = 10\tan^{-1}\left(\frac{a}{b}\right) - 2\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left((a+bi)^{4n}\right) \tag{22}$$

$$\pi r^5 \prod_{n=1}^{\infty}\left(1-\frac{r^{10}}{4n^2}\right) = 2\prod_{n=1}^{\infty}\left(1-\frac{r^8}{(2n-1)^2}\right) \tag{23}$$

$$\pi r^4 \prod_{n=1}^{\infty}\left(1-\frac{r^8}{4n^2}\right) = 2\prod_{n=1}^{\infty}\left(1-\frac{r^{10}}{(2n-1)^2}\right) \tag{24}$$

4. Matrix-M iterative method:

$$M = \begin{pmatrix} 10 & -12 & 14 & -16 & 19 \\ 11+8i & -11+10i & -9-13i & 16-7i & 5+19i \\ 11-8i & -11-10i & -9+13i & 16+7i & 5-19i \\ 33+24i & 17+30i & 2+29i & -8+23i & -14+14i \\ 33-24i & 17-30i & 2-29i & -8-23i & -14-14i \end{pmatrix} \tag{25}$$

$$U_{n+1} = U_n - 0.01M \cdot F(U_n) \quad ,n \in \mathbb{N} \tag{26}$$

$$U_1 = \begin{pmatrix} 0.8 & 0.1+0.8i & 0.1-0.8i & -1+0.5i & -1-0.5i \end{pmatrix}^T \tag{27}$$

$$U_n \to \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix}^T \tag{28}$$

$$F = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \end{pmatrix}^T \tag{29}$$

$$f_1(u_1,u_2,u_3,u_4,u_5) = u_1u_2u_3u_4u_5 + 1 \tag{30}$$

$$f_2(u_1,u_2,u_3,u_4,u_5) = u_1(u_2+u_3+u_4+u_5) + u_2(u_3+u_4+u_5) + u_3(u_4+u_5) + u_4u_5 \tag{31}$$

$$\begin{aligned} f_3(u_1,u_2,u_3,u_4,u_5) = {} & u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4 + u_1u_2u_5 + u_1u_3u_5 \\ & + u_2u_3u_4 + u_1u_4u_5 + u_2u_4u_5 + u_3u_4u_5 \end{aligned} \tag{32}$$

$$f_4(u_1,u_2,u_3,u_4,u_5) = u_1u_2u_3u_4 + u_1u_2u_3u_5 + u_2u_3u_4u_5 + u_1u_3u_4u_5 + u_1u_2u_4u_5 \tag{33}$$

$$f_5(u_1,u_2,u_3,u_4,u_5) = u_1u_2u_3u_4u_5 - 1 \tag{34}$$

5. Newton Method

$$G(x)=\frac{4x^5+3x^4+1}{5x^4+4x^3} \tag{35}$$

$$x_{n+1}=G(x_n) \quad ,x_1=1\Rightarrow x_n\to z_1=r \tag{36}$$

$$x_{n+1}=G(x_n) \quad ,x_1=\frac{1}{10}+i\Rightarrow x_n\to z_2 \tag{37}$$

$$x_{n+1}=G(x_n) \quad ,x_1=\frac{1}{10}-i\Rightarrow x_n\to z_3 \tag{38}$$

$$x_{n+1}=G(x_n) \quad ,x_1=-1+\frac{i}{2}\Rightarrow x_n\to z_4 \tag{39}$$

$$x_{n+1}=G(x_n) \quad ,x_1=-1-\frac{i}{2}\Rightarrow x_n\to z_5 \tag{40}$$

$$H(x)=\frac{4x^5+1}{5x^4-1} \tag{41}$$

$$x_{n+1}=H(x_n) \quad ,x_1=1\Rightarrow x_n\to \frac{1}{z_1}=\frac{1}{r} \tag{42}$$

$$x_{n+1}=H(x_n) \quad ,x_1=\frac{1}{10}-i\Rightarrow x_n\to \frac{1}{z_2} \tag{43}$$

$$x_{n+1}=H(x_n) \quad ,x_1=\frac{1}{10}+i\Rightarrow x_n\to \frac{1}{z_3} \tag{44}$$

$$x_{n+1}=H(x_n) \quad ,x_1=-1-\frac{i}{2}\Rightarrow x_n\to \frac{1}{z_4} \tag{45}$$

$$x_{n+1}=H(x_n) \quad ,x_1=-1+\frac{i}{2}\Rightarrow x_n\to \frac{1}{z_5} \tag{46}$$

