# INERTIAL ENERGY

BSc. Louai Hassan Elzein Bashier

Khartoum, Sudan.
Postal code:11123
E-mail: louaielzein@gmail.com

# Abstract

This paper is prepared to show that as a consequence of induced inertial force due to the acceleration of a system of particles or rigid body, an additional kind of energy will arise in that body or system of particles beside the kinetic energy of it. The Lagrange of this body will acquire an additional term to represent this energy. The paper shows multiple cases of object motion in which this energy is present and shows the derivation of its energy expressions.

## 1. Theoretical Background

The encyclopaedia Britannica defines the inertial force as "any force invoked by an observer to maintain the validity of Isaac Newton's second law of motion in a reference frame that is rotating or otherwise accelerating at a constant rate.[1]" The inertial force is the main force behind the occurrence of the inertial energy that which we are going to show its existence in this paper.

## 2. Analysis

### 2.1. Curvilinear Motion:

#### 2.1.1. *Rigid body in rotational motion*:

Referring to Figure 1, a rigid body "*A*" of mass $m$ is free to rotate relative to its center of mass CM as it is also can be simultaneously free to rotate relative to any arbitrary point $O$. Thus it is pivoted at these two points. If an external force $\mathbf{F}$ acts on its center of mass such that it causes it to angularly accelerate with angular acceleration $\boldsymbol{\alpha_R} = d\boldsymbol{\omega}/dt$ relative to the axis of rotation $O$, then the torque will be obtained by (assuming both axis of rotation, $O$ and CM, are parallel):

$$\begin{aligned}\boldsymbol{\tau_R} &= \boldsymbol{r}_O \times \left(m\frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}_O\right) \ , \\ &= I_O \alpha_R \boldsymbol{e} \ .\end{aligned}$$

where $\boldsymbol{r}_o$ is the vector position of the center of mass of the rigid body relative to the axis of rotation $O$. $m$ is the total mass of the rigid body. $I_o$ is the moment of inertia of this single mass around the pivot point $O$. $\boldsymbol{e}$ is a unit vector perpendicular to the plane of the motion. Therefore, an inertial force $\mathbf{F}_i$ will occur at any element mass $m_i$ of the rigid body. This inertial force is known as Euler force[2]

$$\mathbf{F}_i = \mathbf{F}_{Euler} = -m_i\frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}_i$$

where $\boldsymbol{\omega}$ is the angular velocity of rotation of the center of mass of the rigid body relative to the axis of rotation $O$ and $\boldsymbol{r}_i$ is the vector position of the point where the acceleration is measured relative to the axis of the rotation $O$. Thus we have

$$\boldsymbol{\tau_i} = \boldsymbol{\rho}_i \times \mathbf{F}_i \qquad (1)$$

where $\boldsymbol{\tau_i}$ is the torque of the element mass $m_i$ about the center of mass of the rigid body and $\boldsymbol{\rho}_i$ is the vector position of the element mass relative to the center of mass of the rigid body. Hence the total torque, $\boldsymbol{\tau_{-R}}$, due to the Euler force

$$\begin{aligned}\boldsymbol{\tau_{-R}} &= \sum_i \boldsymbol{\rho}_i \times \mathbf{F}_i \ , \\ &= \sum_i \boldsymbol{\rho}_i \times \left(-m_i\frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}_i\right) \ ,\end{aligned}$$

From Figure 1, we have $\boldsymbol{r}_i = \boldsymbol{r}_O + \boldsymbol{\rho}_i$. Hence one can write

$$\begin{aligned}
\boldsymbol{\tau_{-R}} &= \sum_i \boldsymbol{\rho}_i \times \left( -m_i \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}_O - m_i \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{\rho}_i \right) \ , \\
&= -\sum_i \boldsymbol{\rho}_i \times \left( m_i \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}_O \right) - \sum_i \boldsymbol{\rho}_i \times \left( m_i \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{\rho}_i \right) \ , \\
&= -\left( \sum_i m_i \boldsymbol{\rho}_i \times \frac{d\boldsymbol{\omega}}{dt} \right) \times \boldsymbol{r}_O - \sum_i \boldsymbol{\rho}_i \times \left( \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{\rho}_i \right) m_i \ ,
\end{aligned}$$

