INERTIAL ENERGY

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Abstract

This paper is prepared to show that as a consequence of induced inertial force due to the acceleration of a mass-body that has internal structure, an additional kind of energy will accumulate in that body beside the kinetic energy of it. The Lagrange of this body will acquire an additional term to represent this energy. The paper shows multiple cases of object motion in which this energy is present and shows the derivation of its energy expressions.

1. Theoretical Background

The encyclopaedia Britannica explain the inertial force as "any force invoked by an observer to maintain the validity of Isaac Newton's second law of motion in a reference frame that is rotating or otherwise accelerating at a constant rate.[1]" The inertial force is the main force behind the occurrence of the inertial energy that which we are going to show its existence in this paper.

2. Analysis

All coming cases are consider to be happening in the free space such that the potential energy is null.

2.1. Rectilinear Motion:

Referring to Figure 1-(A), the frame of reference S' represents a body with internal structure of masses m_1 and m_2 which are bind together with a spring[1] of constant k. The S'-frame in Figure 1-(A) is at rest with respect to an observer O on an inertial frame of reference S. Both of the frames, S and S', are positioned in free space. In Figure 1-(B), the S'-frame accelerated uniformly from rest with acceleration a with respect to the S-frame and along its $+x$-axis. The S'-frame will invoke the inertial force, therefore the mass m_2 –which is a part of the S'-frame– will accelerate[2] with uniform acceleration a along the $-x'$-axis of the S'-frame, causing the spring to compress and hence charged it with an elastic potential energy $\frac{1}{2}k(L-l)^2$. Meanwhile, the S'-frame with all of its constituents will also accelerate along the $+x$-axis with respect to the S-frame with uniform acceleration a –see Figure 1-(C). After accelerating for a time t the whole body will end up moving with velocity u along the $+x$-axis with respect to the S-frame and in addition to that the internal constituents of the body, m_1 and m_2, will vibrate[3] with respect to each other with energy $\frac{1}{2}k(L-l)^2$ –see Figure 1-(D). Therefore, one can write the Lagrange of this body

$$L = T - V + \Omega \tag{1}$$

where $T = \frac{1}{2}Mu^2$ is the total kinetic energy of the body. $M = m_1 + m_2$ is the total mass of the body. $u = at$ is the final velocity of the body with respect to the S-frame. V is the potential energy (here $V = 0$). Ω is the mentioned *inertial energy* and in this specific case its value is equivalent to $\frac{1}{2}k(L-l)^2$.

2.2. Curvilinear Motion:

Referring to Figure 2, a rigid body "A" of mass m is free to rotate about its center of mass CM as it can be simultaneously rotate around any arbitrary point O. Thus it is pivoted at these two points. If an external force $\mathbf{F}$ acts

[1] The mass of the spring is negligible.

[2] Under the influence of inertial force $F = m_2 a$.

[3] This vibratory motion will add nothing to the total linear momentum of the body. The total momentum is conserved.

on its center of mass such that it causes the rigid body to angularly accelerate with angular acceleration $d\boldsymbol{\omega}/dt$ relative to the axis of rotation O then an inertial force $\mathbf{F}_i$ will occur at any element mass m_i on the rigid body. This inertial force is known by Euler force[2]

$$\mathbf{F}_i = \mathbf{F}_{Euler} = -m_i \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}_i$$

where $\boldsymbol{\omega}$ is the angular velocity of rotation of the center of mass of the rigid body relative to the axis O and $\boldsymbol{r}_i$ is the vector position of the point where the acceleration is measured relative to the axis of the rotation. Thus we have

$$\boldsymbol{\tau_i} = \boldsymbol{\rho}_i \times \boldsymbol{F}_i \tag{2}$$

where $\boldsymbol{\tau_i}$ is the torque of the element mass m_i about the center of mass of the rigid body and $\boldsymbol{\rho}_i$ is the vector position of the element mass relative to the center of mass of the rigid body. Hence the total torque due to the Euler force

$$\begin{aligned}\boldsymbol{\tau_r} &= \sum_i \boldsymbol{\rho}_i \times \boldsymbol{F}_i \ , \\ &= \sum_i \boldsymbol{\rho}_i \times \left(-m_i \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}_i \right) \ ,\end{aligned}$$

