Quaternions and Elliptical Space
(*Quaternions et Espace Elliptique*[1])

GEORGES LEMAÎTRE[2]

Pontifical Academician[3]
The Pontifical Academy of Sciences of Vatican City

RICHARD L. AMOROSO (Trans.)[4]

Noetic Advanced Studies Institute
Escalante Desert, Beryl, UT 84714 USA
amoroso@noeticadvancedstudies.us

Translators Forward: This is generally a literal translation, terms such as *versor* and *parataxis* in use at the time are not updated to contemporary style, rather defined and briefly compared in an appendix. The decision to translate Lemaître's 1948 essay arose not merely because of personal interest in Lemaître's 'persona', Physicist - Priest, but from an ever increasing interest in Quaternions and more recent discovery of correspondence with Octonions in additional dimensional (XD) brane topological phase transitions (currently most active research arena in all physics), and to make this particular work available to readers interested in any posited historical value during the time quaternions were still considered a prize of some merit, which as well-known, were marginalized soon thereafter (beginning mid1880's) by the occluding dominance of the rise of vector algebra; indeed, Lemaître himself states in his introduction: "*Since elliptic space plays an increasingly important part … I have thought that an exposition … could present some utility even if the specialists … must bear the judgment that it contains nothing really new*", but also because the author feels quaternions (likely in conjunction with octonions), extended into elliptical and hyperbolic XD spaces, especially in terms of the ease they provide in simplifying the Dirac equation, will be essential facilitators in ushering in post-standard model physics of unified field mechanics. The translator comes to realizes that 3rd regime Natural Science (Classical $\rightarrow$ Quantum $\rightarrow$ Unified Field Mechanics) will be described by a reformulated M-Theoretic topological field theory, details of which will be best described by Quaternion-Octonion correspondence.

QUATERNIONS AND ELLIPTICAL SPACE

The author applies the notion of quaternions, as practiced by Klein in the Erlangen program, to determine the fundamental properties of elliptical space[5]**.**

Keywords: Elliptical space, Klein's Erlangen program, Quaternions

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[1] Original (*) note: Submitted February 8, 1948*, Pontificia Academia Scientiarum,* ACTA, Vol. XII, No. 8, pp. 57-80 [1,2].
[2] Georges Lemaître [3].
[3] The Pontifical Academy of Sciences of Vatican City, established in 1936 by Pope Pius XI, promotes the progress of mathematical, physical, and natural sciences and the study of related epistemological problems.
[4] In the work of translation, effort was focused on *meaning*, and not maintaining more literal phraseology of 70 years ago, for example *certainly* might be substituted for *in all certainty.*
[5] Translation of Lemaître's original Latin abstract, termed *Summarium*.

1. Introduction

Quaternions were invented in 1843 by Sir William Rowan Hamilton. It is hard to imagine with what enthusiasm, and also with what confusion this awesome idea was developed by its author.

In the “Introduction to Quaternions” published in London (MacMillan, 1873) by P. Kelland and P.G. Tait, the first author declares: “*The first work of Sir Wm. Hamilton*” Lectures on Quaternions (1852), “*was very dimly and imperfectly understood by me and I dare say by others*”. He added that the *Elements of Quaternions* (1865) [Hamilton] and even exposits that most of the work of his co-author P.G. Tait: *An Elementary Treatise on Quaternions* cannot be regarded as elementary.

The book itself in which these remarks were certainly taken in an elementary character, he even exaggerated in this direction, by presenting demonstrations of too familiar theorems for which the use of a new type of calculation does not seem to be justified.

However, the influence of Hamilton’s discovery was very great. Not only did vector calculus, with its fecund notions of scalar product and vector product emerge, but also the development of elliptic geometry by Cayley, Clifford, etc., seems to have been strongly influenced by the new calculus, as shown by the title of one of these works: “Preliminary sketch on bi-quaternions” (1873).

I do not propose to disentangle the dense history of these discoveries, but by studying elliptic or spherical space, it appeared to me that quaternions provide extremely simple and elegant notations from which the properties of this space immediately flows.

Since elliptic space plays an increasingly important part in cosmogonic[6] theories, I have thought that an exposition which presupposes in the reader only elementary knowledge of analytic geometry could present some utility even if the specialists in the fields of algebra, geometry and history of science of the last century, must bear the judgment that it contains nothing really new.

For the history of the question, the reader may refer to treatises on geometry and particularly to the work of V. Blaschke, *Nicht Euklidische Geometrie und Mechanik* (Teubner 1942), which has more than one point in common with the present exposition but is addressed to a completely different category of readers.

2. Vectors

A vector will be represented by the geometric point of view and algebraic perspective.

In algebraic terms, the vector is obtained from the body of the real numbers, called a *scalar*, by introducing their new symbols not contained in the set of real numbers, and generally designated by the letters i, j, k.

Except for these three letters, whose employment is enshrined in use, we will assume that any Latin letter denotes a scalar, that is to say a real number.

A vector will be represented by

$$xi + yj + zk. \tag{1}$$

The addition of vectors and the multiplication by a scalar will be obtained by the ordinary rules of calculation as if i, j, k were numbers. The result of these operations will still be a vector.

Geometrically, the symbols i, j, k represents a basis, that is to say three vectors of unit length not situated in the same plane. We shall assume that this basis is orthogonal, that is, the three vectors i, j, k are mutually perpendicular.

Then, the three scalars x, y, z are the components, or orthogonal vector projections, onto the three vectors of the basis.

The components of the sum of two vectors are the sums of the components of these vectors.