But from the properties of the center of mass we knew that $\sum_i m_i \boldsymbol{\rho}_i = 0$. Therefore, we obtain

$$\boldsymbol{\tau_{-R}} = -\sum_i \boldsymbol{\rho}_i \times \left( \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{\rho}_i \right) m_i \ , \tag{2}$$

Equation(2) can be mathematically simplified by doing the cross product using the Cartesian coordinate system and converting the summation to integration over the whole body[3][4]. Therefore one will find

$$\boldsymbol{\tau_{-R}} = I_{CM} \alpha_R(-\boldsymbol{e}) \ . \tag{3}$$

where $I_{CM}$ is the moment of inertia of the rigid body relative to its center of mass and $\alpha_R(-\boldsymbol{e}) = -d\boldsymbol{\omega}/dt$ is the angular acceleration of the rigid body relative to its center of mass where it is equal in magnitude to the rigid body angular acceleration relative to the axis of rotation $O$ —assuming both axis of rotation, $O$ and CM, are parallel. Therefore, an *inertial torque* $\boldsymbol{\tau_{-R}}$ will act over the rigid body (where $\boldsymbol{\tau_{-R}} \neq 0$) such that it will cause the rigid body to rotate relative to its center of mass[1] CM in a direction counter to the direction of rotation of the center of mass of the rigid body relative to the axis of rotation $O$, hence the rigid body will accumulate an *additional* rotational kinetic energy due to the interaction of the active torque $\boldsymbol{\tau_R}$ and the inertial torque $\boldsymbol{\tau_{-R}}$. Therefore, one can write the Lagrange of this body

$$L = T_R - V + T_{-R} \tag{4}$$

where $T_{-R}$ is the mentioned *inertial energy* where here it is happened due to the rotational motion. Thus, we have

$$T_{-R} = \tfrac{1}{2} I_{CM} \left\langle -\boldsymbol{\omega}, -\boldsymbol{\omega} \right\rangle \tag{5}$$

where $-\boldsymbol{\omega} = \alpha_R(-\boldsymbol{e})t$ is the instantaneous angular velocity of the rigid body relative to its center of mass.
The kinetic energy $T_R$ of the rigid body relative to the axis of rotation $O$[5][6]

$$T_R = \tfrac{1}{2} m \left\langle \boldsymbol{\omega} \times \boldsymbol{r}_O, \boldsymbol{\omega} \times \boldsymbol{r}_O \right\rangle + \tfrac{1}{2} I_{CM} \left\langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \right\rangle \tag{6}$$

[1] The observation of this phenomenon can be obtain easily by rotating a metallic disk pivoted at its center or by rotating a vessel containing ice cubes floating on water and can be exercise using your hand.

where $\boldsymbol{\omega} \times \boldsymbol{r}_O$ is the tangential velocity of the center of mass of the rigid body relative to the axis of rotation $O$. $V$ is the potential energy and $\boldsymbol{\Omega}$ is an arbitrary rotational angular velocity relative to the center of mass of the rigid body due to the application of any arbitrary external force over any point of the rigid body other than the center of mass of it. Therefore, the Lagrange becomes

$$L = \tfrac{1}{2}m \langle \boldsymbol{\omega} \times \boldsymbol{r}_O, \boldsymbol{\omega} \times \boldsymbol{r}_O \rangle + \tfrac{1}{2}I_{CM} \langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle + \tfrac{1}{2}I_{CM} \langle -\boldsymbol{\omega}, -\boldsymbol{\omega} \rangle - V \ , \tag{7}$$

Since $\boldsymbol{\omega}$ and $\boldsymbol{r}_O$ are mutually orthogonal, so that

$$L = \tfrac{1}{2}m \langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle \langle \boldsymbol{r}_O, \boldsymbol{r}_O \rangle + \tfrac{1}{2}I_{CM} \langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle + \tfrac{1}{2}I_{CM} \langle -\boldsymbol{\omega}, -\boldsymbol{\omega} \rangle - V \ , \tag{8}$$

