New Principles of Differential Equations I

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Abstract
This is the first part of the total paper. Since the theory of partial differential equations (PDEs) has been established nearly 300 years, there are many important problems have not been resolved, such as what are the general solutions of Laplace equation, acoustic wave equation, Helmholtz equation, heat conduction equation, Schrodinger equation and other important equations? How to solve the problems of definite solutions which have universal significance for these equations? What are the laws of general solution of the mth-order linear PDEs with n variables (n,$m \geq 2$)? Is there any general rule for the solution of a PDE in arbitrary orthogonal coordinate systems? Can we obtain the general solution of vector PDEs? Are there very simple methods to quickly and efficiently solve the exact solutions of nonlinear PDEs? And even general solution? Etc. These problems are all effectively solved in this paper. Substituting the results into the original equations, we have verified that they are all correct.

keywords: concise general solution; series general solution; exact solutions; transformational equations; problems of definite solutions.

Abbreviations: concise general solution (CGS); series general solution (SGS); independent variable transformational equations (IVTEs), dependent variable transformational equations (DVTEs); symmetric vector partial differential equations (SVPDEs); corresponding scalar equation (CSE); independent variable transformation vector equation (IVTVE).

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1 Introduction

In the establishment period of PDE theory, because of the general solution of the one-dimensional wave equation solved by d'Alembert, the mathematicians at that time believed that the general solutions of PDEs existed universally. Since there was no substantive breakthrough, with Cauchy's advice, they had to turn their attention to a variety of problems of definite solutions.

Using the new defines, laws and methods presented in this paper, the general solutions of many important PDEs had been solved for the first time, such as the Laplace equation, the wave equation, the Helmholtz equation, heat equation and so on, the exact solutions of relevant Cauchy problems have been solved too. In some cases, the general solutions and the exact solutions of the Cauchy problem for the Poisson equation and the Schrödingerequation have been solved. The types and numbers of the PDEs whose general solution could be solved by the new analytic method system are far more than the sum of the other methods can solve, and the solving process is very clear and concise.

The new theory also further reveals two major flaws and errors in the existing theory:

1. The new theory found that the general solutions of many PDEs have two forms: concise general solutions and series general solutions, in theory, infinite series solutions of a PDE can be obtained by its series general solution. The general solution of the one-dimensional wave equation introduced by mainstream textbooks and professional books is not a general solution in fact. It is only a special case of real general solution which can deduce the Fourier series solution. According to the obtained general solutions, we find why the state and the change of many natural phenomena that can be described by PDEs are both infinite. In this paper we find the relationship between the general solution and the series solution of PDEs, and point out that the alleged general solution of 1D wave equation obtained in current textbooks and professional books is not the general solution in fact. We find that any series solutions of PDEs can be obtained by its series general solution theoretically and show the root cause why the states and changes of some natural phenomena described by PDEs are all infinity.

2. In the theory of PDEs, almost all of the textbooks and professional books directly or indirectly declare that the number of the arbitrary functions in the general solution of mth-order PDEs is m, but no related rigorous proof up to now. In this paper we find a singularity of general solutions of Helmholtz equation the first time, namely the number of arbitrary functions in the general solutions is more than 2.

In Chapter 2 of this paper we present three types transformational equations: independent variable transformational equations (IVTEs), dependent variable transformational equations (DVTEs), independent variable transformation vector equation (IVTVE), and get a law of partial differential equations solution in various orthogonal coordinate system. The general solution of vector wave equation in Cartesian, cylindrical and spherical coordinate systems have been solved for the first time. We point out that the general solutions or particular solutions of various symmetric vector partial differential equations can be obtained similarly in any orthogonal coordinate system, such as vector Helmholtz equation, the magnetic vector potential equation and so on.

The laws of the general solution of mth-order linear partial differential equations with n variables have been studied deeply in Chapter 3 ($n, m \geq 2$).We have solved some typical nonlinear partial differential equations general solutions, particular solutions or solitary wave solutions in Chapter 4 and other relevant chapter, such as Emden-Fowler equation, Klein-Gordon equation, sine-Gordon equation, Burgers equation, KdV equation, etc. Based on the large number of results obtained in this book, we find that the general solutions of some PDEs have similarities, and find the roots of these similarities by the concepts of general equations and restricted equations.

1. General solutions and exact solutions of Cauchy problem of mathematical physics equations

In recent decades, for solving partial differential equations (PDEs) many analytic methods [1-3] and numerical methods [4-6] have been developed rapidly, the solitary wave solutions [7-9] plays an important role in nonlinear PDEs (NLPDEs), the existence [10, 11], uniqueness [12, 13], and stability [14, 15] of the PDEs solution have been well studied. The formulas of differential equations general solution have the same important value and significance as the algebraic equations Root Formulas, although the exact or numerical solutions of many PDEs have been found, species of PDEs which have the general solution are extremely rare yet.