3. Directions

It is customary to designate vectors by Greek letters. We will however deviate somewhat from this traditional notation by reserving Greek letters for the unitary vectors alone, That is to say to the vectors for which the sum of the squares of the components is equal to one.

We thought it necessary to introduce a shorter designation for the expression "unit vector". The term "direction" was deemed appropriate. Indeed, since a vector is a directed quantity, the unitary vector, whose magnitude is fixed once and for all, only indicates the direction thus the term direction is well suited to it.

4. Quaternions

The main idea of Hamilton has been to define the law of multiplying the symbols i, j, k in such a way that all the rules of the algebraic calculation remain valid except for one: The commutative property of multiplication. He thus founded the noncommutative algebra.

In this algebra, the value of a bridge product depends

[6] Cosmogony - theories of the origin of the universe. Cosmogony distinguishes itself from cosmology in that it allows theological argument.

on the order of the factors.

Starting from the multiplication table of two of the symbols *i* and *j*,

$$i^2 = -1,\ j^2 = -1,\ ij = -ji = k \tag{2}$$

we can easily deduce from these formulas (by the application of ordinary rules of the calculus, taking care to respect the order in which the factors present themselves) that the analogous formulas obtained by circularly permuting the letters *i, j, k,*

$$k^2 = -1,\ jk = -kj = i,\ ki = -ik = j \tag{3}$$

are valid.

Applying these rules of computation to the product of a direction of components *x, y, z*, by another direction alpha of component *x', y', z'* we obtain

$$\begin{aligned}\alpha\alpha' = -&(xx' + yy' + zz') \\ &+(yz' - zy')i \\ &+(zx' - xz')j \\ &+(xy' - yx')k.\end{aligned} \tag{4}$$

This expression is formed with a scalar part and a vector part.

The usage has prevailed of calling the scalar product (dot product) the changed scalar part of sign, while the vector part is still what we call the vector product of the two vectors.

This aggregate of a scalar and a vector is called a quaternion.

5. Quaternion Conjugates

We can replace the three basis vectors *i, j, k* by another basis of opposite chirality, that is to say, presenting with the former the same relations as the right hand with the left hand.

Such a basis is

$$i' = -i,\ j' = -j,\ k' = -k\,. \tag{5}$$

The relations that exist between the i', j', k' are analogous to those which exist between *i, j, k*. But the factors are transposed, that is to say, written in the reverse order.

For example,

$$k = ij = -ji \tag{6}$$

we deduce

$$k' = j'i' = -i'j'. \tag{7}$$

Suppressing the prime inflections as useless, would indicate that the quaternion conjugate is the same quaternion but referred to in the basis of opposite chirality. The conjugated quaternion will thus be obtained while retaining the scalar part and by changing the sign of the vector part or, if the quaternion is written as a product of quaternions, by multiplying the conjugates of the written factors in the reverse order.

6. Versors[7]

The product of a quaternion with a quaternion conjugate is a scalar, which is called the norm of the quaternion.

The norm of the product of two quaternions, Q and Q' is the product $\mathrm{QQ}'QQ'$. But $\mathrm{Q}'Q'$ the product of Q' by the quaternion conjugate Q' is the norm N' of Q', likewise $\mathrm{N} = \mathrm{Q}Q$ is the norm of Q. The norm of the product of NN' is thus the product of the norm of the factors.

A quaternion whose norm is equal to one is called a 'versor'. The product of two versors is a versor.

One direction may be considered as a quaternion. It is a quaternion whose scalar part is zero.

Moreover, it is a versor. For if, in the formula of the product of two directions, we first make $\alpha' = \alpha$, this product is equal to minus one, so that the directions may be considered as roots of minus one.

[7] *Versor*: - Rotations of Unit Quaternions. Term introduced by Hamilton in developing quaternions. Versor is sometimes used synonymously with "unit quaternion" with no reference to rotations. An algebraic parametrization of rotations. In classical quaternion theory, a versor is a quaternion of norm one (unit quaternion). Each versor has the form $q = \exp(ar\pi) = \cos a + r\sin a,\ r^2 = -1,\ \ a \in [0,\pi]$, where the $r^2 = -1$ condition means that r is a 3D unit vector. In case $a = \pi/2$, the versor is termed a right versor. The corresponding 3D rotation has the angle $2a$ about the axis *r* in axis–angle representation. The word is derived from Latin *versare* "to turn" or *versor* "the turner").

The conjugate of the vector is this vector of changed sign, the norm, product of the vector by the conjugate vector, is thus the square of the changed sign, that is to say, plus one. A direction is therefore a versor.

If u is the scalar and $v\gamma$ the magnitude v and direction γ the vector of a versor V, we will have

$$V = u + v\gamma \tag{8}$$

with

$$u^2 + v^2 = 1 \tag{9}$$

We can therefore write

$$u = \cos c, \quad v = \sin c \tag{10}$$

and with

$$V = \cos c + \gamma \sin c \tag{11}$$

If

$$V = \alpha\alpha' \tag{12}$$

- $\cos c$ is the scalar product of the two directions α and α' while the vector product is a vector of magnitude $\sin c$ and of direction γ.

We can thus interpret geometrically γ and c, saying that γ is a direction perpendicular to the plane of the two directions α and α', and that c is the supplement of the angle formed by these directions, that is, the external angle of these two directions.

Conversely, each versor is the product of two directions located in a plane perpendicular to the versor vector closing an angle, in the proper direction, equal to $\pi - c$.

The formulas of analytic geometry furnish an algebraic equivalent of these geometrical notions. They make it possible to establish the result which we have just obtained even if we take a purely algebraic point of view.