Because the instantaneous rotational angular velocity $\boldsymbol{\omega}$ relative to the axis of rotation $O$ and the instantaneous rotational angular velocity $-\boldsymbol{\omega}$ relative to the center of mass —caused by the inertial torque— are equal in magnitude, then one can write

$$L = \tfrac{1}{2}(m \langle \boldsymbol{r}_O, \boldsymbol{r}_O \rangle + I_{CM}) \langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle + \tfrac{1}{2}I_{CM} \langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle - V \ ,$$

The quantity in the parenthesis, $m \langle \boldsymbol{r}_O, \boldsymbol{r}_O \rangle + I_{CM}$, is nothing other than the parallel axis theorem. Hence, the Lagrange becomes

$$L = \tfrac{1}{2}I_O \langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle + \tfrac{1}{2}I_{CM} \langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle - V \ . \tag{9}$$

where $I_O$ is the moment of inertia relative to the axis of rotation $O$ which is a perpendicular distance $\boldsymbol{r}_O$ from the centre of mass. The term $\tfrac{1}{2}I_O \langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle$ is the total rotational energy due to the rotational motion.

2.1.2. *Rigid body in spin motion*:

In rotational motion we found that the vector position $\boldsymbol{r}_O$ of the center of mass of the rigid body relative to the axis of rotation $O$ is irrelevant to the build up of the inertial energy. In the case of the spin motion we have $\boldsymbol{r}_O = 0$, therefore we expect the same result. Referring to Figure 2, a rigid body "$B$" of mass $m$ is free to rotate relative to its center of mass CM. When an external force $\mathbf{F}$ acts on the rigid body at any point other than its center of mass, then this force will create a torque

$$\boldsymbol{\tau}_{\boldsymbol{s}} = I_{CM}\alpha_S \boldsymbol{e}$$

where $I_{CM}$ is the moment of inertia of the rigid body relative to its center of mass CM and $\alpha_S \boldsymbol{e} = d\boldsymbol{\Omega}/dt$ is the angular acceleration of the rigid body relative to its center of mass and $\boldsymbol{\Omega}$ is the angular velocity of rotation of the rigid body relative to its center of mass (it is the same angular velocity mentioned in equation(6)). Therefore, Euler force $\mathbf{F}_i$ will occur at any element mass $m_i$ of that rigid body and causes an inertial torque $\boldsymbol{\tau}_{\text{-}s}$ such that it will spin relative to its center of mass in a direction counter to the direction of the original motion. Hence from Figure 2 we have

$$\begin{aligned} \boldsymbol{\tau}_{\text{-}s} &= \sum_i \boldsymbol{\rho}_i \times \mathbf{F}_i \ , \\ &= \sum_i \boldsymbol{\rho}_i \times \left( -m_i \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{\rho}_i \right) \ , \end{aligned} \tag{10}$$

Therefore, we obtain

$$\boldsymbol{\tau}_{-\boldsymbol{s}} = I_{CM}\alpha_S(-\boldsymbol{e}) \ . \tag{11}$$

where $I_{CM}$ is the moment of inertia of the rigid body relative to its center of mass and $\alpha_S(-\boldsymbol{e}) = -d\boldsymbol{\Omega}/dt$ is the angular acceleration of the rigid body relative to its center of mass due to the inertial torque $\boldsymbol{\tau}_{-\boldsymbol{s}}$. Hence, the inertial energy due to spin motion

$$T_{-S} = \tfrac{1}{2}I_{CM}\left\langle -\boldsymbol{\Omega}, -\boldsymbol{\Omega}\right\rangle \tag{12}$$

where $-\boldsymbol{\Omega} = \alpha_S(-\boldsymbol{e})t$ is the instantaneous angular velocity of the rigid body relative to its center of mass and $T_{-S}$ is inertial energy due to the spin motion. The kinetic energy $T_S$ of the rigid body relative to the center of mass

$$T_S = \tfrac{1}{2}I_{CM}\left\langle \boldsymbol{\Omega}, \boldsymbol{\Omega}\right\rangle \tag{13}$$