6. Newton-Julia set for: $p(z) = z^5 + z^4 - 1$.

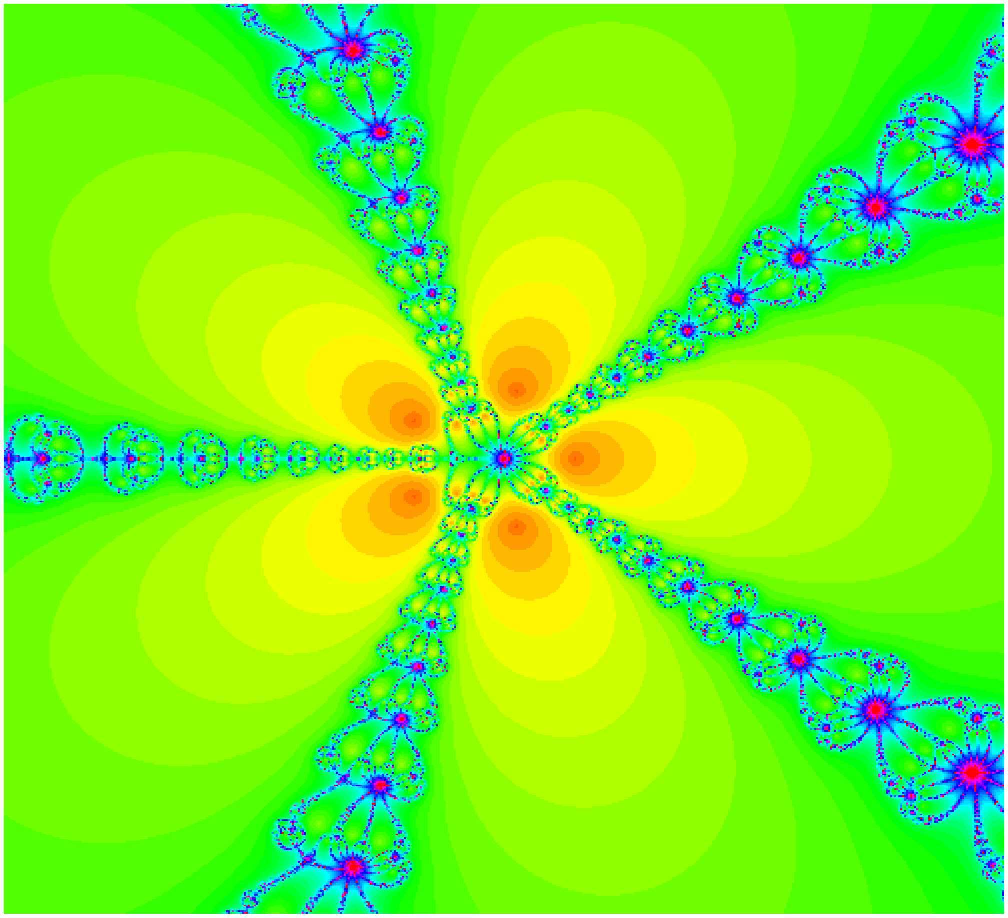

Figure 3.

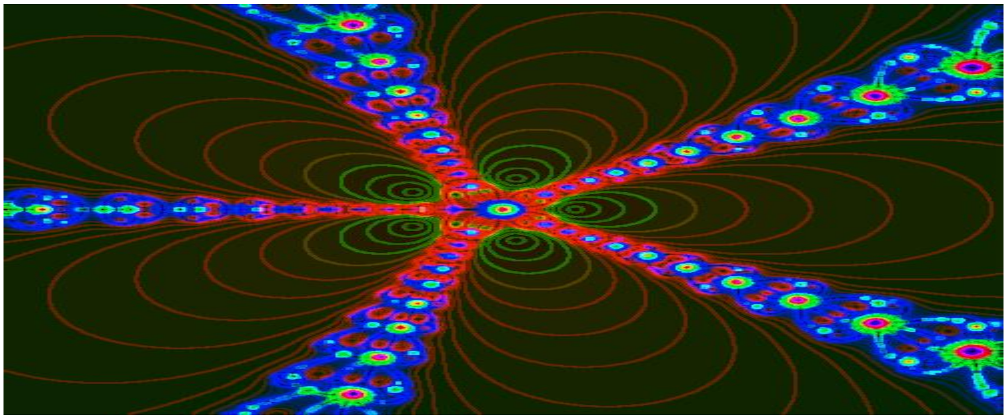

Figure 4.