The product of two directions is a direction only when the product scalar is zero, that is to say when the two directions are perpendicular.

If α and β are perpendicular, that is to say if

$$\alpha\beta = -\beta\alpha \tag{13}$$

then this product is equal to a direction γ which is perpendicular to an α and a β.

7. Exponential Notation

It is very useful to represent a versor using the notation

$$V = e^{c\gamma} \tag{14}$$

which we will explain.

The exponential defines itself by its development in a series of powers

$$e^{c\gamma} = \sum_{n=0}^{\infty} \frac{c^n}{n!}\gamma^n \tag{15}$$

which can be decomposed into

$$\sum_{m=0}^{\infty} \frac{c^{2m}}{(2m)!}\lambda^{2m} + \sum_{m=0}^{\infty} \frac{c^{2m+1}}{(2m+1)!}\lambda^{2m+1} \tag{16}$$

As

$$\gamma^{2m} = (-1)^m \tag{17}$$

and

$$\gamma^{2m+1} = (-1)^m \gamma \tag{18}$$

and that the development of the cosine and sine are respectfully

$$\cos c = \sum_{m=0}^{\infty} \frac{(-)^m c^{2m}}{(2m)!} \tag{19}$$

and

$$\sin c = \sum_{m=0}^{\infty} \frac{(-)^m c^{2m+1}}{(2m+1)!} \tag{20}$$

one obtains

$$e^{c\gamma} = \cos c + \gamma \sin c \tag{21}$$

Clearly, until we have cause, the exponential continues in the same direction, that is to say for the same root of minus one, we can use the rules of the calculus of exponentials and in particular the law

$$e^{c\gamma}e^{c'\gamma}=e^{(c+c')\gamma} \tag{22}$$

Note also that if α is perpendicular to γ we have

$$\alpha e^{c\gamma}=e^{-c\gamma}\alpha \tag{23}$$

indeed, the first member is

$$\alpha\cos c+\alpha\gamma\sin c=\alpha\cos c-\gamma\alpha\sin c=\left(\cos c-\gamma\sin c\right)\alpha. \tag{24}$$

8. The Erlangen Program[8]

In our presentation of spherical and elliptical geometry, we will adopt the point of view proposed by Klein in the Erlangen program.

The geometry is then specified when we give, for every pair of points, a certain expression called the distance invariant. Two pairs of points for which the distance invariant has the same values are then considered as congruent or superimposable.

A transformation which transforms any pair of points into a pair of points invariant of distance is called a displacement and the study of groups of displacements is reduced to the study of groups of transformations which leave the invariant of distance invariant.

The distance itself must be a function of the distance invariant, such that the length of a line segment divided into two partial segments is the sum of the lengths of these segments.

The length of the segment is defined as the distance between its ends.

As for the straight line, we shall consider it as an axis of rotation, such as a locus of points left invariant by a displacement.

9. The Distance Invariant

We will assume that each point of the spherical space is specified by a versor V.

If V and V' are the versors representing two points we will define the distance invariant of this pair of points by the scalar

$$\mathrm{I}=\frac{1}{2}\left(\mathrm{V}V'+\mathrm{V}'V\right) \tag{25}$$

In this expression V and V' denote the conjugates of V and V'.

These definitions suffice to define the geometry in the sense of the Erlangen program.

Although this is not necessary for the rest of the exposition, we interspersed here some remarks which have no other purpose than to show how we were led to choose this point of departure.

If u is the scalar and x, y, z the components of the vector of the versor V, we have

$$x^2+y^2+z^2+u^2=1 \tag{26}$$

which can be considered as a hyper-sphere of radius one or spherical space. This shows how a pourer can characterize a point of the spherical space.

Similarly, if the primed letters designate the analogous quantities for the versor V', the distance invariant is

$$\mathrm{I}=xx'+yy'+zz'+uu' \tag{27}$$

an expression which generalizes the expression of the cosine to four dimensions as a function of the angle of the direction of the cosine boundaries.

We can therefore predict that the distance invariant will be the cosine of the distance.

10. Parataxis[9]

A first group of displacements is obtained by multiplying the representative versor of the various points of the space by a fixed versor. We shall call these displacements of the parataxies, parataxies on the left if the multiplication is made on the left, parataxies on the right if it is made on the right.

Let us designate

[8] Erlangen Program - Method of characterizing geometries based on group theory and projective geometry introduced by Felix Klein in 1872 as *Vergleichende Betrachtungen über neuere geometrische Forschungen* (Comparative considerations on recent geometric researches [4]) named after the University Erlangen-Nürnberg, where Klein was given a professorship.

[9] Paratactic- A Parameter whose association/ arrangement/juxtaposition is only implied.

$$e^{a\alpha}, \quad e^{a'\alpha'} \tag{28}$$

two arbitrary points of space, Let $e^{c\gamma}$ be the fixed versor and $e^{b\beta}$ and $e^{b'\beta'}$ the two points in which the points $e^{a\alpha}$ and $e^{a'\alpha'}$ are transformed by a parataxis to the left, we shall have

$$e^{b\beta} = e^{c\gamma}e^{a\alpha} \qquad e^{b'\beta'} = e^{c\gamma}e^{a'\alpha'}. \tag{29}$$

If I' denotes the distance invariant after transformation, we have to verify that $\mathrm{I}' = \mathrm{I}$. It becomes

$$2\mathrm{I} = e^{b\beta}e^{-b'\beta'} + e^{b'\beta'}e^{-b\beta}. \tag{30}$$

For the conjugates

$$e^{-b\beta} = e^{-a\alpha}e^{-c\gamma} \tag{31}$$

and

$$e^{-b'\beta'} = e^{-a'\alpha'}e^{-c\gamma} \tag{32}$$

obtained by taking the product of the conjugates in the reverse order it becomes

$$\mathrm{I}' = e^{c\gamma}\mathrm{I}e^{-c\gamma} \tag{33}$$

which reduces to I since I is a scalar that can equally well be inscribed as head of the product.