Therefore, the Lagrange

$$\begin{aligned} L &= T_S - V + T_{-S} \ , \\ &= \tfrac{1}{2}I_{CM}\left\langle \boldsymbol{\Omega}, \boldsymbol{\Omega}\right\rangle + \tfrac{1}{2}I_{CM}\left\langle -\boldsymbol{\Omega}, -\boldsymbol{\Omega}\right\rangle - V \ . \end{aligned} \tag{14}$$

where the term $\frac{1}{2}I_{CM}\left\langle \boldsymbol{\Omega}, \boldsymbol{\Omega}\right\rangle$ is the one that appears in equation(8). Because the instantaneous rotational angular velocity $\boldsymbol{\Omega}$ relative to the center of mass and the instantaneous rotational angular velocity $-\boldsymbol{\Omega}$ relative also to the center of mass —caused by the inertial torque— are equal in magnitude, then one can write equation(14)

$$L = I_{CM}\left\langle \boldsymbol{\Omega}, \boldsymbol{\Omega}\right\rangle - V \tag{15}$$

The term $I_{CM}\left\langle \boldsymbol{\Omega}, \boldsymbol{\Omega}\right\rangle$ is the total rotational energy due to spin motion and this term is nothing other than the parallel axis theorem where the parallel axis is passing through the center of mass itself or in another words it is zero perpendicular distance from the centre of mass.

The sum of equations(8) and (14) gives the complete Lagrange for the rigid body in curvilinear motion —rotating and spinning at the same time. Therefore,

$$L = T - V + N \tag{16}$$

where $T = T_R + T_S$, the total rotational kinetic energy, and $N = T_{-R} + T_{-S}$, the total inertial energy. Hence, we can write

$$\begin{aligned} L &= (T_R + T_S) - V + (T_{-R} + T_{-S}) \ , \\ &= (T_R + T_{-R}) + (T_S + T_{-S}) - V \ , \\ &= \tfrac{1}{2}m\left\langle \boldsymbol{r}_O, \boldsymbol{r}_O\right\rangle\left\langle \boldsymbol{\omega}, \boldsymbol{\omega}\right\rangle + \tfrac{1}{2}I_{CM}\left\langle -\boldsymbol{\omega}, -\boldsymbol{\omega}\right\rangle + \tfrac{1}{2}I_{CM}\left\langle \boldsymbol{\Omega}, \boldsymbol{\Omega}\right\rangle + \tfrac{1}{2}I_{CM}\left\langle -\boldsymbol{\Omega}, -\boldsymbol{\Omega}\right\rangle - V \ , \\ &= \tfrac{1}{2}I_O\left\langle \boldsymbol{\omega}, \boldsymbol{\omega}\right\rangle + I_{CM}\left\langle \boldsymbol{\Omega}, \boldsymbol{\Omega}\right\rangle - V \ . \end{aligned} \tag{17}$$

For the sake of completion we have to mention that the centripetal force and its counter the centrifugal force are also contribute to this Lagrange while the Coriolis force has no contribution due to the absence of the radial vector velocity[2].

## 2.2. Rectilinear Motion:

### 2.2.1. *System of particles in translational motion*:

Referring to Figure 3-(A), the frame of reference $S'$ represents a system of particles with two masses $m_1$ and $m_2$ which are bind together elastically with a bond[3] of elastic constant $k$. The $S'$-frame in Figure 3-(A) is at rest with respect to an observer $O$ on an inertial frame of reference $S$. Both of the frames, $S$ and $S'$, are positioned in free space. In Figure 3-(B), the $S'$-frame accelerated uniformly from rest with acceleration $a$ with respect to the $S$-frame and along its $+x$-axis. The $S'$-frame will invoke the inertial force, therefore the mass $m_2$ –which is a part of the $S'$-frame– will accelerate[4] with uniform acceleration $a$ along the $-x'$-axis of the $S'$-frame, causing the elastic bond to deform and hence charged it with an elastic potential energy $\frac{1}{2}k(L-l)^2$. Meanwhile, the $S'$-frame with all of its constituents will also accelerate along the $+x$-axis with respect to the $S$-frame with uniform acceleration $a$ –see Figure 3-(C). After accelerating for a time $t$ the whole system will end up moving with velocity $u$ along the $+x$-axis with respect to the $S$-frame and in addition to that the internal constituents of the system, $m_1$ and $m_2$, will vibrate[5] with respect to each other with energy $\frac{1}{2}k(L-l)^2$ –see Figure 3-(D). Therefore, one can write the Lagrange of this system of particles

