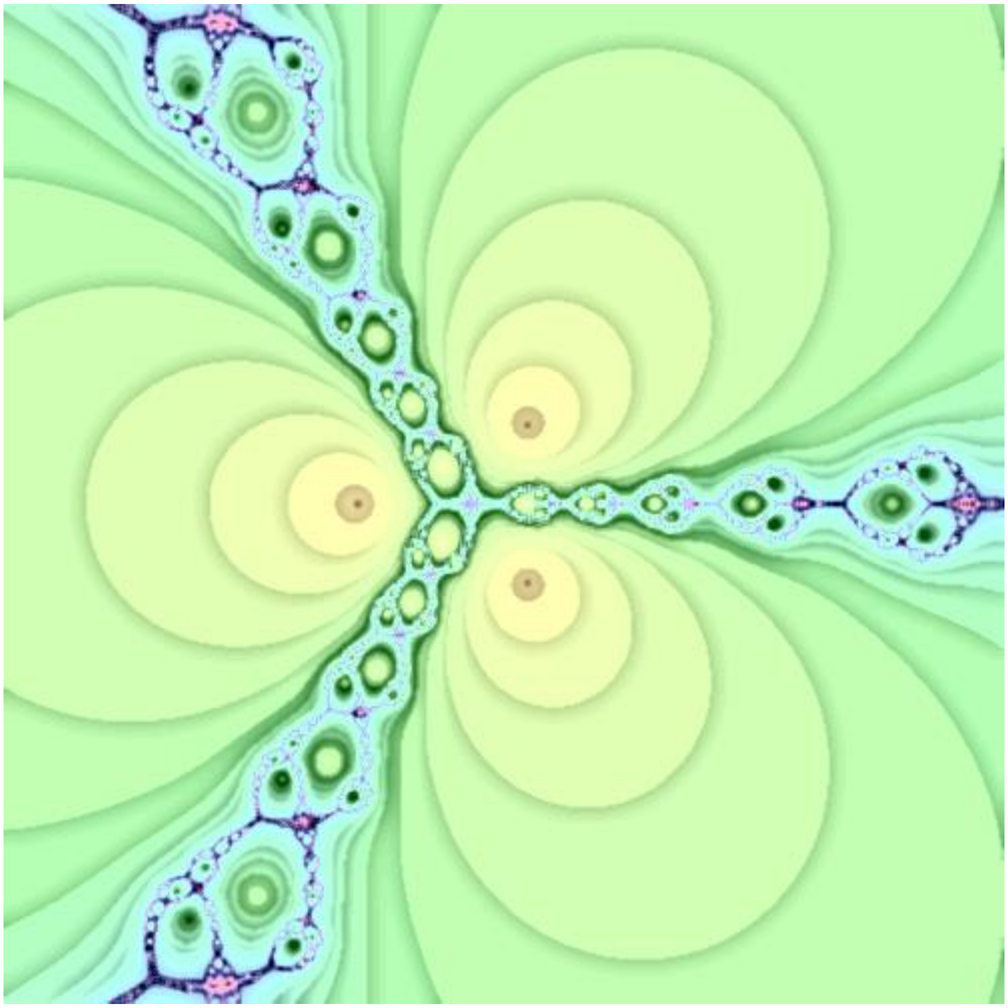
The Cubic : $x^3+x^2+1=0$



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Abstract

This note presents some formulas and fractals related with the equation: $x^3+x^2+1=0$.

1. Introduction. Roots of the equation: $p(x)=x^3+x^2+1=0$.

$$p(x)=x^3+x^2+1=0 \Rightarrow \begin{cases} x_1=r\in\mathbb{R} \\ x_2=z=u+iv\in\mathbb{C} \\ x_3=\overline{z}=u-iv\in\mathbb{C} \end{cases} ,i=\sqrt{-1} \tag{1}$$

$$r=-\frac{1}{3}-\frac{a}{6}-\frac{2}{3a} \tag{2}$$

$$z=u+iv=-\frac{1}{3}+\frac{a}{12}+\frac{1}{3a}+\frac{i\sqrt{3}}{2}\left(-\frac{a}{6}+\frac{2}{3a}\right) \tag{3}$$