From Figure 2, we have $\boldsymbol{r}_i = \boldsymbol{r}_{cm} + \boldsymbol{\rho}_i$ where $\boldsymbol{r}_{cm}$ is the vector position of the center of mass relative to the axis of rotation O. Hence one can write

$$\begin{aligned}\boldsymbol{\tau_r} &= \sum_i \boldsymbol{\rho}_i \times \left(-m_i \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}_{cm} - m_i \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{\rho}_i \right) \ , \\ &= -\sum_i \boldsymbol{\rho}_i \times \left(m_i \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}_{cm} \right) - \sum_i \boldsymbol{\rho}_i \times \left(m_i \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{\rho}_i \right) \ , \\ &= -\left(\sum_i m_i \boldsymbol{\rho}_i \times \frac{d\boldsymbol{\omega}}{dt} \right) \times \boldsymbol{r}_{cm} - \sum_i \boldsymbol{\rho}_i \times \left(\frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{\rho}_i \right) m_i \ ,\end{aligned}$$

But from property of the center of mass we knew that $\sum_i m_i \boldsymbol{\rho}_i = 0$. Therefore, we get

$$\boldsymbol{\tau_r} = -\sum_i \boldsymbol{\rho}_i \times \left(\frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{\rho}_i \right) m_i \tag{3}$$

Equation(3) can be mathematically simplified by doing the cross product using the Cartesian coordinate system[3][4]. Therefore, one finds

$$\boldsymbol{\tau_r} = -I_C \boldsymbol{\alpha} \tag{4}$$

where I_C is the inertia tensor of the rigid body relative to its center of mass and $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$ is the angular acceleration of the rigid body about its center of mass where it is also equal to its angular acceleration around the point O. Therefore, an *inertial torque* $\boldsymbol{\tau_r}$ will occur on the rigid body $(\boldsymbol{\tau_r} \neq 0)$ such that will cause it to rotate around its center of mass[4] CM in a direction counter to the direction of rotation of the rigid body relative to the axis O and the rigid body will accumulate an *additional* rotational kinetic energy due to this inertial torque. Thus we find

$$\Omega = \tfrac{1}{2} I_C \omega_f^2 \tag{5}$$

where $\omega_f = \alpha t$ is the terminal angular velocity of the rigid body relative to its center of mass and Ω is the inertial energy that defined earlier. Hence equation(1) is also valid in curvilinear motion where here the rotation kinetic energy T of the rigid body relative to the axis of rotation O is $\frac{1}{2} m \omega_f^2 r_{cm}^2$ where m is the total mass of the rigid body and $\omega_f r_{cm}$ is its tangential velocity relative to the axis of rotation O.

In this specific sort of motion one can states that: any object accelerated to an axis rotation relative to its attractive inertial mass, immediately becomes activated by inertial energy and acts as an independent force.

2.3. Spin Motion

In curvilinear motion we found that the vector position $\boldsymbol{r}_{cm}$ of the center of mass of the rigid body relative to the center of rotation O is irrelevant to the build up of the inertial energy. In the case of the spin motion we have $\boldsymbol{r}_{cm} = 0$ therefore we expect the same result. Referring to Figure 3, a rigid body "B" of mass m is free to rotate about its center of mass CM. When an external force $\mathbf{F}$ acts on the rigid body then Euler force $\mathbf{F}_i$ will occur at any element mass m_i of it and an inertial torque $\boldsymbol{\tau_r}$ will occur on the rigid body such that causes it to spin around its center of mass in a direction counter to the direction of the original motion. Thus we have

$$\begin{aligned} \boldsymbol{\tau_r} &= \sum_i \boldsymbol{\rho}_i \times \boldsymbol{F}_i \ , \\ &= \sum_i \boldsymbol{\rho}_i \times \left(-m_i \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{\rho}_i \right) \ , \end{aligned} \tag{6}$$

Therefore, we again will obtain

$$\boldsymbol{\tau_r} = -I_C \boldsymbol{\alpha} \ . \tag{7}$$

and

$$\Omega = \tfrac{1}{2} I_C \omega_f^2 \tag{8}$$

In this specific sort of motion the body accumulates inertial energy as potential energy instead of rotational kinetic energy –although the expression is similar to the expression of the rotational kinetic energy– and that happens due to the

[4]The observation of this phenomenon can be obtained by rotating a metallic disk pivoted at its center or by rotating a vessel containing ice cubes floating on water and can be exercise using your hand.

reaction of the active torque $\boldsymbol{\tau} = I_C\boldsymbol{\alpha}$ and the inertial torque. The inertial torque will not hinder the original motion of the body and the inertial energy will be carried with the body similar to the case of the linear motion that has been mentioned above.