From the fact that the product of two versors is a versor, the parataxis on the left form a group.

For the parataxis on the right, we will have the same

$$\begin{aligned} e^{b\beta} &= e^{a\alpha}e^{c\gamma} \\ e^{b'\beta'} &= e^{a'\alpha'}e^{c\gamma} \end{aligned} \tag{34}$$

and so

$$2I' = e^{a\alpha}e^{c\gamma}e^{-c\gamma}e^{-a'\alpha'} + e^{a'\alpha'}e^{c\gamma}e^{-c\gamma}e^{-a\alpha} = 2\mathrm{I}. \tag{35}$$

The parataxies with straight lines are thus also displacements and form a group of displacements.

11. Homogeneity of Space

A versor whose vector is zero reduces to the scalar one. We will call the corresponding point the origin.

Any point can be transformed at the origin by a right or left parataxy. It is enough to take for the symbol of the parataxis the versor conjugated to the symbol of the point to be transported at the origin. For $e^{c\gamma} = e^{-a\alpha}$ we have $e^{b\beta} = 1$.

It follows from this that the space considered is homogeneous since there are displacements which carry every point at the origin.

12. Rotation

If a parataxy is performed successively on a left and a right parataxis having as a symbol the versor conjugated to that of the parataxis on the left, evidently a displacement is obtained. That is to say a transformation which preserves the distance invariant. This Transformation transforms any point $e^{a\alpha}$ into a point $e^{b\beta}$ by the formula

$$e^{b\beta} = e^{c\gamma}e^{a\alpha}e^{-c\gamma} \tag{36}$$

If $e^{a\alpha}$ is the origin, $e^{b\beta}$ will also be at the origin. The transformation thus preserves the origin; we shall say that it is a rotation around the origin.

13. Straight Lines

This allows us to define a straight line as an axis of rotation.

The points that are retained by the rotation are included in the expression

$$e^{c'\gamma} \tag{37}$$

where c' can assume an arbitrary value.

This expression for the c' variable is the equation of a line passing through the origin.

By displacing the origin with a parataxis, we obtain the equation of a line passing through the point in which the parataxy has transformed the origin.

14. Straight Parataxies

The parataxies of the same species (i.e. all to the right or all to the left) of fixed direction γ, for any parameter c form a group, subgroup of the group of parataxies which results from

$$e^{c\gamma}e^{c'\gamma} = e^{(c+c')\gamma} \tag{38}$$

such that the two parataxies of parameter c and c' carried out successively in any order are equivalent to a single parataxy of parameter

$$c'' = c + c' \tag{39}$$

This particular group retains the straight line $e^{c\gamma}$ (c variable), that is to say, transforms the points of this line into points of the same line.

This group will retain (even if it is a left parataxy), all straight lines

$$e^{c\gamma}e^{x\xi} \tag{40}$$

(c variable, γ, x, ξ fixed).

For different values of x and ξ but the same value of γ, these lines are called parataxies (left).

Similarly, the straight lines

$$e^{x\zeta}e^{c\gamma} \tag{41}$$

(c single variable) are preserved by right parataxies and are right paratactic.

15. Distances

Consider two left parataxis of direction γ and parameters c and c', carried out successively.

The first transforms the origin into the point $e^{c\lambda}$. The second transforms this point into a point in $e^{(c+c')\gamma}$.

When we have three points in a straight line, the length of the total segment must be the sum of the lengths of the partial segments. The length of a segment is the distance of the extremities, that is to say a function of the distance invariant for these two points. The invariant is $\cos c$ and $\cos c'$ for the partial segments and $\cos c''$ for the total segment.

c, c', c'' are functions of the distance invariants and since

$$c'' = c + c' \tag{42}$$

are additive functions.

For a suitable choice of the unit of length, c, c' and c'' are the distances themselves.

16. Perpendicular Lines

Consider two straight lines passing through the origin, as

$$e^{x\xi} \tag{43}$$

for the x variable and

$$e^{y\eta} \tag{44}$$

for the y variable.

Suppose further that the directions ξ and η are perpendicular to one another; we intend to show that the two straight lines are perpendicular.

This may seem obvious, but in reality, this must be demonstrated. Indeed, the directions have been introduced without reference to the spherical space and to its distance invariant.

We shall define the right angle, as in Euclid, by the condition that the angle is equal to the adjacent angle obtained by extending one of the sides. In other words, there must be a displacement (a rotation) which transforms the first right angle into the second and the second right angle into the angle opposite the first.

The calculation is very elementary, but we give it in detail by way of example of this type of calculation.

Since the directions ξ and η are perpendicular, there exists a direction ζ such that

$$\zeta = \xi\eta = -\eta\xi \ . \tag{45}$$

Consider then the rotation

$$e^{b\beta} = e^{\frac{\pi}{1}\zeta}e^{a\alpha}e^{-\frac{\pi}{1}\zeta} \tag{46}$$

which transforms any point $e^{a\alpha}$ into $e^{b\beta}$. We must show that if we set $e^{a\alpha} = e^{x\xi}$, we obtain $e^{b\beta} = e^{x\eta}$ and if we set $e^{a\alpha} = e^{y\eta}$, we obtain $e^{b\beta} = e^{-y\xi}$.

