SERIES REPRESENTATION OF POWER FUNCTION

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Abstract. In this paper described numerical expansion of natural-valued power function x^n, in point $x = x_0$ where $n,\ x_0$ - natural numbers. Applying numerical methods, that is calculus of finite differences, namely, discrete case of Binomial expansion is reached. Received results were compared with solutions according to Newton's Binomial theorem and MacMillan Double Binomial sum. Additionally, in section 4 exponential function's e^x representation is shown. In Application 3 generalized calculus of finite differences, based on expression (1.9) is shown.

Keywords. Power function, Monomial, Polynomial, Power series, Finite difference, Derivative, Differential calculus, Differentiation, Binomial coefficient, Newton's Binomial Theorem, Exponential function, Pascal's triangle, Series expansion, Double Binomial Sum, Perfect cube, Diophantine equations, Multinomial theorem

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1. Introduction

Let basically describe Newton's Binomial Theorem and Fundamental Theorem of Calculus, that is most widely used theorems about power's expansion and some their

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properties. In elementary algebra, the binomial theorem (or binomial expansion) describes the algebraic expansion of powers of a binomial. The theorem describes expanding of the power of $(x+y)^n$ into a sum involving terms of the form ax^by^c where the exponents b and c are nonnegative integers with $b+c=n$, and the coefficient a of each term is a specific positive integer depending on n and b. The coefficient a in the term of ax^by^c is known as the binomial coefficient. The main properties of the Binomial Theorem are next

Properties 1.1. *Binomial theorem properties*

- *The powers of x go down until it reaches $x_0 = 1$ starting value is n (the n in $(x+y)^n$)*
- *The powers of y go up from 0 ($y^0 = 1$) until it reaches n (also n in $(x+y)^n$)*
- *The n-th row of the Pascal's Triangle (see* [1]*) will be the coefficients of the expanded binomial.*
- *For each line, the number of products (i.e. the sum of the coefficients) is equal to $x+1$*
- *For each line, the number of product groups is equal to 2^n*

According to the Binomial theorem, it is possible to expand any power of $x+y$ into a sum of the form (see [2], [4])

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{1.2}$$

Let us to expand monomial $x^n : n \in \mathbb{N}_1$ using Binomial theorem (1.2)

Lemma 1.3. *Power function could be represented as discrete integral of its first order finite difference*

$$x^n = \int_0^{x-1} \Delta[x^n] \Delta h \tag{1.4}$$

$$= \sum_{k=0}^{x-1} \underbrace{nk^{n-1}\Delta h + \binom{n}{2} k^{n-2} (\Delta h)^2 + \cdots + \binom{n}{n-1} k (\Delta h)^{n-1} + (\Delta h)^n}_{\Delta[x^n]\Delta h}$$

We can reach the same result using Fundamental Theorem of Calculus, we have, respectively

$$x^n = \int_0^x nt^{n-1} dt = \sum_{k=0}^{x-1} \int_k^{k+1} nt^{n-1} dt = \sum_{k=0}^{x-1} (k+1)^n - k^n \tag{1.5}$$

Hereby, let be lemma

Lemma 1.6. *First order finite difference of power x^n could be reached by binomial expansion of the form*

$$\Delta[x^n] = \sum_{k=1}^{n} \binom{n}{k} x^{n-k} (\Delta h)^{k-1} \tag{1.7}$$

Otherwise, let be a table of finite differences (see also [6], eq. 7) $\Delta^k[x^3],\ x \in \mathbb{N}_1,\ k \in [1,3] \subseteq \mathbb{N}_1$ over x from $[0,7] \subseteq \mathbb{N}$

(1.8)

x	x^3	$\Delta[x^3]$	$\Delta^2[x^3]$	$\Delta^3[x^3]$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Figure 1: Difference table of $x^3,\ x \in \mathbb{N}$ up to 3^{rd} order, [6], eq. 7

Note that increment Δh is set to be $\Delta h = 1$ and $k > 2$-order difference is taken regarding to [10]. From figure (1) could be observed next regularities (sequence A008458 in OEIS, [9])

$$\begin{gathered}
\Delta(x_0^3) = x_1^3 - x_0^3 = 1 + 3! \cdot 0 \\
\Delta(x_1^3) = x_2^3 - x_1^3 = 1 + 3! \cdot 0 + 3! \cdot 1 \\
\Delta(x_2^3) = x_3^3 - x_2^3 = 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 \\
\vdots \\
\Delta(x_m^3) = x_{m+1}^3 - x_m^3 = 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + \cdots + 3! \cdot m
\end{gathered}$$