3. Conclusion

Throughout space there is energy. This energy accumulates in non-inertial frames due to the reaction of active forces and inertial forces over the mass-bodies.

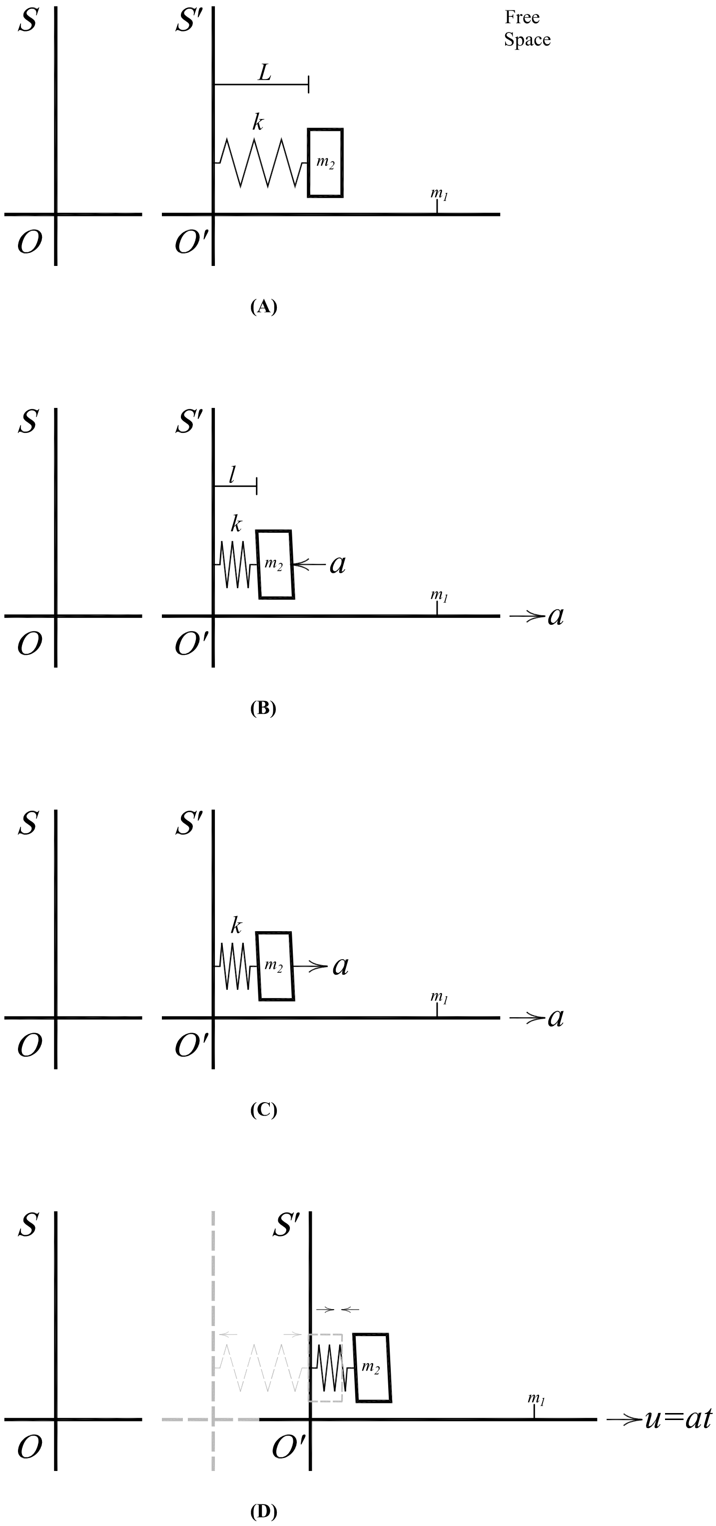

Figure 1: The accumulation of inertial energy in rectilinear motion.