In the first case, we have

$$e^{b\beta} = \frac{1}{2}(1+\zeta)e^{x\xi}(1-\zeta) = \frac{1}{2}(1+\zeta)(1-\zeta)\cos x + \frac{1}{2}(1+\zeta)\xi(1-\zeta)\sin x \quad . \qquad (47)$$

but

$$(1+\zeta)(1-\zeta) = 2 \qquad (48)$$

and

$$(1+\zeta)\xi(1-\zeta) = (1+\zeta)^2 \xi = 2\zeta\xi = 2\eta \qquad (49)$$

it becomes therefore

$$e^{b\beta}\cos x + \eta \sin x = e^{x\eta} . \qquad (50)$$

In the second case

$$e^{b\beta} = \frac{1}{2}(1+\zeta)e^{y\eta}(1-\zeta) = e^{-x\xi} \qquad (51)$$

the calculation is the same, γ replacing ξ, but

$$\zeta\eta = -\xi . \qquad (52)$$

17. Left Rectangles

Let $e^{c\gamma}$ be a fixed point; then for *x* and y variables with ξ perpendicular to η then

$$e^{c\gamma}e^{x\xi} \qquad (53)$$

and

$$e^{c\gamma}e^{y\eta} \qquad (54)$$

represent two straight lines perpendicular to one another.

In particular, if $\eta = \gamma$, the second

$$e^{(c+y)\gamma} \qquad (55)$$

is a line which passes through the origin, it is the line joining the origin to the point $e^{c\gamma}$. The first line is the left parataxis to the line $e^{x\xi}$ passing through the fixed point, $e^{c\gamma}$.

As the line $e^{x\xi}$ is also perpendicular to the line $e^{(c+y)\gamma}$, we see that the two paratactics $e^{x\xi}$ and $e^{c\gamma}$ $e^{x\zeta}$ have a common perpendicular.

Let us make a right parataxis of symbol $e^{x'\xi}$ (x' fixed) the two lines $e^{x\xi}$ and $e^{c\gamma}$ $e^{x\xi}$, left parataxies, are each transformed into themselves and the common perpendicular moves while maintaining the same length.

The figure formed by the two parataxies and the two common perpendiculars is therefore a rectangle in the sense that it is a quadrilateral whose angles are right with the opposite sides equal to each other. But it is not a plane figure, but a left rectangle (in the English sense of "skew").

18. Clifford Surfaces

The Clifford surface is called, the place where the paratactic lines have the same straight line, called the axis of the surface, and such that the perpendicular common to the axis has the same length as the radius of the surface.

Let us first consider left parataxies; the surface points of the Clifford axis

$$e^{x\xi} \qquad (56)$$

(*x* variable) are

$$e^{c\gamma}e^{x\xi} \qquad (57)$$

where *c* is the radius of the surface and where *x* is variable as well as the direction γ which can represent all directions perpendicular to ξ.

For right parataxies, it would be the same

$$e^{x'\xi}e^{c\gamma'} \qquad (58)$$

The two expressions are equal

$$x' = x, \;\; xe^{c\gamma'} = e^{-x\xi}e^{c\gamma}e^{x\xi} \qquad (59)$$

that is to say

$$\gamma' = e^{-x\xi}\gamma e^{x\xi} . \qquad (60)$$

This shows that the place of the paratactic on the right is the same as that of the paratactic on the left.

The Clifford surface is the locus of points at constant distance c from the axis of the surface. It is a regulated surface which admits two systems of generators, the paratactic ones to the left and to the right of the axis of the area.

If one performs paratactic displacements which retain the axis, the Clifford surface transforms into itself, the generators of one system are transformed into themselves and the generators of the other system are interchanged.

Two pairs of generators of each of the two systems thus form parallelograms, the angles are equal or additional and the opposite sides are equal.

The angle of these parallelograms is easily calculated; in fact, the two generators passing through the point $e^{c\gamma}$ are

$$e^{c\gamma} e^{x\xi} \tag{61}$$

(x variable) and

$$e^{x'\xi} \;\; e^{c\gamma} \tag{62}$$

(x' variable). A parataxis with the left symbol $e^{-c\gamma}$ brings the vertex of the angle to the origin, the lines are transformed into

$$e^{x\xi} \tag{63}$$

and

$$e^{-c\gamma} \;\; e^{x'\xi} \;\; e^{c\gamma} \tag{64}$$

which are transformed one into the other by a rotation of the angle 2c.

Perhaps this last point is not perfectly clear, we shall return to it in an instant after having studied the plane.

19. Conjugate Lines

In the particular case where $c = \pi / 2$ we have

$$e^{c\gamma} = \gamma \tag{65}$$

and therefore, since γ and ξ are assumed to be perpendicular

$$e^{x'\xi}\gamma = \gamma e^{-x'\xi} \tag{66}$$

the two paratactic ones, the one on the right and the one on the left are therefore identical. Their points correspond to

$$x' = -x \tag{67}$$

Consider any point on the line

$$\mathrm{V}' = e^{x'\xi}\gamma \tag{68}$$

(that is to say, a particular value of the variable x') and any point on the line

$$\mathrm{V} = e^{x\xi} \tag{69}$$

These two straight lines are right parataxies for the exceptional case $c = \pi/2.$

We will show that these two points are the same distance $\pi/2$, that is to say that their distance invariant is zero. In fact

$$\begin{aligned} &\mathrm{I} = \frac{1}{2}\left(\mathrm{V}V' + \mathrm{V}'V\right) = \\ &\frac{1}{2}e^{x\xi}\left(-\gamma\right)e^{-x'\xi} + \frac{1}{2}e^{-x'\xi}\gamma e^{-x\xi} = . \\ &\frac{1}{2}\left[-e^{(x+x')\xi}\gamma + e^{(x'+x)\xi}\gamma\right] = 0 \end{aligned} \tag{70}$$

It would be easy to show that the straight line joining V and V', that is to say, any line intersecting the two straight lines v and v' (for x and x' variables) is perpendicular to these two straight lines. But no doubt we have given sufficient examples of these calculations.