Hereby, applying compact sigma notation each first order finite difference $\Delta[x^3]$ could be represented as

$$\Delta[x^3] = \sum_{k=1}^{x} jk + \frac{1}{x},\ x = x_0 \in \mathbb{N},\ x \neq 0 \tag{1.9}$$

The backward difference, respectively, is

$$\nabla[x^3] = \sum_{k=0}^{x-1} jk + \frac{1}{x},\ x = x_0 \in \mathbb{N},\ x \neq 0$$

where $j = 1 \cdot 2 \cdot 3 = 3!$ (also according to [6], [7]) and $\Delta h = 1$ by definition. Note that Mathematica code of expression (1.9) available in Application 1 as well as backward difference equation. Since, by lemma (1.3) discrete integral of first difference is used to reach expansion of x^n and we have an equality between binomial difference from lemma (1.6) and (1.9)

$$\Delta[x^3] = \sum_{k=1}^{3} \binom{3}{k} x^{3-k} (\Delta h)^{k-1} \equiv \sum_{k=1}^{x} jk + \frac{1}{x},\ x = x_0 \in \mathbb{N},\ \Delta h = 1,\ x \neq 0 \tag{1.10}$$

Then we have right to substitute (1.9) into (1.4) instead binomial expansion and represent x^3 as summation over k from 0 to $x - 1$. Let be example for x^3, that is the sum of sequence A008458 in OEIS

$$x^3 = (1 + 3! \cdot 0) + (1 + 3! \cdot 0 + 3! \cdot 1) + (1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2) + \cdots$$

$$\cdots + (1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + \cdots + 3! \cdot (x-1)) \tag{1.11}$$

$$x^3 = x + (x-0) \cdot 3! \cdot 0 + (x-1) \cdot 3! \cdot 1 + (x-2) \cdot 3! \cdot 2 + \cdots$$

$$\cdots + (x - (x-1)) \cdot 3! \cdot (x-1)$$

Provided that x is natural. Note that increment Δh in equation (1.9) is set to be $\Delta h = 1$ and not displayed. Using compact sigma notation on (1.9), we received

$$x^3 = x + j \sum_{m=0}^{x-1} mx - m^2 = \sum_{m=0}^{x-1} j \cdot m \cdot x - j \cdot m^2 + 1 \tag{1.12}$$

Also, the next relations hold

$$x^3 = \sum_{m=0}^{x-1} \left(1 + j \sum_{u=0}^{m} u\right) \equiv j \sum_{m=1}^{x-1} \left(mx - m^2 + \frac{x}{j(x-1)}\right) \Bigg|_{x>1}, \quad x \in \mathbb{N} \tag{1.13}$$

Property 1.14. *Let be each term of (1.12) multiplied by x^{n-3} to reach x^n, then let be a sets $\mathfrak{S}(x) := \{1,\ 2,\ \ldots,\ x\} \subseteq \mathbb{N},\ \mathfrak{C}(x) := \{0,\ 1,\ \ldots,\ x\} \subseteq \mathbb{N},\ \mathfrak{U}(x) := \{0,\ 1,\ \ldots,\ x-1\} \subseteq \mathbb{N}$, let be central part of (1.12) written as $T(x,\ \mathfrak{U}(x))$ where $x \in \mathbb{N}$ is variable and $\mathfrak{U}(x)$ is iteration set of (1.12), then we have equality*

$$T(x,\ \mathfrak{U}(x)) \equiv T(x,\ \mathfrak{S}(x)),\ x \in \mathbb{N} \tag{1.15}$$

Let be right part of (1.12) denoted as $U(x,\ \mathfrak{C}(x))$, then

$$U(x,\ \mathfrak{C}(x)) \equiv U(x,\ \mathfrak{S}(x)) \equiv U(x,\ \mathfrak{U}(x)) \tag{1.16}$$

Other words, changing iteration sets of (1.12) by $\mathfrak{C}(x),\ \mathfrak{S}(x),\ \mathfrak{U}(x)$ doesn't change its value.