$$\overline{z}=u-iv \tag{4}$$

$$a=\left(116+12\sqrt{93}\right)^{1/3} \tag{5}$$

2. Some Relations

$$r+z+\overline{z}=r+2u=-1 \tag{6}$$

$$rz+r\overline{z}+z\overline{z}=2ru+u^2+v^2=0 \tag{7}$$

$$rz\overline{z}=r\left(u^2+v^2\right)=-1 \tag{8}$$

$$u^3-3uv^2+u^2-v^2+1=0 \tag{9}$$

$$3u^2-v^2+2u=0 \tag{10}$$

3. Graphics

$$p(x)=x^3+x^2+1=0 \tag{11}$$

$$q(x)=x^3p\left(\frac{1}{x}\right)=x^3+x+1 \tag{12}$$

$$\mathrm{Re}\left(p(x+iy)\right)=x^3-3xy^2+x^2-y^2+1 \tag{13}$$

$$\mathrm{Im}\left(p(x+iy)\right)=3x^2y-y^3+2xy \tag{14}$$

$$\mathrm{Re}\left(q(x+iy)\right)=x^3-3xy^2+x+1 \tag{15}$$

$$\mathrm{Im}\left(q(x+iy)\right)=3x^2y-y^3+y \tag{16}$$

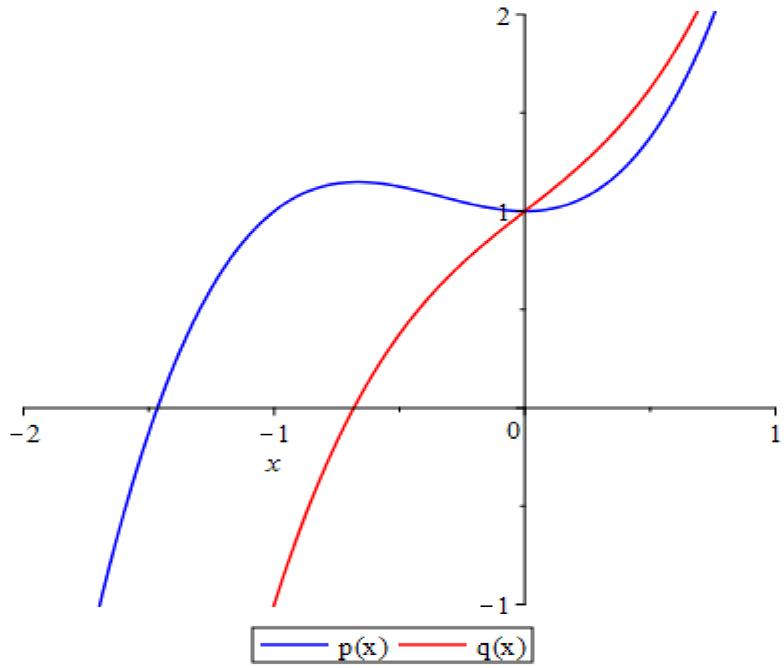


Figure 1.

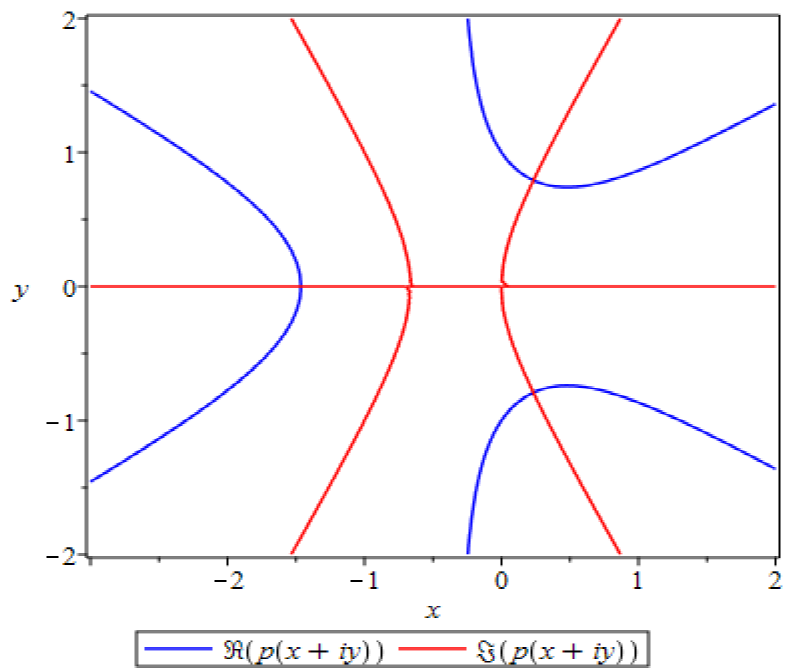


Figure 2.