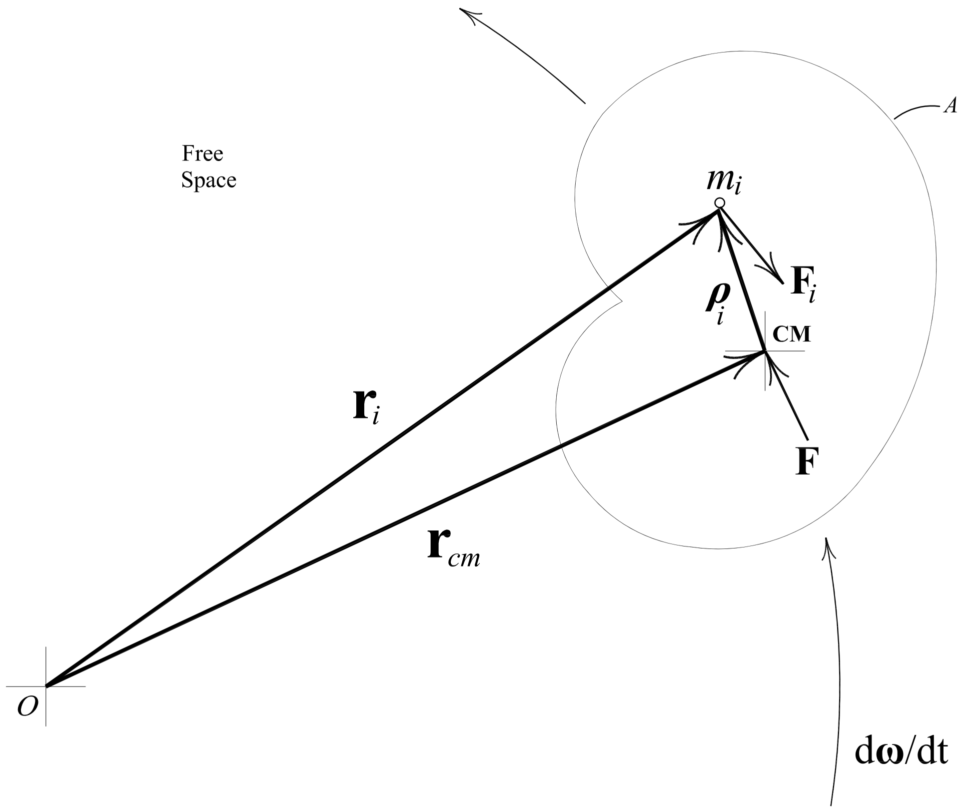

Figure 2: The accumulation of inertial energy in curvilinear motion.

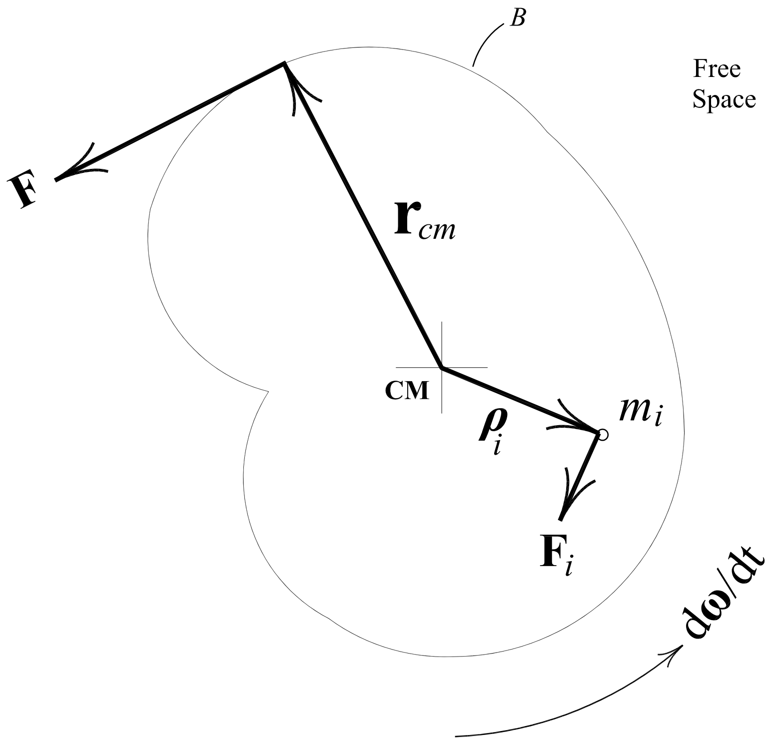

Figure 3: The accumulation of inertial energy in spin motion.

References

[1] "inertial force", *Encyclopaedia Britannica Online.* 2017. Web. https://www.britannica.com/science/inertial-force.

[2] David Morin, *Introduction to classical mechanics: with problems and solutions*, Cambridge University Press 2008, p. 469.

[3] Daniel Fleisch, *A Student's Guide to Vectors and Tensors*, Cambridge University Press 2012, p. 163 to 164.

[4] G. E. Hay, *Vector and Tensor Analysis*, Dover Publications, Inc., in 1953, p. 80 to 82.