Proof. Let be a plot of $jkx^{n-2} - jk^2x^{n-3} + x^{n-3}$ by k over $\mathbb{R}^+_{\leq 10}$, given $x = 10$ and $n = 10$

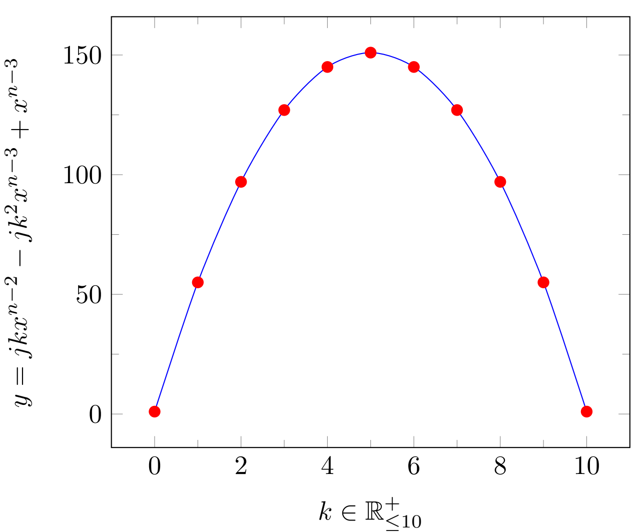

Figure 2. Plot of $jkx^{n-2} - jk^2x^{n-3} + x^{n-3}$ by k over $\mathbb{R}^+_{\leq 10}$, $x = 10,\ n = 10$

Obviously, being a parabolic function, it's symmetrical over $\frac{x}{2}$, hence equivalent $T(x,\ \mathfrak{U}(x)) \equiv T(x,\ \mathfrak{S}(x)),\ x \in \mathbb{N}$ follows. Reviewing central part of (1.12) and denote $u(t) = tx^{n-2} - t^2x^{n-3}$, we can conclude, that $u(0) \equiv u(x)$, then equality of $U(x,\ \mathfrak{C}(x)) \equiv U(x,\ \mathfrak{S}(x)) \equiv U(x,\ \mathfrak{U}(x))$ immediately follows.
This completes the proof. □

Note that for right part of (1.12) property (1.14) holds only in case (1.15). Let analyse property (1.14), we can see that for each row corresponded to variable $x = 1,\ 2,\ 3, ...,\ \mathbb{N}$, raised to power $n = 3$ and represented by means of (1.12) the numbers distribution is quite similar to distribution of Pascal's triangle items, more detailed, let be an example of triangle[1] built using right part of expression (1.12) up to $x = 4$, with set $n = 3$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 7 & & 1 & & \\ & 1 & & 13 & & 13 & & 1 & \\ 1 & & 19 & & 25 & & 19 & & 1 \end{array} \tag{1.17}$$

Figure 3. Triangle generated by right part of (1.12) given $n = 3$ up to $x = 4$, (sequence A287326 in OEIS, [9])

One could compare Figure (3) with Pascal's triangle [1]

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Figure 4. Pascal's triangle up to $n = 4$, sequence A007318 in OEIS, [9], [1]

Let us approach to show a few properties of triangle (1.17)

Properties 1.18. *Properties of (1.17). For each $(n, m) \in \mathbb{N}$ holds*

- *Summation of each n-th row from 0 to $n - 1$ returns perfect cube n^3*
- *Induction. Summation of each n-th row multiplied by n^{m-3} from 0 to $n-1$ returns n^m*
- *Summation of each n-th row from 0 to n returns $n^3 + n^0$*
- *Induction. Summation of each n-th row multiplied by n^{m-3} from 0 to n returns $n^m + n^{m-3}$*
- *Each second item of $2(n-1)$-th row returns backward difference $n^3 - (n-1)^3,\ n = 1, 2, 3, ...$*
- *Distribution of (1.17) items is similar to Pascal's triangle*
- *Each k_1 item of n-th row is generated by $k_0 + k_1 + 5$ of $n - 1$-th row*

[1]Note that each $x = 1,\ 2,\ 3, ...,\ \mathbb{N}$ row's item of triangle is generated by $jmx - jm^2 + 1$ over $m \in [0,\ x] \subseteq \mathbb{N}$

- *From (1.12) follows that for each item $k \neq 1$ of n-th row holds* $(k-1) \bmod 6 = 0$
- *Each column is generated by* $x^k \cdot (1 + (6 \cdot k - 1) \cdot x)/(1-x)^2$

1.1. **Discrete Binomial Theorem.** Review the triangle (3), let define the items of n-th row of triangle as