$$\begin{aligned} L &= T - V + N, \\ &= \tfrac{1}{2}Mu^2 + \tfrac{1}{2}k(L-l)^2 - V. \end{aligned} \tag{18}$$

where $T = \frac{1}{2}Mu^2$ is the total kinetic energy of the whole system. $M = m_1 + m_2$ is the total mass of the system. $u = at$ is the final velocity of the whole system with respect to the $S$-frame. $L - l$ is the deformation in elastic bond due to the interaction of active force and inertial force over the system. $V$ is the potential energy. $N$ is the mentioned *inertial energy* and in this specific case its value is equivalent to $\frac{1}{2}k(L-l)^2$.

### 2.2.2. *Rigid body in translational motion*:

Referring to Figure 4, a rigid body "$C$" of mass $m$ is free to translational move relative to the frame of reference $S$. When an external force $\mathbf{F}$ acts on the rigid body at its center of mass CM, then the body will move in straight line in the direction of the active force $\mathbf{F}$ with uniform acceleration $a\boldsymbol{e} = d\boldsymbol{v}/dt$ relative to the $S$-frame; where "$a$" is the magnitude of the translational acceleration

[2] We do not interest in centrifugal and Coriolis forces in this search.
[3] The mass of the bond is negligible.
[4] Under the influence of inertial force $F = m_2 a$.
[5] This vibratory motion will add nothing to the total linear momentum of the body. The total momentum is conserved.

of the rigid body. $\boldsymbol{e}$ is a unit vector parallel to the plane of the motion. $\boldsymbol{v}$ is the translational velocity of the rigid body. Therefore,

$$\mathbf{F} = ma\boldsymbol{e}$$

Since the motion of the rigid body under influence of the active force $\mathbf{F}$ with respect to $S$-frame is non-inertial then an inertial force $\mathbf{F}_i$ will occur at every element mass $m_i$ of the rigid body. Hence, the total inertial torque $\boldsymbol{\tau}_{inertial}$ due to this inertial force will be

$$\begin{aligned}
\boldsymbol{\tau}_{inertial} &= \sum_i \boldsymbol{\rho}_i \times \mathbf{F}_i \ , \\
&= \sum_i \boldsymbol{\rho}_i \times \left( -m_i \frac{d\boldsymbol{v}}{dt} \right) \ , \\
&= -\sum_i m_i \boldsymbol{\rho}_i \times \frac{d\boldsymbol{v}}{dt} \ , \qquad (19)
\end{aligned}$$

But from the properties of the center of mass we knew that $\sum_i m_i \boldsymbol{\rho}_i = 0$. Therefore, we obtain

$$\boldsymbol{\tau}_{inertial} = 0 \ . \qquad (20)$$

Thus, the inertial torque does *not* present in the rigid body translational motion. To find the total inertial force $\mathbf{F}_{inertial}$, one have

$$\begin{aligned}
\mathbf{F}_{inertial} = -\sum_i m_i \frac{d\boldsymbol{v}}{dt} \ &= \ -m\frac{d\boldsymbol{v}}{dt} \ , \\
&= \ ma(-\boldsymbol{e}) \ . \qquad (21)
\end{aligned}$$

where $a(-\boldsymbol{e}) = -d\boldsymbol{v}/dt$ is the translational acceleration of the rigid body relative to the $S$-frame due to the inertial force. Therefore, the inertial energy, $N$, due to the translational motion of the rigid body

$$N = \tfrac{1}{2}m \left\langle -\boldsymbol{v}, -\boldsymbol{v} \right\rangle \qquad (22)$$

where $\boldsymbol{v}$ is the instantaneous translational velocity relative to the $S$-frame. The kinetic energy, $T$, of the rigid body

$$T = \tfrac{1}{2}m \left\langle \boldsymbol{v}, \boldsymbol{v} \right\rangle$$

Therefore, the Lagrange

$$\begin{aligned}
L &= T - V + N \ , \\
&= \tfrac{1}{2}m \left\langle \boldsymbol{v}, \boldsymbol{v} \right\rangle - V + \tfrac{1}{2}m \left\langle -\boldsymbol{v}, -\boldsymbol{v} \right\rangle \ , \qquad (23)
\end{aligned}$$