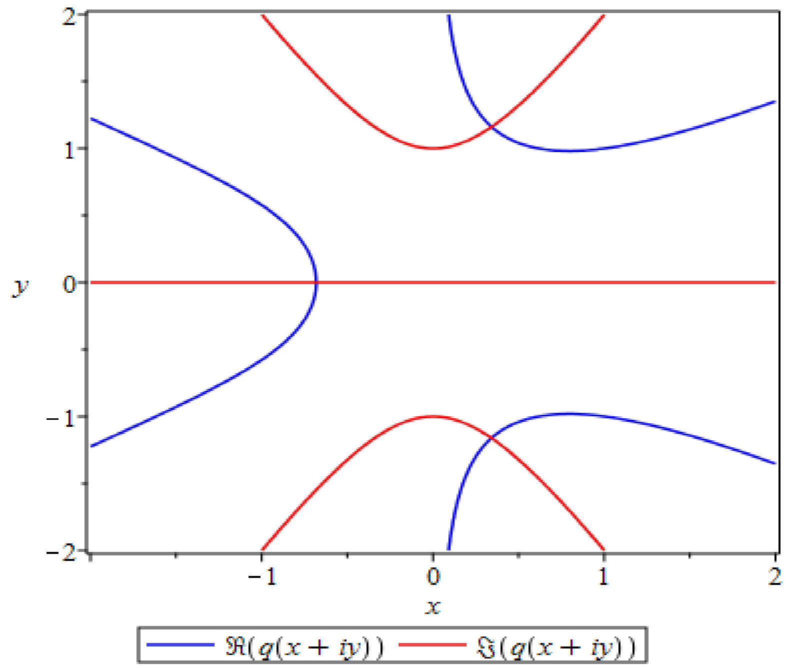


Figure 3.

4. Recurrences

$$f(x)=\frac{2x^3+x^2-1}{2x+3x^2} \tag{17}$$

$$x_{n+1}=f(x_n)\ ,x_1=-1\Rightarrow \lim_{n\to\infty} x_n=r \tag{18}$$

$$x_{n+1}=f(x_n)\ ,x_1=\frac{1+i}{2}\Rightarrow \lim_{n\to\infty} x_n=z \tag{19}$$

$$x_{n+1}=f(x_n)\ ,x_1=\frac{1-i}{2}\Rightarrow \lim_{n\to\infty} x_n=\overline{z} \tag{20}$$

$$u_{n+1}=\frac{1+8u_n^2+16u_n^3}{2+16u_n+24u_n^2}\quad ,u_1=0\Rightarrow \lim_{n\to\infty} u_n=u \tag{21}$$

$$v_{n+1}=\frac{31+4v_n^2+96v_n^4+328v_n^6}{8v_n+128v_n^3+384v_n^5}\quad ,v_1=1\Rightarrow \lim_{n\to\infty} v_n=v \tag{22}$$

5. Representations

$$r=-\sqrt[3]{1+\sqrt[3/2]{1+\sqrt[3/2]{1+\sqrt[3/2]{1+\dots}}}} \qquad (23)$$

$$r=-\frac{1}{3}-\frac{1}{3}\sqrt[3]{29+3\sqrt[3]{29+3\sqrt[3]{29+\dots}}} \qquad (24)$$

$$r=-\frac{1}{3}-\frac{1}{3}\sqrt{3+\frac{29}{\sqrt{3+\frac{29}{\sqrt{3+\dots}}}}} \qquad (25)$$

$$z=\frac{i}{\sqrt{1+\frac{i}{\sqrt{1+\frac{i}{\sqrt{1+\frac{i}{\sqrt{1+\dots}}}}}}}} \qquad (26)$$

$$\frac{1}{z}=-i\sqrt{1+z}=-i\sqrt{1+\frac{i}{\sqrt{1+\frac{i}{\sqrt{1+\frac{i}{\sqrt{1+\dots}}}}}}} \qquad (27)$$

$$z=-\frac{1}{3}+\frac{1}{6}\sqrt[3]{29+3\sqrt[3]{29+3\sqrt[3]{29+\dots}}}+i\sqrt{-\frac{1}{6}+\frac{1}{12}\sqrt[3]{839+3\sqrt[3]{839+3\sqrt[3]{839+\dots}}}} \qquad (28)$$

$$\frac{1}{z}=\frac{1}{2\sqrt[3]{1+\sqrt[3/2]{1+\sqrt[3/2]{1+\sqrt[3/2]{1+\dots}}}}}+i\sqrt{\frac{1}{2}+\frac{1}{4}\sqrt[3]{29+3\sqrt[3]{29+3\sqrt[3]{29+\dots}}}} \qquad (29)$$

$$\overline{z}=-i\sqrt{1-\frac{i}{\sqrt{1-\frac{i}{\sqrt{1-\frac{i}{\sqrt{1-\dots}}}}}}} \qquad (30)$$

$$\frac{1}{\overline{z}}=i\sqrt{1-\frac{i}{\sqrt{1-\frac{i}{\sqrt{1-\frac{i}{\sqrt{1-\dots}}}}}}} \qquad (31)$$

$$r=-2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\ldots}}}}=[-2;1,1,6,1,3,5,4,22,1,\ldots] \tag{32}$$

$$\begin{aligned} z=i+\cfrac{1}{2+2i+\cfrac{1}{2-i+\cfrac{1}{2-2i+\cfrac{1}{2-3i+\ldots}}}}= \\ =[i;2+2i,2-i,2-2i,2-3i,3+i,-1-2i,-2i,1-2i,-1-2i,\ldots] \end{aligned} \tag{33}$$

$$\begin{aligned} \bar{z}=-i+\cfrac{1}{2-2i+\cfrac{1}{2+i+\cfrac{1}{2+i+\cfrac{1}{-i+\ldots}}}}= \\ =[-i;2-2i,2+i,2+i,-i,2+2i,3-2i,-1-2i,-2i,-3i,\ldots] \end{aligned} \tag{34}$$