Definition 1.19.

$$\left\langle {n \atop k} \right\rangle := 6nk - 6k^2 + 1, \qquad 0 \leq k \leq n$$

Since the, summation of each n-th row from 0 to $n-1$ returns perfect cube n^3, then for each given number $x \in \mathbb{N}$ holds

$$x^n = \sum_{k=0}^{x-1} \left\langle {x \atop k} \right\rangle \cdot x^{n-3} \tag{1.20}$$

Multinomial case for $n = 4$ and $u, j, \ldots, m \in \mathbb{N}$.

$$(u + j + \cdots + m)^4 = \sum_{k=0}^{u+j+\cdots+m-1} \left\langle {u + j + \cdots + m \atop k} \right\rangle u + \left\langle {u + j + \cdots + m \atop k} \right\rangle j + \cdots \\ \cdots + \left\langle {u + j + \cdots + m \atop k} \right\rangle m \tag{1.21}$$

Note that (1.20) has also a recurrence relation of the form, which holds for each natural x

$$\left\langle {x \atop 0} \right\rangle x^{n-3} + \left\langle {x \atop 1} \right\rangle x^{n-3} + \cdots + \left\langle {x \atop x-1} \right\rangle x^{n-3} - x^n = 0$$

2. Derivation of high power

In this section are reviewed the ways to change obtained in previous annex expression (1.12) to higher power i.e $n > 3 \in \mathbb{N}$. Examples are shown for case $\Delta h = 1,\ x = x_0 \in \mathbb{N}$, applying operator of finite difference. By means of Fundamental Theorem of Calculus, we know that (1.4) holds, let rewrite it

$$x^n = \int\limits_0^x nt^{n-1} dt = \sum_{k=0}^{x-1} \int_k^{k+1} nt^{n-1} dt = \sum_{k=0}^{x-1} (k+1)^n - k^n$$

According to property (1.14), expression (1.12) could be written as

$$x^3 = x + j \sum_{m \in \mathfrak{U}(x)} mx - m^2 \equiv x + j \sum_{m \in \mathfrak{S}(x)} mx - m^2 \\ \equiv x + j \sum_{m \in \mathfrak{C}(x)} mx - m^2$$

In next equations we will use full notation of sets $\mathfrak{U}(x), \mathfrak{S}(x), \mathfrak{C}(x)$ with sigma operator to avoid curiosity. Let derive the relationships between sums of forward and backward differences, that is

$$x^n = \sum_{k=0}^{x-1} (k+1)^n - k^n \equiv \sum_{k=1}^{x} k^n - (k-1)^n \tag{2.1}$$

We could observe that iteration sets of right and left parts of (2.1) are $\mathfrak{U}(x) = \{0,\ 1, \ldots,\ x-1\}$ and $\mathfrak{S}(x) = \{1,\ 2,\ \ldots,\ x\}$, respectively. Since the property (1.14) holds, from forward difference we can derive expansion of x^3 in point $x = x_0 \in \mathbb{N}$, applying (1.12)

$$x^3 = \sum_{k=0}^{x-1} (k+1)^3 - k^3 = \sum_{k=0}^{x-1} \left((k+1)^3 - k - j \sum_{m=0}^{k} mk - m^2 \right) \tag{2.2}$$

$$\equiv \sum_{k=0}^{x-1} \left((k+1) + j \sum_{m=0}^{k} \left(m(k+1) - m^2 \right) - k^3 \right)$$

Otherwise, let be derived expansion of x^3 in point $x = x_0 \in \mathbb{N}$ by means of backward finite difference, that is right part of (2.1), applying (1.12). Note that if property (1.14) holds, then we have right to change iteration set of (1.12) regarding to (2.1) and substitute (1.12) into (2.1) instead $(x+1)^n,\ x^n$, hereby

$$x^3 = \sum_{k=1}^{x} k^3 - (k-1)^3 = \sum_{k=1}^{x} \left(k + j \sum_{m=1}^{k} \left(mk - m^2 \right) - (k-1)^3 \right) \tag{2.3}$$

$$-x^3 = \sum_{k=1}^{x} \left(k^3 - k + 1 + j \sum_{m=0}^{k-1} \left(m \cdot k + m - m^2 \right) \right)$$