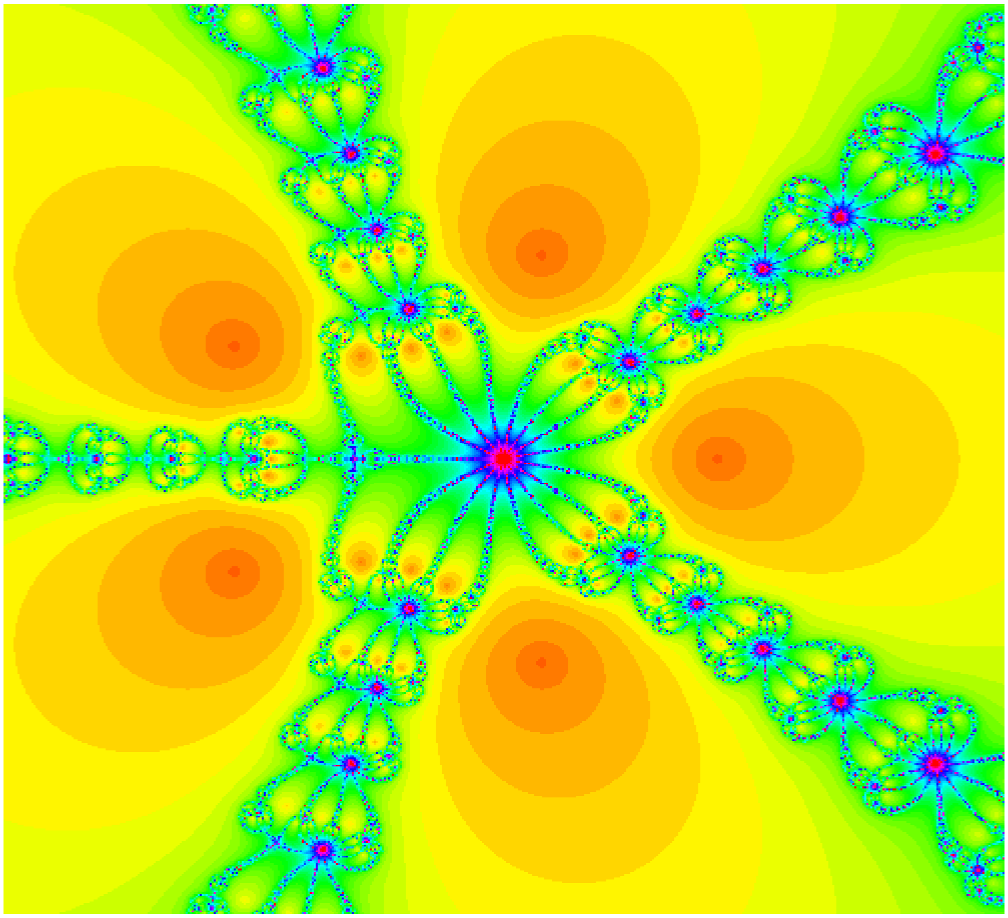

Figure 5.

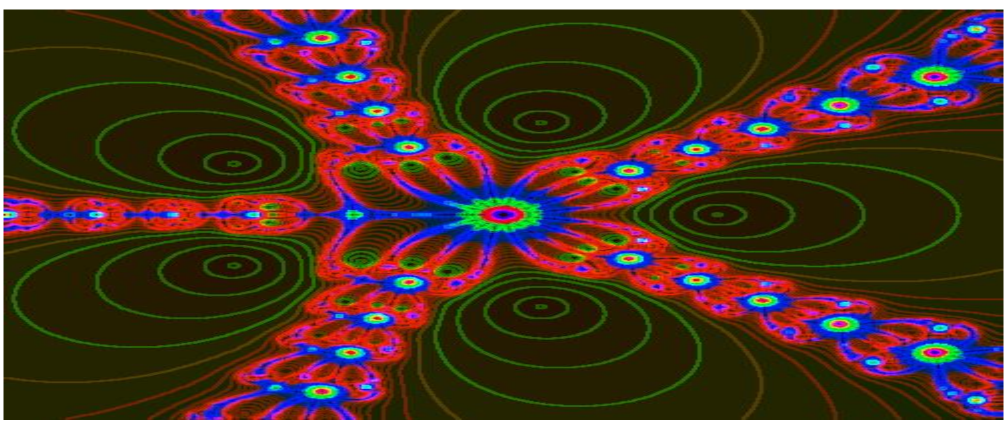

Figure 6.