$$z=-\frac{1+r}{2}+i\frac{\sqrt{3r^2+2r-1}}{2} \tag{35}$$

$$z=i\sqrt{r+r^2+(1+r)i\sqrt{r+r^2+(1+r)i\sqrt{r+r^2+\ldots}}} \tag{36}$$

6. Pi constant

$$\pi=8\tan^{-1}\left(\frac{u}{v}\right)+4\tan^{-1}\left(\frac{1+u-v}{1+u+v}\right) \tag{37}$$

$$\pi=-2i\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(2\left(\frac{z-1}{z+1}\right)^{2n+1}+\left(\frac{z}{2+z}\right)^{2n+1}\right) \tag{38}$$

$$\pi=2\sqrt{3}\sum_{n=0}^{\infty}z^n\sum_{k=[n/3]}^{[n/2]}\binom{k}{n-2k}\frac{3^{-k}}{2k+1} \tag{39}$$

$$\pi=2\tan^{-1}\left(\frac{1-u^2-v^2}{2u}\right)-i\ln\left(\frac{u^2+(1+v)^2}{u^2+(1-v)^2}\right)+4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}z^{2n+1} \tag{40}$$

$$\pi=8\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}z^{2n+1}-4\tan^{-1}\left(\frac{2u-1+u^2+v^2}{2u+1-u^2-v^2}\right)-2i\ln\left(\frac{u^2+(1+v)^2}{u^2+(1-v)^2}\right) \tag{41}$$

7. Fractals

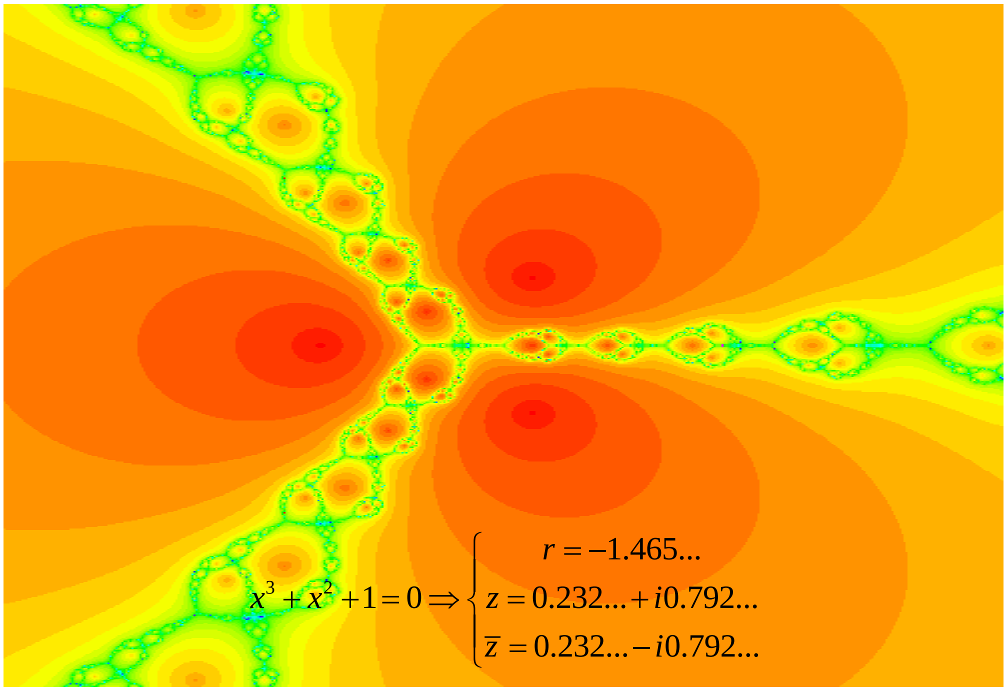
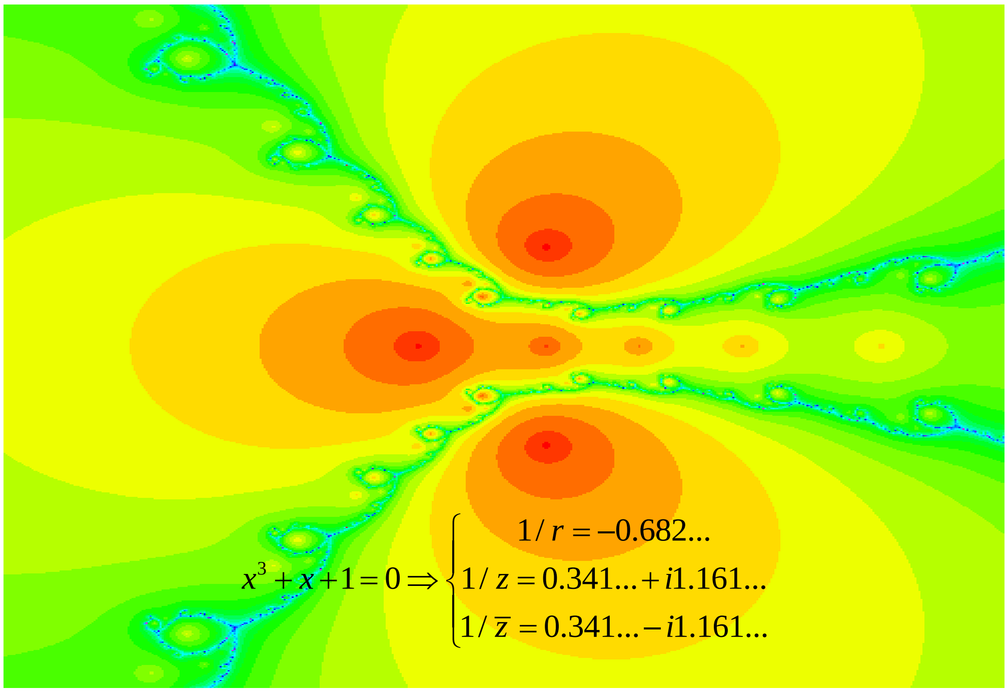