Multiplying each term of (2.2) by x^{n-3}, we receive expansion of $f(x) = x^n,\ \forall (x = x_0,\ n) \in \mathbb{N}$

$$x^n = \sum_{k=0}^{x-1} \left((k+1)^{n-2} + j \sum_{m=1}^{k} \left(m(k+1)^{n-2} - m^2 (k+1)^{n-3} \right) - k^n \right) \tag{2.4}$$

$$\equiv \sum_{k=0}^{x-1} \left((k+1)^n - k^{n-2} + j \sum_{m=1}^{k} \left(m \cdot k^{n-2} - m^2 \cdot k^{n-3} \right) \right)$$

By means of main property of the power function $x^n = x^k \cdot x^{n-k}$, from equation (2.3) for $f(x) = x^n,\ (x = x_0,\ n) \in \mathbb{N}$ we receive

$$x^n = \sum_{k=1}^{x} \left(k^{n-2} + j \sum_{m=1}^{k} \left(m \cdot k^{n-2} - m^2 \cdot k^{n-3} \right) - (k-1)^n \right) \tag{2.5}$$

$$-x^n = \sum_{k=1}^{x} \left(k^n - (k+1)^{n-2} + j \sum_{m=0}^{k-1} \left(m(k+1)^{n-2} - m^2 (k+1)^{n-3} \right) \right)$$

Similarly as (2.5) reached, from the expression (1.13) for $f(x) = x^n,\ n \in \mathbb{N}$, we derive expansion

$$x^n = \sum_{m=0}^{x-1} \left(x^{n-3} + j \sum_{u=0}^{m} u x^{n-3} \right),\ (n,\ x) \in \mathbb{N} \tag{2.6}$$

$$\equiv j \sum_{k=0}^{x-1} \left(k \cdot x^{n-2} - k^2 \cdot x^{n-3} + \frac{x^{n-2}}{j(x-1)} \right),\ x \in \mathbb{N}_{\neq 1},\ n \in \mathbb{N}$$

In case of $f(x) = x^n,\ x \geq 0,\ (x,\ n) \in \mathbb{N}$ expression (1.12) could be written as follows

$$x^n = x^{n-2} + j \sum_{k=0}^{x-1} k \cdot x^{n-2} - k^2 \cdot x^{n-3} \tag{2.7}$$

$$= \sum_{k=0}^{x-1} j \cdot k \cdot x^{n-2} - j \cdot k^2 \cdot x^{n-3} + x^{n-3}$$

Let be summation $(u_1 \pm u_2 \pm \cdots \pm u_k)$ raised to power $n \geq 3 \in \mathbb{N}$, then applying (1.12) we could have an expression for continuous summation to power, provided that summation takes natural value

$$(u_1 \pm \cdots \pm u_k)^3 = \sum_{g \leq k}^{k} u_g + j \sum_{m=0}^{u_1 \pm \cdots \pm u_k - 1} \left(m \cdot \sum_{g \leq k}^{k} u_g - m^2 \right)$$

Hereby, above expression is discrete case of Multinomial theorem, given $n = 3$. Similarly as (2.7) the summation to natural n-th power could be reached by

$$(u_1 \pm \cdots \pm u_k)^n = \left(\sum_{g \leq k}^{k} u_g \right)^{n-2} + j \sum_{m=0}^{u_1 \pm \cdots \pm u_k - 1} \left(m \left(\sum_{g \leq k}^{k} u_g \right)^{n-2} - m^2 \left(\sum_{g \leq k}^{k} u_g \right)^{n-3} \right)$$

Note that continuous summation raised to power $n \in \mathbb{N}$ could be reached similarly by means of (2.2, 2.3, 2.4, 2.5, 2.6) replacing $x^{n-3},\ x^{n-2}$ by

$$\left(\sum_{g \leq k}^{k} u_g \right)^a,\ a \in \{n-2,\ n-3\},\ n \in \mathbb{N} \tag{2.8}$$

3. Binomial Theorem Representation

Newton's Binomial theorem says that it is possible to expand any power of $x+y$ into a sum of the form (see [2], [4])

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{3.1}$$

Since for each $(x = x_0,\ n) \in \mathbb{N}$ we have an equality (2.7)

$$x^n = \sum_{k=0}^{x-1} j \cdot k \cdot x^{n-2} - j \cdot k^2 \cdot x^{n-3} + x^{n-3}$$

Then we have right to substitute (2.7) into (1.2) instead x^{n-k} and y^k. Let be definition

Definition 3.2.