The paratactic lines for $c = \pi/2$ are said to be conjugate or absolute polar.

20. The Plane

The plane can be defined as the locus of straight lines perpendicular to the same straight line

$$e^{c\gamma},\; e^{x\xi} \tag{71}$$

(x variable).

The points of the plane are thus represented by the versors

$$\mathrm{V} = e^{c\gamma} e^{y\eta} \tag{72}$$

y is arbitrary and η also but perpendicular to ξ.

We shall show that the plane is the locus of points situated at distance $\pi/2$ from a point

$$\mathrm{V}' = e^{c\gamma}\xi \tag{73}$$

called the center of the plane.

We must verify that the distance invariant of the two points V and V' is zero. We have

$$\begin{gathered} \mathrm{I} = \frac{1}{2}\left(\mathrm{V}V' + \mathrm{V}'V\right) = \\ \frac{1}{2}e^{c\gamma}e^{y\eta}\left(-\xi\right)e^{-c\gamma} + \frac{1}{2}e^{c\gamma}\xi e^{-y\eta}e^{-c\gamma} \end{gathered} \tag{74}$$

which is null, since for $\xi\eta = -\eta\xi$ we have

$$\xi e^{-y\eta} = e^{y\eta}\xi \tag{75}$$

Or we can put in the equation of the plane, the versor V' representing the center.

If

$$\zeta = \xi\eta \tag{76}$$

We have

$$\begin{gathered} \mathrm{V} = \mathrm{V}'\left(-\xi\right)e^{y\eta} = \\ \mathrm{V}'\left(-\xi\cos y - \zeta\sin y\right) = \mathrm{V}'\chi. \end{gathered} \tag{77}$$

It is easy to realize that χ is a direction that is arbitrary. Indeed, it is the direction whose orthogonal projections on the directions $-\xi$ and $-\zeta$ are respectively $\cos y$ and $\sin y$. χ is therefore in the plane of ξ and ζ forming an angle y with $-\xi$. But η is an arbitrary direction perpendicular to ξ and since y is arbitrary, χ is also arbitrary.

In particular, if the center is at the origin, we see that the directions represent the points of a plane, that is to say of a sphere of radius $\pi/2$ centered on the origin.

As the familiar theorems which show that the angles at the center are measured by the intercepted arc on the sphere apply without modification, it follows that the angle of two straight lines from the center $e^{x\xi}$ and $e^{y\eta}$ is the distance of the two directions ξ and η.

When the versors are reduced to directions, the distance is reduced to the scalar product of the two directions. The angle of the two straight lines is therefore the angle of the directions of these lines.

In particular, with the rotation

$$e^{b\beta} = e^{c\gamma}e^{a\alpha}e^{-c\gamma} \tag{78}$$

we have for $a = b = \pi/2$

$$\beta = e^{c\gamma}\alpha e^{-c\gamma} \tag{79}$$

and if α is perpendicular to γ

$$\beta = e^{2c\gamma}\alpha = \alpha\cos 2c + \gamma\alpha\sin 2c \tag{80}$$

β has thus rotated by an angle $2c$ in the plane perpendicular to γ.

This completes the justification of the end of section 18.

21. Antipodal Points

When x varies from zero to 2π, the expression

$$e^{x\xi} \tag{81}$$

represents successively the various points of a straight line, partly from the origin, and on returning there to traverse in the same order the points already traversed. In fact,

$$e^{(x+2\pi)\xi} = e^{x\xi}. \tag{82}$$

The line is therefore a closed line whose length is equal to 2π.

If we consider all the straight lines passing through the origin, that is, when we consider different values of the direction ξ, we see that for $x = \pi$ all these lines pass through the point -1.

This point is called the antipode of the origin.

If we consider similarly straight lines passing through a point $e^{c\gamma}$ we would see that all these lines pass through the point $-e^{c\gamma}$ the antipode of $e^{c\gamma}$.

The antipode points are thus represented by versors from opposite signs, every straight line passing through such a point also passes through the antipode of this point.

22. Elliptical Space

If, instead of the invariant of distance I, we had taken as invariant distance, I^2 or the absolute value of I, then two versors V and $\mathrm{V}' = -\mathrm{V}$ would have as distance invariant plus one. Instead of considering them as representing distinct points of space, the antipodes, they should be considered as two representations of one and the same point of space.

Apart from this circumstance concerning the disappearance of the antipodes, all the formulas established for the spherical space remain valid for the new space.

This is called the *elliptical* space.

Some authors nevertheless call it a simply elliptic space so as to leave to the term "*elliptical space*" a generic meaning which applies to both of the spaces considered as various "*form*s" of the elliptical space.

23. Representations of Elliptical Euclidean Space

First of all, we note that infinitely small figures of elliptical space can, in the limit, be considered as Euclidean figures.

This already appears in the fact that the angle of warping of a left rectangle is equal to the dimension; It therefore tends to zero if this side is infinitely small and then the rectangle becomes a plane and the geometry Euclidean.