7. Newton-Julia set for: $q(z) = z^5 - z - 1$.

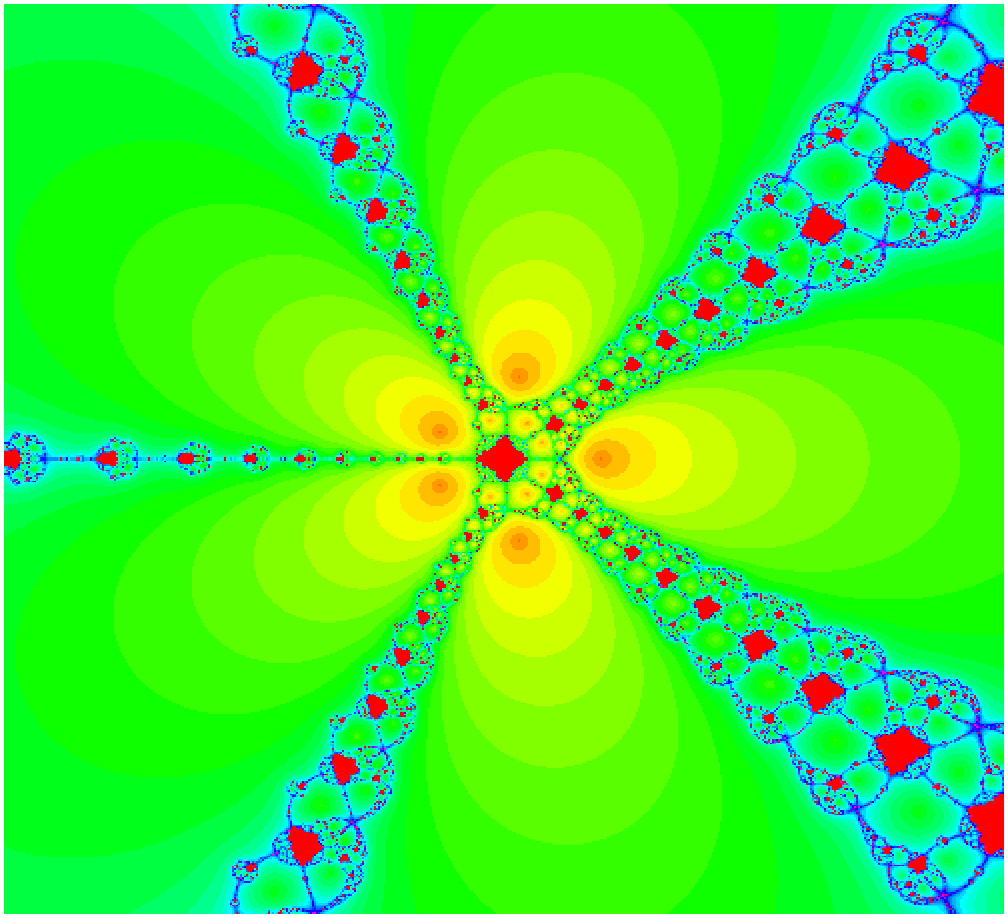

Figure 7.

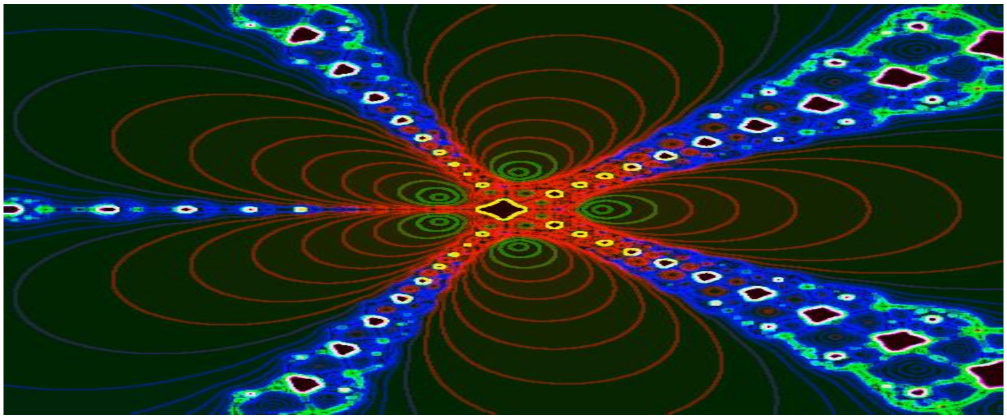

Figure 8.