$$\mathbf{G}^n(x,\ k) := j \cdot k \cdot x^{n-2} - j \cdot k^2 \cdot x^{n-3} + x^{n-3}$$

Hereby, by means of (1.2) one could derive the value of $(x+y)^n$ at $x = x_0,\ y = y_0,\ (x_0,\ y_0) \in \mathbb{N}$

$$(x+y)^n = \sum_{k=0}^{n} \left(\binom{n}{k} \sum_{m=0}^{x-1} \mathbf{G}^{n-k}(x,\ m) \cdot \sum_{r=0}^{y-1} \mathbf{G}^k(y,\ r) \right) \tag{3.3}$$

Note that Wolfram Mathematica Code of expression (3.4) available in Application 1.

3.1. **Comparison of (2.7) to Binomial Theorem and MacMillan Double Binomial sum.** In this subsection let compare (2.7) to Binomial Theorem and MacMillan Double Binomial sum[2] ([3], eq. 12) hereby follow changes hold

- While expansion taken by Binomial theorem or MacMillan Double Binomial sum, the summation (1.2) is done over $k = 0, 1, 2..., \mathbb{N}$ from $0,\ 1$ to n, where n - power. Reviewing (2.7) we can observe that summation is taken over k from 0 to x, where x - variable, provided that natural-valued.
- According to property (1.14) the iteration sets of (2.7) and (1.12) could be changed to (1.15), (1.16) without changes in result.
- Order of polynomial provided by Binomial theorem and MacMillan Double Binomial sum is n, for (2.7) is $n-2$

Additionally, let us approach to apply (2.7) in calculus of q-differences, introduced by Jackson, 1908, [13]. By definition (3.2) we have

$$\Delta_q[x^n] = \sum_{k=0}^{x-1} \mathbf{G}^n(xq,\ k) - \mathbf{G}^n(x,\ k) \tag{3.4}$$

And high order

$$\Delta_q^u[x^n] = \sum_{k=0}^{u} \left((-1)^k \binom{u}{k} \sum_{m=0}^{x-1} \mathbf{G}^n(xq^{n-k},\ m) \right) \tag{3.5}$$

By difference of powers property

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... + a^2b^{n-3} + ab^{n-2} + b^{n-1})$$

we could receive divided difference as well, hence for each $a = b + h$

$$\frac{\Delta_h[a^n]}{h} = \sum_{k=1}^{n} \left(\sum_{m=0}^{x-1} \mathbf{G}^{n-k}(a,\ m) \cdot \sum_{r=0}^{y-1} \mathbf{G}^{k-1}(b,\ r) \right) \tag{3.6}$$

4. e^x Representation

Since the exponential function $f(x) = e^x,\ x \in \mathbb{R}$ is defined as infinite summation of $\frac{x^n}{n!},\ n = 1,\ 2,\ ...,\ \infty$ over n (see [5]). Then (2.7) could be applied, hereby

$$e^x = \sum_{m=0}^{\infty} \left(j \sum_{k=1}^{x} \left(\frac{kx^{m-2} - k^2x^{m-3} + \frac{x^{m-3}}{j}}{m!} \right) \right),\ x \in \mathbb{N} \tag{4.1}$$

Note that Wolfram Mathematica Code of expression (4.1) available in Application 1.

[2]MacMillan Double Binomial sum is the other one numerical expansion of the form

$$\sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{k-j} j^n \binom{k}{j} \binom{x}{k} = x^n$$

5. Conclusion and future research

In this paper expansion (2.7) is shown, as well as other combinations from section 2. Comparison of (2.7) to Binomial Theorem is shown in subsection (3.1). Additionally, (2.7) is applied on power's q-difference, see expression (3.4). Exponential function's e^x representation, applying (2.7) is shown in section 4. In Application 1 Wolfram Mathematica 11 codes of most expressions from sections 1 and 2 are shown and attached. In Application 2 extended version of Triangle (3) is shown. In future research the main aim is to extend finite difference (1.9) to infinitesimal case and derive new derivative. Note that in Application 3 generalized calculus of finite differences, based on (1.9) is shown.

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This paper dedicated to God's Word, Inspiring every act of creation, gives wisdom to each searching mind.