We can also show that when x, y, z and x', y', z' are infinitely small, the invariant of distance I becomes, neglecting the quantities of order higher than the second

$$\mathrm{I} = 1 - \frac{1}{2}\left[\left(x - x'\right)^2 + \left(y - y'\right)^2 + \left(z - z'\right)^2\right] + \ldots \quad (83)$$

as I is the cosine of the distance *r*, it is at the same approximation equal to the Euclidean value

$$r^2 = \left(x - x'\right)^2 + \left(y - y'\right)^2 + \left(z - z'\right)^2 \quad (84)$$

We can use this remark, to represent the totality of the elliptical space, in a sphere of infinitely small radius ε. Let us note that by exception we use this Greek letter, in its traditional sense of an infinitely small scalar.

A point

$$e^{x\xi} \quad (85)$$

may be represented inside the sphere by the point

$$e^{x'\xi} = e^{\varepsilon x \xi} \quad (86)$$

or by neglecting the terms in ε^2 by the point

$$1 + \varepsilon x \xi \quad (87)$$

Since geometry can be considered as Euclidean in the sphere, we shall have, taking ε as units of Euclidean lengths, that a point of the elliptica space $e^{x\xi}$ is represented by a Euclidean vector of direction ξ and length *x*.

We obtain all the points on the line considering all the values of *x* from minus $\pi/2$ to plus $\pi/2$. The extreme points represented on the sphere of radius $(\pi/2)\varepsilon$ are the antipodal points of this sphere and would represent the antipodes of space if we consider the spherical space. As we consider the elliptical space these two points represent two representations of the same point of the elliptical space.

All the points of this space are thus represented inside our Euclidean sphere and the points situated on the frontier of the representation are represented there twice. It is therefore never difficult to follow the representation on a contour which reaches its edge, since all the points on the edge have two representations in such a way that, instead of leaving the sphere, it can always pass to the other representation of the same point and continue to walk towards the interior of the sphere.

24. Representations of Spherical Space

An analogous representation can be used for spherical space. It is now assumed that within the sphere there are two kinds of points. We will say the blue dots and the pink dots. The points of the frontier are not more of one species than the other. We will say that these are mauve points.

We shall suppose that we cannot pass from a pink point to a blue point than through a mauve dot.

In other words, there are, within the sphere, two distinct spaces, the blue space and the pink space, and these two spaces are connected by the purple border, the surface of the sphere.

This representation can be modified in a variety of ways with respect to the topology by making it resemble the projections of the sphere, such as the stereographic projection or the orthogonal projection. But these developments would lead us outside our subject.

* * *

Translators Appendices

Appendix A: Brief Definitions of Terms

1. Versor – An affinor (affine tensor) which effects a rotation of a vector through a right angle. For quaternions, the versor of an axis (or a vector) is a unit vector indicating its direction. In general, a versor defines all of the following: a directional axis; the plane normal to that axis; and an angle of rotation. For quaternions, the versor of an axis (or of a vector) is a unit vector indicating its direction [5].

Rotations of Unit Quaternions. Term introduced by Hamilton in developing quaternions. Versor is sometimes used synonymously with "unit quaternion" with no reference to rotations. An algebraic parametrization of rotations. In classical quaternion theory, a versor is a quaternion of norm one (unit quaternion). Each versor has the form $q = \exp(ar\pi) = \cos a + r\sin a,\ r^2 = -1,\quad a \in [0,\pi]$, where the $r^2 = -1$ condition means that r is a 3D unit vector. In case $a = \pi/2$, the versor is termed a right versor. The corresponding 3D rotation has the angle 2*a* about the axis *r* in axis–angle representation. The word is derived from Latin *versare* "to turn" or *versor* "the turner").

2. Parataxis – Corresponds to a hyperbolic displacement by a half line.

3. Paratactic lines - In elliptic geometry two oblique lines with an infinite set of common perpendiculars of the same length are called Clifford parallels, equidistant or paratactic lines in elliptic geometry if the perpendicular distance between them is constant from point to point.

The concept was first studied by William K. Clifford in elliptic space. Since parallel lines have the property of equidistance, the term *parallel* was taken from Euclidean geometry, but the lines of elliptic geometry are curves with finite length, unlike lines in Euclidean geometry. Quaternion algebra describes the geometry of elliptic space in which Clifford parallelism is made explicit.

4. Hyperbolic geometry – a Lobachevskian, or non-Euclidean geometry, where the Euclidean parallel postulate is replaced with: *For any given line R and point P not on R, in the plane containing both line R and point P there are at least two distinct lines through P that do not intersect R.*

5. Riemannian geometry – or elliptic geometry, is a non-Euclidean geometry regarding space as a sphere and a line like a great circle. Euclid's 5th postulate is rejected and his 2nd postulate modified. Simply, Euclid's 5th postulate states: through a point not on a given line there is only one line parallel to the given line. In Riemannian geometry, there are no lines parallel to the given line. Euclid's 2nd postulate is: a straight line of finite length can be extended continuously without bounds. In Riemannian geometry, a straight line of finite length can be extended continuously without bounds, but all straight lines are of the same length. However, Riemannian geometry allows the other three Euclidean postulates.