Because the instantaneous translational velocity $\boldsymbol{v}$ relative to the $S$-frame and the instantaneous translational velocity $-\boldsymbol{v}$ relative also to the $S$-frame — caused by the inertial force— are equal in magnitude, then one can write equation(23)

$$L = m \left\langle \boldsymbol{v}, \boldsymbol{v} \right\rangle - V \ . \qquad (24)$$

The term $m \left\langle \boldsymbol{v}, \boldsymbol{v} \right\rangle$ represents the total energy of the rigid body in translational motion.

## 3. Conclusion

Throughout space there is energy that arises in non-inertial frames of references. This energy accumulates in rigid bodies and systems of particles due to the interaction between active forces and inertial forces over them.

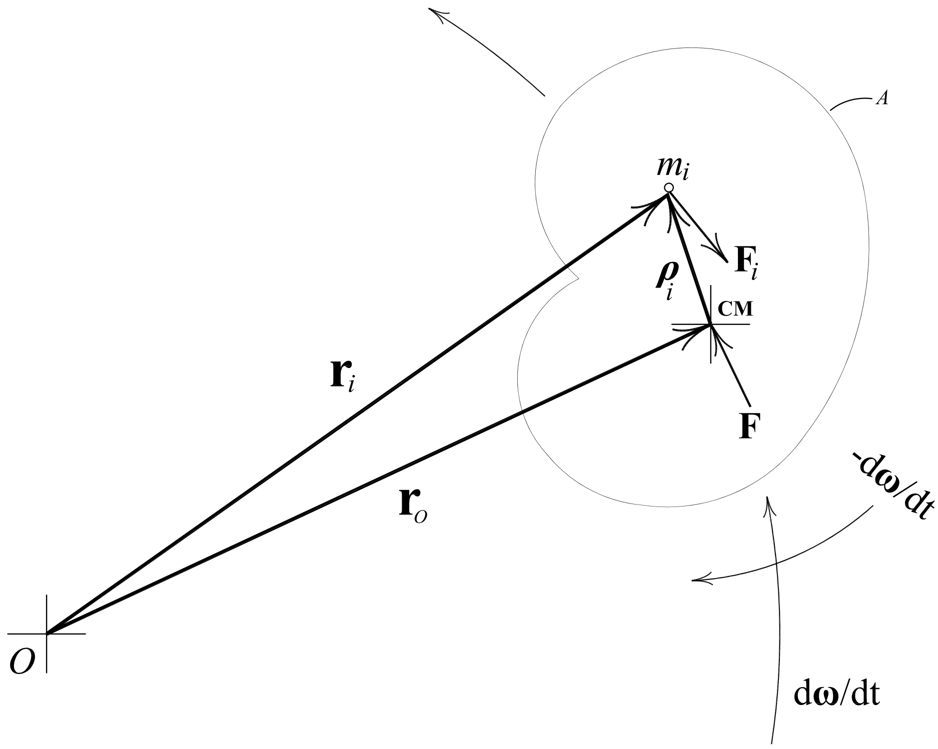


Figure 1: The accumulation of inertial energy in curvilinear motion.