Figure 4.



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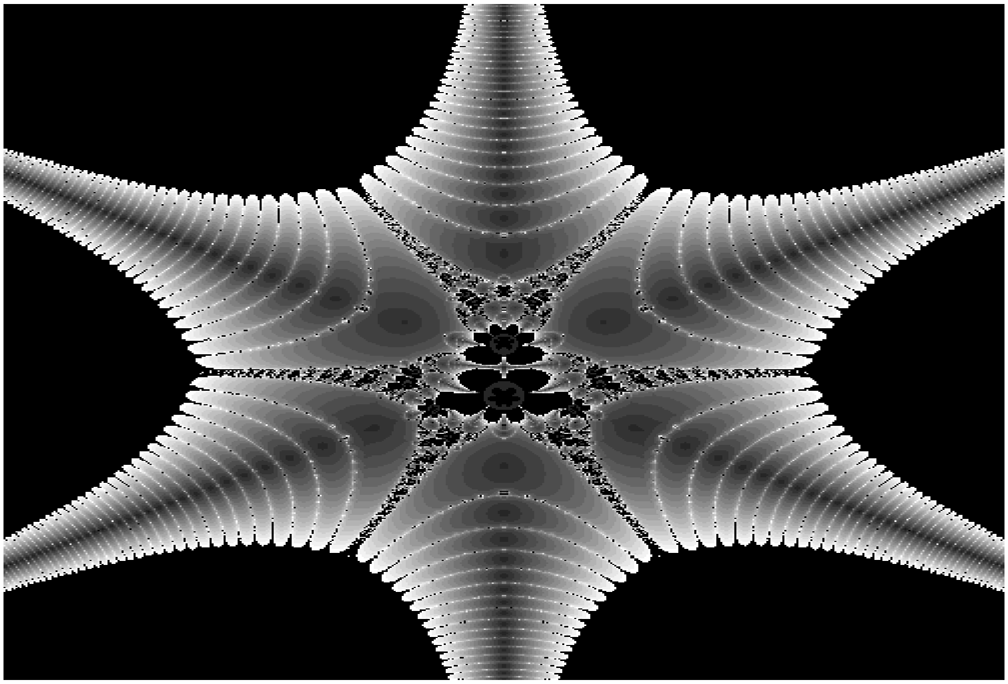

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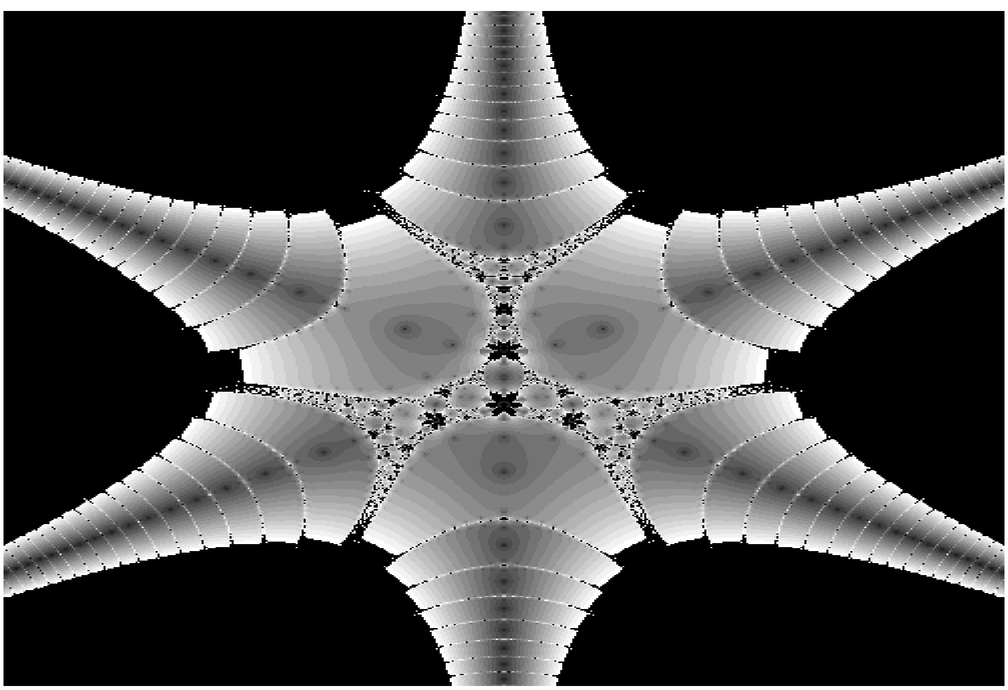