6. Elliptical space - Elliptic space can be constructed in a similar manner to the construction of 3D vector space: One uses directed arcs on great circles of the sphere. As directed line segments are equipollent[10] when they are parallel, of the same length, and similarly oriented, so directed arcs found on great circles are equipollent when they are of the same length, orientation, and great circle. These relations of equipollence produce 3D vector space and elliptic space, respectively. Access to elliptic space structure is provided through the vector algebra of William Rowan Hamilton: he envisioned a sphere as a domain of square roots of minus one. Then Euler's formula $e^{ix} = r(\cos\theta + i\sin\theta)$ where *r* is on the sphere, represents the great circle in the plane perpendicular to *r*. Opposite points *r* and –*r* correspond to oppositely directed circles. In elliptic space, arc length is less than π, so arcs may be parametrized with θ in [0, π) or (–π/2, π/2]

[10] In Euclidean geometry, equipollence is a binary relation between directed line segments. A line segment *AB* from point *A* to point *B* has the opposite direction to line segment *BA*. Two directed line segments are equipollent when they have the same length and direction.

The concept of equipollent line segments originated with Giusto Bellavitis in 1835 [6]. Subsequently the term vector was adopted for a class of equipollent line segments.

7. Hyperbolic versor - Regarding versors, a parameter of rapidity specifying a reference frame change corresponding to the real variable in a one-parameter group of hyperbolic versors. With the further development of special relativity, the action of a hyperbolic versor is now called a Lorentz boost [7].

Appendix B: Erlangan Program

Method of characterizing geometries based on group theory and projective geometry as introduced by Felix Klein in 1872 in *Vergleichende Betrachtungen über neuere geometrische Forschungen* (Comparative considerations on recent geometric researches [4]) named after the University Erlangen-Nürnberg, where Klein was given a professorship.

Appendix C: Lemaître Biographical Note

Georges Henri Joseph Édouard Lemaître (July 1894–June 1966) was a Belgian Catholic priest, astronomer and professor of physics (rare mix in modern times) at the Catholic University of Leuven. As he was a secular priest, he was called *Abbé*, then, after being made a canon, *Monseigneur*.

He is best known for the discovery of the proposed expansion of the universe, still widely misattributed to Hubble. He was the first to derive what is now known as Hubble's law and made the first estimation of what is now called the Hubble constant, published in 1927, two years before Hubble's article; but since it was published in French it was unknown in the US for a time. Lemaître also proposed what is known as the Big Bang theory, which he called his "*hypothesis of the primeval atom*" or the "*Cosmic Egg*" [2].

References (Translators) Annotated / Notes

[1] Lemaître, G. (1948) *Quaternions et espace elliptique*, *Pontificia Academia Scientiarum,* (Acta Pontifical Academy of Sciences), ACTA, Vol. XII, No. 8, 12:57-80; http://www.casinapioiv.va/content/dam/accademia/pdf/acta12/acta12-lemaitre2.pdf.
[2] Coxeter, H.S.M. (1950) also a contributor to elliptic geometry, summarized Lemaître's paper, English synopsis of Lemaître, in Mathematical Reviews, MR0031739 (11,197f; in his review of Lemaître's paper, Coxeter comments on Lemaître's construction of the *versor* (quaternion of unit norm), as being expressible as $v = \cos c + \gamma \sin c = e^{c\gamma}$, where γ is the direction of a unit vector. A versor has four constituents interpreted as coordinates for a point in spherical 3-space, where the distance between points u and v is $\cos^{-1}\left(u\bar{v} + v\underline{u}\right)/2 = c.$ Multiplying all versors on the left or right by a given versor $v = e^{c\gamma}$ is a left or right parataxy called a Clifford translation, which moves every point through a distance $\cos^{-1}\left(u + \bar{v}\right)/2 = c.$ The general displacement, $x \to uxv$ is obtained when the two types of Clifford translation are combined. If u and v are conjugate, point 1 is an invariant rotation.

For versors $e^{c\gamma}$, where angle c varies with direction γ remaining fixed, represent points on a line through point 1, which is the axis of the rotation $x \to e^{a\gamma} x e^{-a\gamma}$. Thus the line $e^{c\gamma}$ is described. The two Clifford parallels to it through any fixed-point u are $ue^{c\gamma}$ and $e^{c\gamma}u.$ When u varies along a line $e^{b\beta}$, the line $ue^{c\gamma}$ generates a Clifford surface, whose points are $e^{b\beta}e^{c\gamma}$ with β and γ fixed. Finally, elliptic space is derived from spherical space by identifying each pair of antipodal points $\pm v$. At the end of his note, Coxeter asks the reader to compare his monograph on 'non-Euclidean geometry' which predates Lemaître's paper by six years; Non-Euclidean geometry, H.S.M. Coxeter The University of Toronto Press, (1942).
[3] Short Lemaître bio, http://dictionary.sensagent.com/georges%20 lemaitre/en-en/.
[4] Felix Klein (1849-1925) published his inauguration paper (1872) *Vergleichende Betrachtungen über neuere geometrische Forschungen* for a professorship at the University of Erlangen (Bavaria, Germany). The paper acquired world-wide fame among mathematicians under the name of *Erlanger Program*; Original German: http://quod.lib.umich.edu/cgi/t/text/pageviewer-idx?c=umhistmath;cc=umhistmath;idno=ABN7632. 0001.001;seq=1; Full English Translation: http://arxiv.org/abs/0807.3161.
[5] Encyc. Math: https://www.encyclopediaofmath.org/index.php/Main_Page.
[6] Crowe, M.J. (1967) A History of Vector Analysis, "Giusto Bellavitis and His Calculus of Equipollences", pp 52–4, University of Notre Dame Press.
[7] Robb, A.A. (1911) Optical Geometry of Motion: A New View, of the Theory of Relativity, Cambridge: Heffer & Sons.