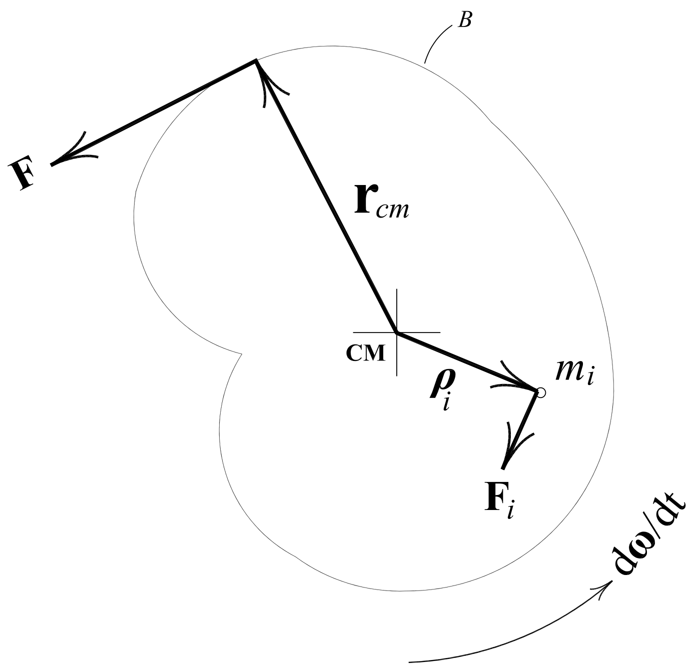


Figure 2: The accumulation of inertial energy in spin motion.