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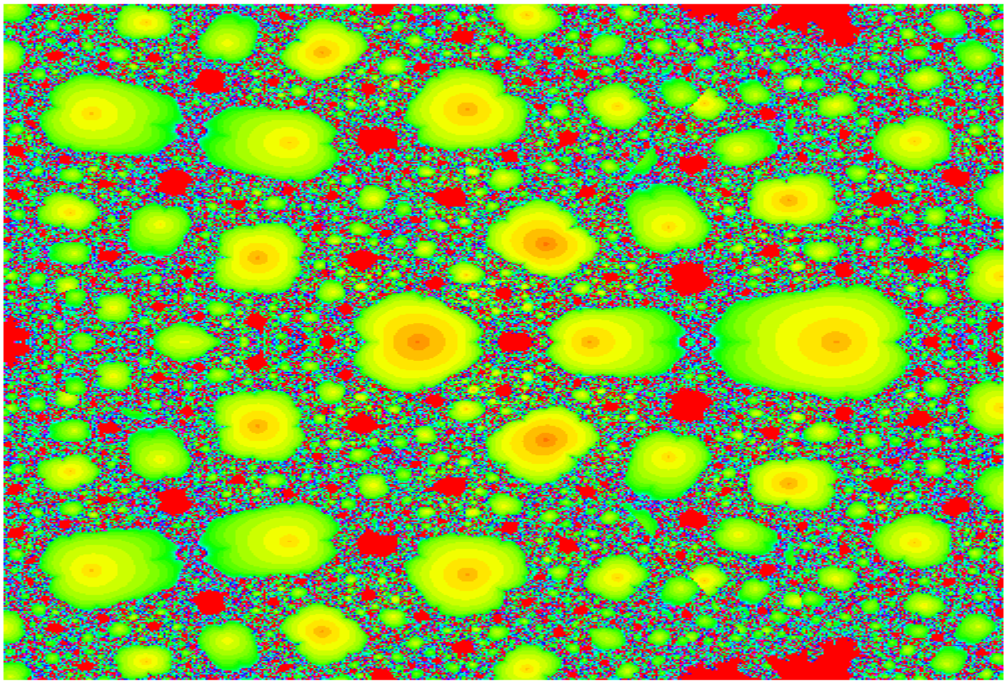

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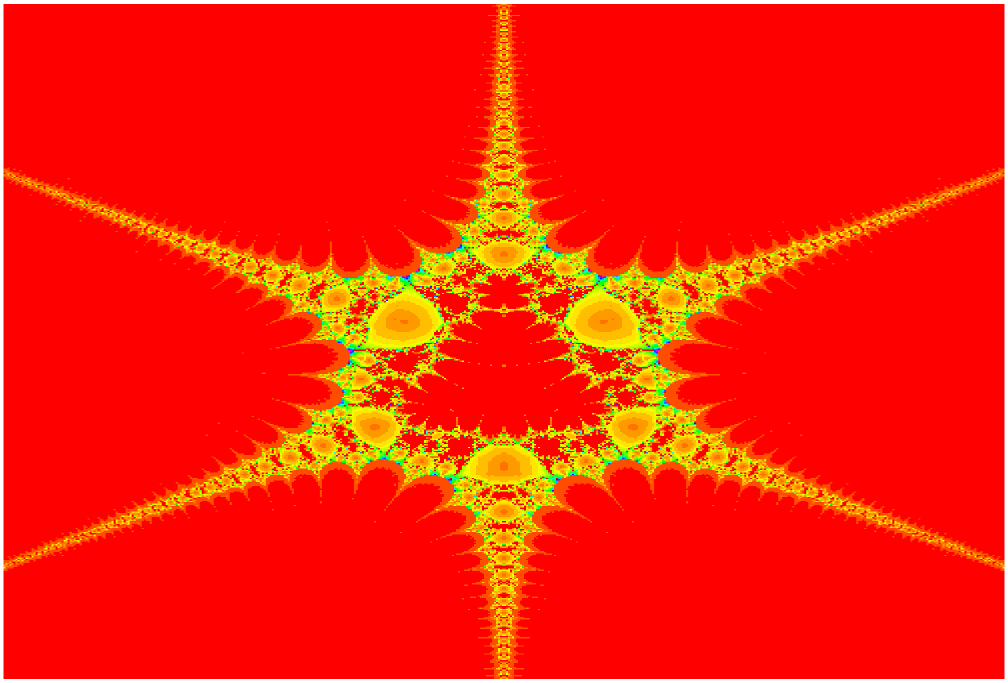