References

[1] Conway, J. H. and Guy, R. K. "Pascal's Triangle." In The Book of Numbers. New York: Springer-Verlag, pp. 68-70, 1996.
[2] Abramowitz, M. and Stegun, I. A. (Eds.). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, pp. 10, 1972.
[3] Weisstein, Eric W. "Power", From Mathworld.
[4] Arfken, G. Mathematical Methods for Physicists, 3rd ed. Orlando, FL: Academic Press, pp. 307-308, 1985.
[5] Rudin, Walter (1987). Real and complex analysis (3rd ed.). New York: McGraw-Hill. p. 1. ISBN 978-0-07-054234-1.
[6] Weisstein, Eric W. "Finite Difference." From Mathworld.
[7] Richardson, C. H. An Introduction to the Calculus of Finite Differences. p. 5, 1954.
[8] N. J. A. Sloane et al., Entry A008458 in [9], 2002-present.
[9] The OEIS Foundation Inc.,The On-Line Encyclopedia of Integer Sequences, 1964-present https://oeis.org/
[10] Bakhvalov N. S. Numerical Methods: Analysis, Algebra, Ordinary Differential Equations p. 59, 1977. (In russian)
[11] Jackson F.H. On q-functions and a certain difference operator. Trans. Roy Soc. Edin. 46, (1908), 253-281.
[12] P. Kolosov et al., Entry A287326 in [9], 2017.
[13] N. J. A. Sloane and Mira Bernstein et al., Entry A007318 in [9], 1994-present.

7. Application 1. Wolfram Mathematica 11 codes of some expressions

In this application the most expression's Wolfram Mathematica version 11 codes are shown, more particular
Equation (1.9), that is forward finite difference of cubes given $\Delta h = 1$

$$\mathsf{In[5] := Sum[6 * k + 1/x, \{k, 1, x\}]}$$

$$\mathsf{Out[5]} = 1 + 3\mathsf{x} + 3\mathsf{x}^2$$

Backward finite difference of cubes, next after (1.9), given $\Delta h = 1$

$$\mathsf{In[6] := Sum[6 * k + 1/x, \{k, 0, x - 1\}]}$$

$$\mathsf{Out[6]} = 1 - 3\mathsf{x} + 3\mathsf{x}^2$$

Generating formula of Triangle (1.17), Figure 3

$$\mathsf{In[6] := Column[Table[k * n * 6 - k^2 * 6 + 1, \{n, 0, 10\}, \{k, 0, n\}], Center]}$$

Right part of (1.11)

$$\mathsf{In}[7] := \mathsf{Sum}[k * x * 6 - k^2 * 6 + 1, \{k, 0, x - 1\}]$$

$$\mathsf{Out}[7] = x^3$$

Central part of (1.12)

$$\mathsf{In}[8] := \mathsf{Sum}[1 + 6 * \mathsf{Sum}[u, \{u, 0, m\}], \{m, 0, x - 1\}]$$

$$\mathsf{Out}[8] = x^3$$

Right part of (1.12)

$$\mathsf{Out}[9] := 6 * \mathsf{Sum}[m * x - m^2 + x/(6 * (x - 1)), \{m, 1, x - 1\}]$$

$$\mathsf{Out}[9] = x^3$$

Expression (2.2) first part

$$\mathsf{In}[10] := \mathsf{Sum}[(k + 1)^3 - k - 6 * \mathsf{Sum}[m * k - m^2, \{m, 1, k\}], \{k, 0, x - 1\}]$$

$$\mathsf{Out}[10] = x^3$$

Expression (2.2) second part

$$\mathsf{In}[11] := \mathsf{Sum}[k + 1 + 6 * \mathsf{Sum}[m * (k + 1) - m^2, \{m, 0, k\}] - k^3, \{k, 0, x - 1\}]$$

$$\mathsf{Out}[11] = x^3$$

Expression (2.3) first part

$$\mathsf{In}[12] := \mathsf{Sum}[k + 6 * \mathsf{Sum}[m * k - m^2, \{m, 1, k\}] - (k - 1)^3, \{k, 1, x\}]$$

$$\mathsf{Out}[12] = x^3$$

Expression (2.3) second part

$$\mathsf{In}[13] := \mathsf{Sum}[k^3 - k - 1 - 6 * \mathsf{Sum}[m * k + m - m^2, \{m, 0, k - 1\}], \{k, 1, x\}]$$

$$\mathsf{Out}[13] = -x^3$$

Expression (2.5) first part

$$\mathsf{In}[14] := \mathsf{Sum}[k^{(n-2)} + 6 * \mathsf{Sum}[m * k^{(n-2)} - m^2 * k^{(n-3)}, \{m, 0, k\}] - (k - 1)^n, \{k, 1, x\}]$$

$$\mathsf{Out}[14] = -0^n - \mathsf{HarmonicNumber}[-1 + x, -n] + \mathsf{HarmonicNumber}[x, -n]$$