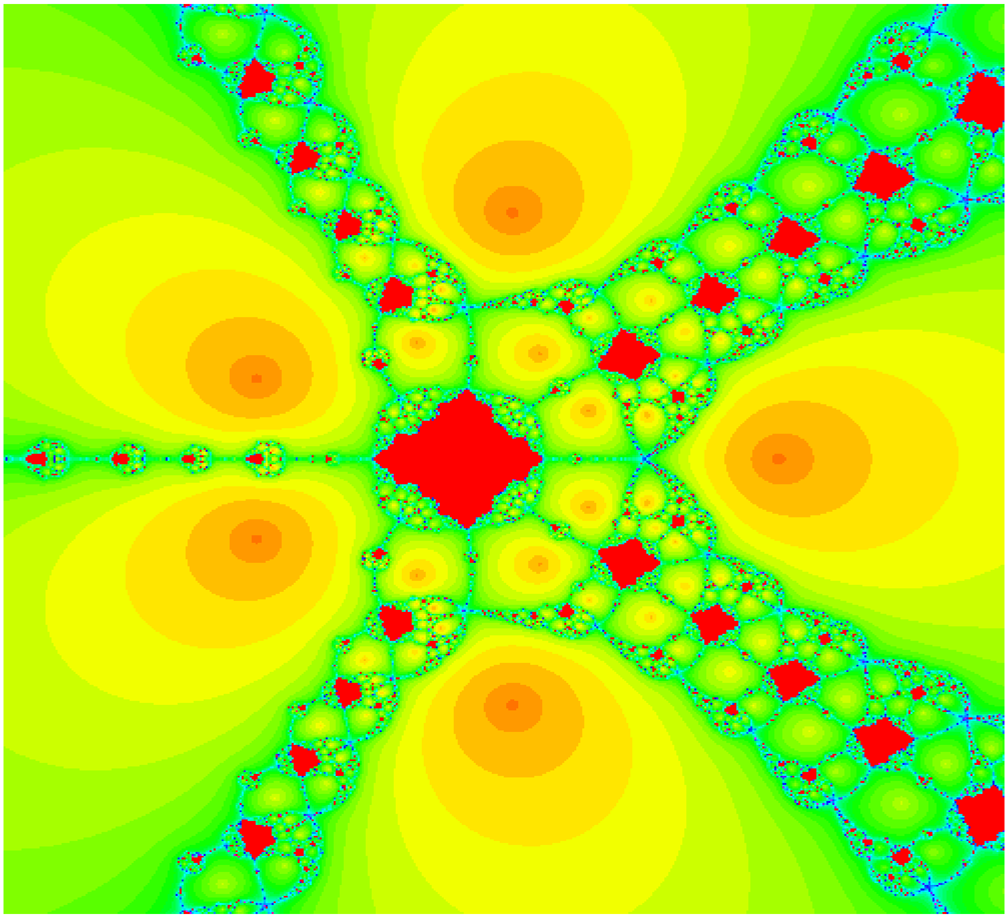

Figure 9.

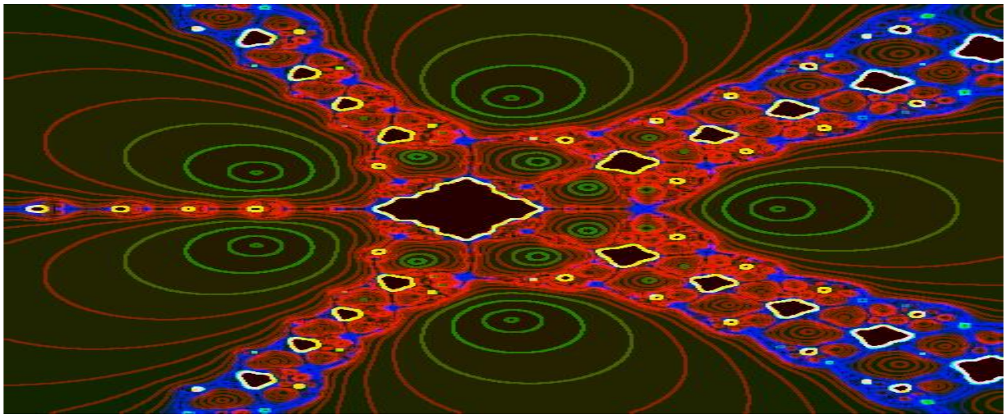

Figure 10.

References

1. HUBBARD, J., SCHLEICHER, D., AND SUTHERLAND, S. How to find all roots of complex polynomials by Newton's method. Invent. Math. 146 no.1 (2001),1-33.
2. D.S. ALEXANDER, A History of Complex Dinamics from Schroder to Fatou and Julia, Vieweg&Sohn, Wiesbaden/Braunschweig, 1994.
3. F. HAESELER, H.O. PEITGEN, Newton's method and complex dynamical systems, Acta Appl. Math. 13 (1988), 3-58.
4. H.O. PEITGEN and P.H. RICHTER, The Beauty of Fractals, Springer-Verlag ,1986
5. J. MILNOR, Dynamics in One Complex Variable: Introductory Lectures, Vieweg, 1999.