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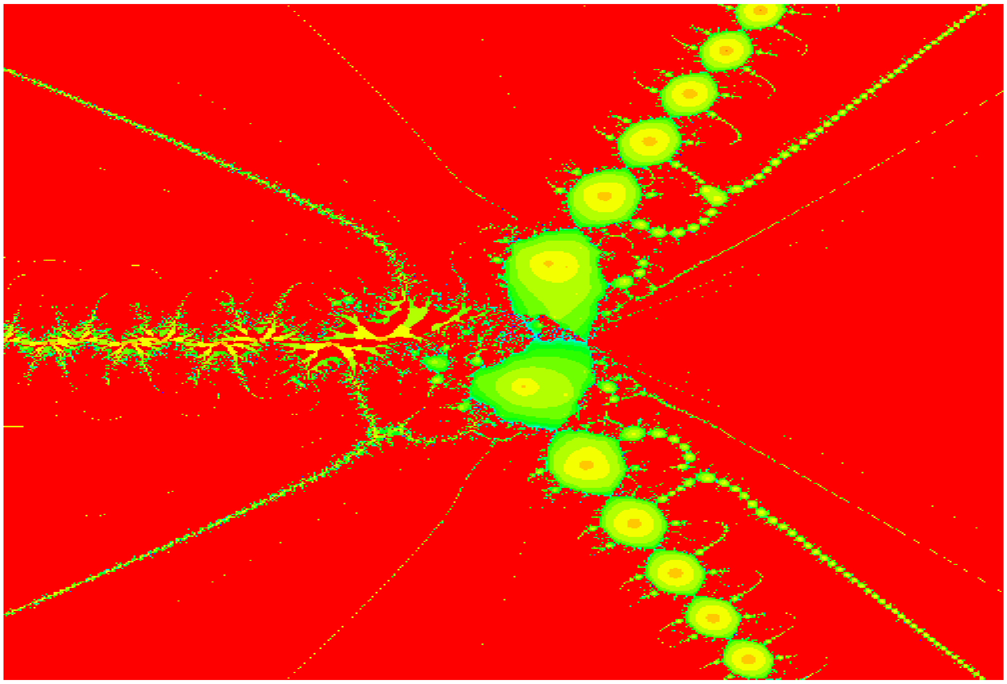

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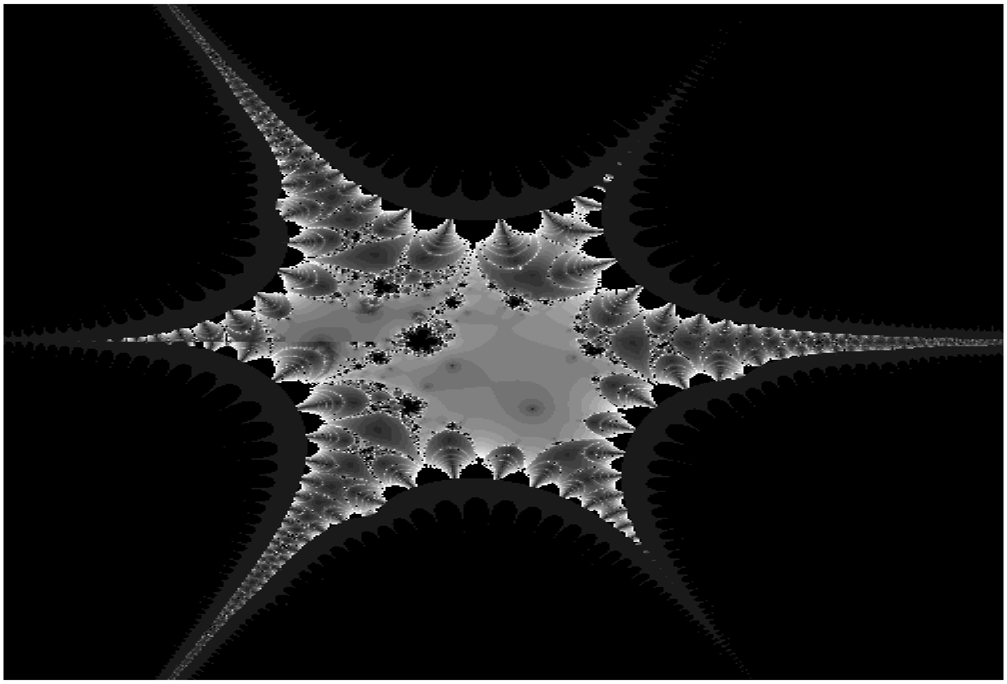

Figure 11.