Expression (2.6) first part

$$\mathsf{In}[15] := \mathsf{Sum}[x^{(n-3)} + 6 * \mathsf{Sum}[u * x^{(n-3)}, \{u, 0, m\}], \{m, 0, x - 1\}]$$

$$\mathsf{Out}[15] = x^n$$

Second part of (2.7)

$$\mathsf{In}[16] := \mathsf{Sum}[6 * m * x^{(n-2)} - 6 * m^2 * x^{(n-3)} + x^{(n-3)}, \{m, 0, x - 1\}]$$

$$\mathsf{Out}[16] = x^n$$

Second part of (2.7) with set $\mathfrak{S}(x)$

$$\mathsf{In}[17] := \mathsf{Sum}[6 * m * x^{(n-2)} - 6 * m^2 * x^{(n-3)} + x^{(n-3)}, \{m, 1, x\}]$$

$$\mathsf{Out}[17] = x^n$$

Expression (3.3) "Newton's Binomial Theorem representation"

$$\mathsf{In}[18] := \mathsf{Sum}[\mathsf{Binomial}[n, k] *$$

$$\mathsf{Sum}[6 * m * x^{(n-k-2)} - 6 * m^2 * x^{(n-k-3)} + x^{(n-k-3)}, \{m, 0, x - 1\}] *$$

$$\mathsf{Sum}[6 * r * y^{(k-2)} - 6 * r^2 * y^{(k-3)} + y^{(k-3)}, \{r, 0, y - 1\}], \{k, 0, n\}]$$

$$\mathsf{Out[18] = (x + y)^n}$$

Expression (4.1) e^x representation

```
In[19] := Sum[6 * Sum[(k * x^(m-2) - k^2 * x^(m-3) + x^(m-3)/6)/m!, {k, 1, x}], {m, 0, Infinity}]
```

$$\mathsf{Out[19] = E^x}$$

Templates of all the programs available online at this link.

8. Application 2. An extended version of triangle (3)

1
1 1
1 7 1
1 13 13 1
1 19 25 19 1
1 25 37 37 25 1
1 31 49 55 49 31 1
1 37 61 73 73 61 37 1
1 43 73 91 97 91 73 43 1
1 49 85 109 121 121 109 85 49 1
1 55 97 127 145 151 145 127 97 55 1
1 61 109 145 169 181 181 169 145 109 61 1

Figure 5. Extended version of Triangle (3) generated from (1.11) given $n = 3$ up to $x = 12$, sequence A287326 in OEIS, [9]

9. Application 3. Generalized calculus of finite differences based on (1.9)

Consider the equation (1.9)

$$\Delta[x^3] = \sum_{k=1}^{x} jk + \frac{1}{x},\ x = x_0 \in \mathbb{N},\ x \neq 0$$

Let extend it up to difference of x^4, we use the power's property $a^n = a^{n-1}a$, then

$$\Delta[x^4] = (x+1)^3(x+1) - x^3x = (x+1)^3x + (x+1)^3 - x^3x$$

Particularizing above expression, we have

$$\Delta[x^4] = \underbrace{(x+1)^3x - x^3x}_{\Delta[x^3]x} + (x+1)^3 = x\sum_{k=1}^{x}\left(jk + \frac{1}{x}\right) + (x+1)^3$$

This way, the general finite difference relation could be

$$\Delta[x^n] = \Delta[x^{n-1}]x + (x+1)^{n-1}$$

Let set recurrence relation of the form

$$\Delta[x^n] = \Delta[x^{n-k}]x^k + \sum_{j=0}^{k-1}(x+1)^{n-k+j} \cdot x^{k-j+1}$$

Then, substituting $k = n - 3$, (1.9) could be applied

$$\Delta[x^n] = \Delta[x^3]x^{n-3} + \sum_{j=0}^{n-4}(x+1)^{j+3} \cdot x^{n-2-j}$$

$$= x^{n-3}\sum_{k=1}^{x} 3! \cdot k + \frac{1}{x} + \sum_{j=0}^{n-4}(x+1)^{j+3} \cdot x^{n-2-j}$$