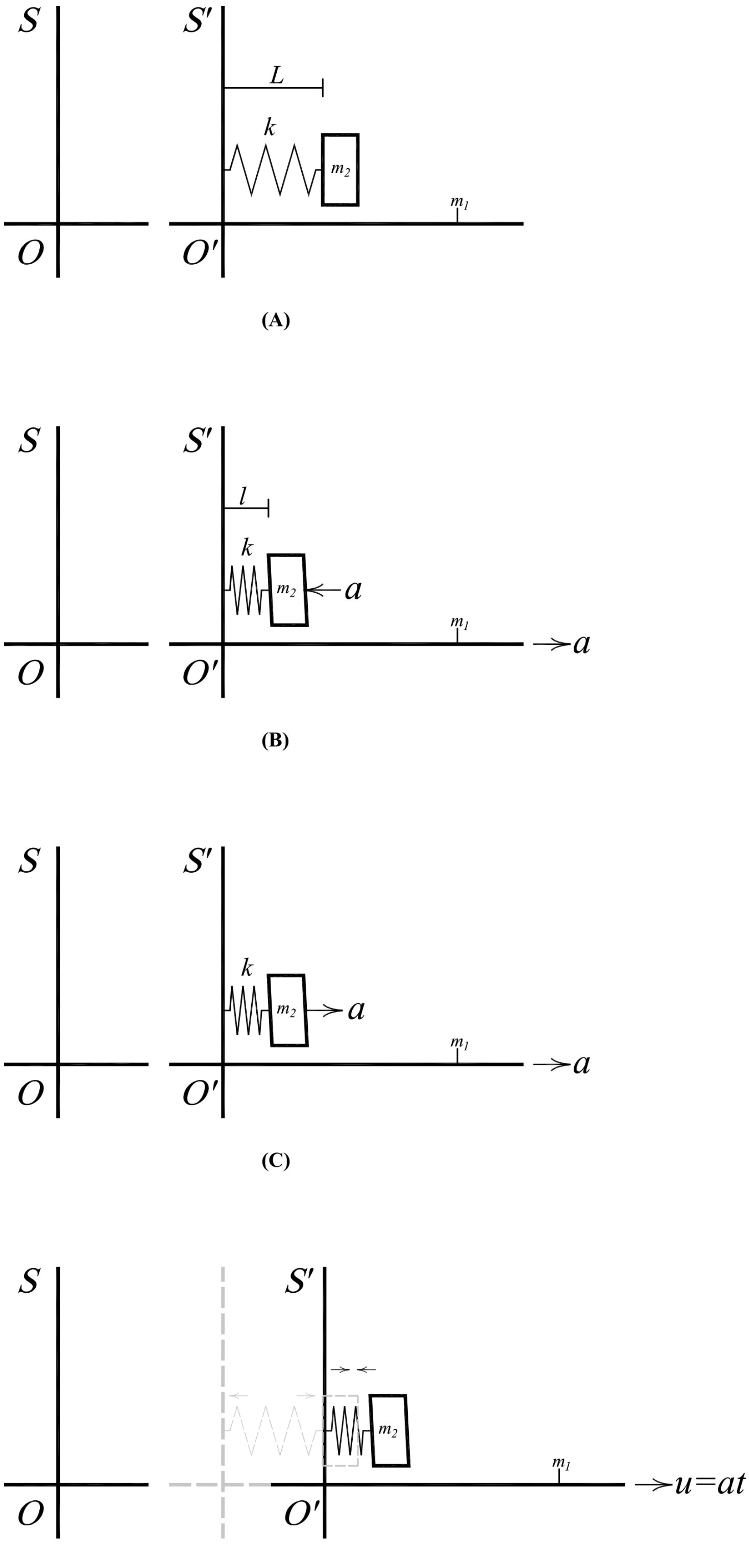


Figure 3: The accumulation of inertial energy in rectilinear motion. Case of the system of particles.

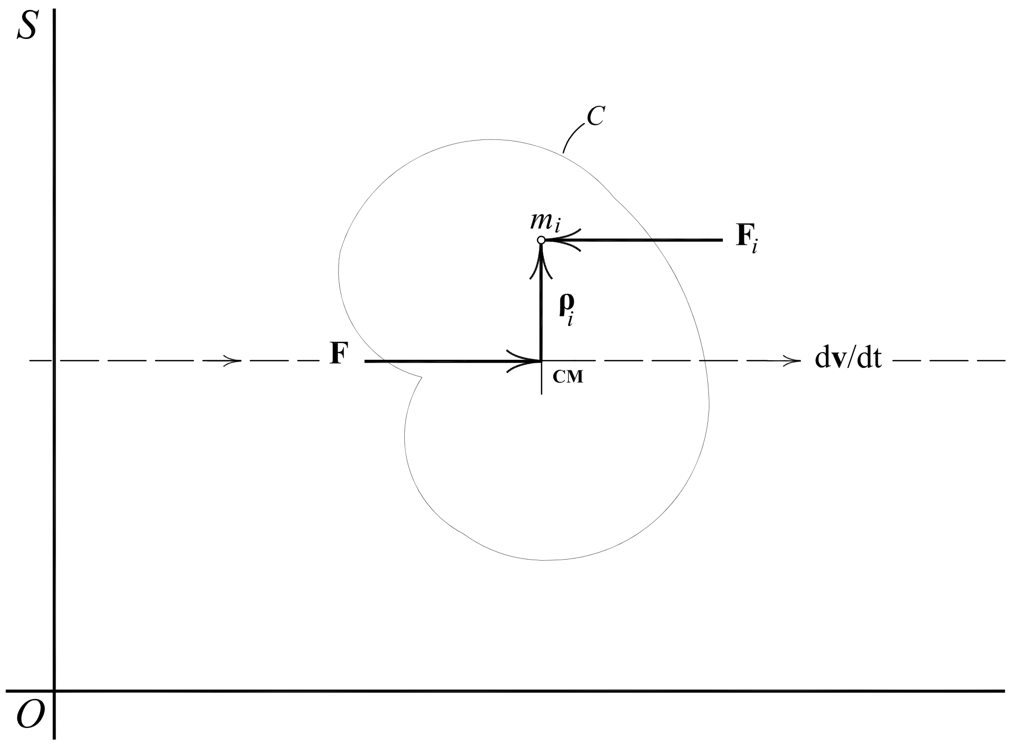


Figure 4: The accumulation of inertial energy in rectilinear motion. Case of the rigid body.

## References


[1] "inertial force", *Encyclopaedia Britannica Online.* 2017. Web. https://www.britannica.com/science/inertial-force.

[2] David Morin, *Introduction to classical mechanics: with problems and solutions*, Cambridge University Press 2008, p. 469.

[3] Daniel Fleisch, *A Student's Guide to Vectors and Tensors*, Cambridge University Press 2012, p. 163 to 164.

[4] G. E. Hay, *Vector and Tensor Analysis*, Dover Publications, Inc., in 1953, p. 80 to 82.

[5] Paul, Burton, *Kinematics and Dynamics of Planar Machinery*, Prentice Hall, in 1979.

[6] Uicker, John J.; Pennock, Gordon R.; Shigley, Joseph E., *Theory of Machines and Mechanisms* (4th ed.), Oxford University Press, in 2010.