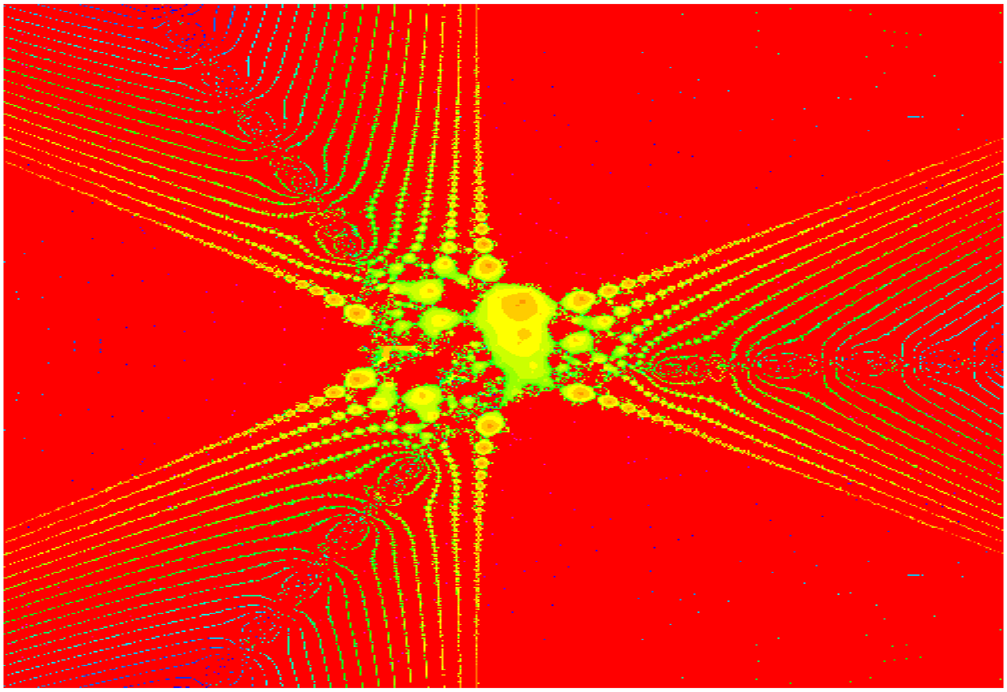

Figure 12.

8. The sequence $Z_n = U_n + iV_n$

$$Z_{n+1} = \frac{i}{\sqrt{1+Z_n}} \quad , Z_1 = 0 \Rightarrow \lim_{n\to\infty} Z_n = z = u + iv \tag{42}$$

$$\{Z_n : n \in \mathbb{N}\} = \left\{ 0, i, \frac{i}{\sqrt{1+i}}, \frac{i}{\sqrt{1+\frac{i}{\sqrt{1+i}}}}, \frac{i}{\sqrt{1+\frac{i}{\sqrt{1+\frac{i}{\sqrt{1+i}}}}}}, \ldots \right\} \tag{43}$$

If $Z_n = U_n + iV_n$ then

$$U_{n+1} = \sqrt{\frac{\sqrt{(1+U_n)^2 + V_n^2} - 1 - U_n}{2\left((1+U_n)^2 + V_n^2\right)}} \tag{44}$$

$$V_{n+1}=\sqrt{\frac{\sqrt{\left(1+U_n\right)^2+V_n^2}+1+U_n}{2\left(\left(1+U_n\right)^2+V_n^2\right)}} \tag{45}$$

$$U_1=V_1=0 \tag{46}$$

$$\lim_{n\to\infty}U_n=u \quad , \quad \lim_{n\to\infty}V_n=v \tag{47}$$

If $Z_n=U_n+iV_n=R_ne^{i\theta_n}$ then

$$R_{n+1}=\left(1+R_n^2+2R_n\cos\theta_n\right)^{-1/4} \tag{48}$$

$$\theta_{n+1}=\frac{\pi}{2}-\frac{1}{2}\tan^{-1}\left(\frac{R_n\sin\theta_n}{1+R_n\cos\theta_n}\right) \tag{49}$$

$$R_1=0\,,\,\theta_1=0 \tag{50}$$

$$\lim_{n\to\infty}R_n=\left|z\right|=\sqrt{u^2+v^2} \quad , \quad \lim_{n\to\infty}\theta_n=\tan^{-1}\left(\frac{v}{u}\right) \tag{51}$$

9. Integral formula

$$i\int_0^1\frac{3+x^2}{1+x^2+x^3}dx-\int_0^1\frac{4-x^2+2xi}{\left(3-4x^2\right)+i\left(5x-x^3\right)}dx+\int_0^1\frac{3-x^2}{1-x^2-x^3i}dx+$$
$$+i\int_0^1\frac{2+x^2+2xi}{\left(3x-x^2-x^3\right)+i\left(1-2x-3x^2\right)}dx=2\pi z=\frac{2\pi i}{\sqrt{1+\frac{i}{\sqrt{1+\frac{i}{\sqrt{1+...}}}}}} \tag{52}$$

References

1. M. Abramowitz and I.A. Stegun. Handbook of Mathematical Functions. Dover Publications, New York , 1970.