Since mathematical physics equations (MPEs) are very important in PDEs, their progress is always been noticed especially [16, 17]. In the professional books of MPEs only one dimensional wave equation [18, 19] and some linear PDEs (LPDEs) with two variables [20] have general solution. Using the new analytic methods proposed in this chapter, the general solutions of many important PDEs had been solved for the first time, the exact solutions of relevant problems of definite solutions have been solved too.

1.1. New principles and methods I

We first study the laws of one multivariate function is a composite function of another. In $\mathbb{R}^n$ space ($n \geq 2$), assuming $u(x_1, \cdots, x_n), v(x_1, \cdots, x_n)$ are both smooth functions, and u is a composite function of v

$$u(x_1, \cdots, x_n) = f(v), (f'_v \neq 0), \tag{1}$$

where f is an unary smooth function, according to the laws of differential and total differential

$$\begin{aligned} du &= f'_v dv = f'_v v_{x_1} dx_1 + f'_v v_{x_2} dx_2 + \cdots + f'_v v_{x_n} dx_n \\ &= u_{x_1} dx_1 + u_{x_2} dx_2 + \cdots + u_{x_n} dx_n. \end{aligned}$$

So

$$u_{x_i} = f'_v v_{x_i}. \tag{2}$$

By (2), we obtain further

$$u_{x_i x_i} = f'_v v_{x_i x_i} + f''_v v_{x_i}^2, u_{x_i x_j} = f'_v v_{x_i x_j} + f''_v v_{x_i} v_{x_j}. \tag{3}$$

Higher order laws may be deduced analogously. We set

$$v(x_1, \cdots, x_n) = k_1x_1 + k_2x_2 + \cdots + k_nx_n + k_{n+1}, \tag{4}$$

where k_i are all arbitrary constants($i = 1, 2, \cdots, n+1$), then

$$v_{x_i} = k_i, v_{x_ix_i} = v_{x_ix_j} = 0, (i \neq j), (i, j \in \{1, 2, \cdots, n\}). \tag{5}$$

By (3) and under the condition of (4), we have

$$u_{x_i} = k_i f'_v, u_{x_ix_i} = k_i^2 f''_v, u_{x_ix_j} = k_ik_j f''_v \tag{6}$$

Using mathematical induction we can get

$$u_{x_i}^{(m)} = k_i^m f_v^{(m)}, u_{x_ix_j}^{(pq)} = k_i^p k_j^q f_v^{(p+q)}, (0 \leq m < \infty, 0 \leq p, q < \infty), \tag{7}$$

$$u_{x_1x_2\cdots x_n}^{(i_1i_2\cdots i_n)} = k_1^{i_1} k_2^{i_n} \cdots k_n^{i_n} f_v^{(m)}, (i_1 + i_2 + \cdots + i_n = m), \tag{8}$$

where

$$\begin{aligned} &f_v^{(m)} \equiv \frac{d^m f}{dv^m}, f_v^{(p+q)} \equiv \frac{d^{p+q} f}{dv^{p+q}}, u_{x_i}^{(m)} \equiv \frac{\partial^m u}{\partial x_i^m}, \\ &u_{x_ix_j}^{(pq)} \equiv \frac{\partial^{p+q} u}{\partial x_i^p x_j^q}, u_{x_1x_2\cdots x_n}^{(i_1i_2\cdots i_n)} \equiv \frac{\partial^m u}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}. \end{aligned} \tag{9}$$

By the above laws, we present a new transformational method to solve the general solutions or exact solutions of some PDEs.

Transformational Method 1. *In the domain D, ($D \subset \mathbb{R}^n$), any established mth-order PDE with n space variables $F(x_1, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots) = 0$, set $v = v(x_1, \cdots, x_n)$ and $u = f(v)$ are both undetermined mth-differentiable functions ($u, v \in C^m(D)$), then substitute $u = f(v)$ and its partial derivatives into $F = 0$*

1. In case of working out $v(x_1, \cdots, x_n)$ and $f(v)$, then $u = f(v(x_1, \cdots, x_n))$ is the solution of $F = 0$,

2. In case of dividing out $f(v)$ and its partial derivative, also working out $v(x_1, \cdots, x_n)$, then $u = f(v(x_1, \cdots, x_n))$ is the solution of $F = 0$, and f is an arbitrary unary mth-differentiable function,

3. In case of dividing out $f(v)$ and its partial derivative, also getting $k = 0$, but in fact $k \neq 0$, then $u = f(v(x_1, \cdots, x_n))$ is not the solution of $F = 0$, and f is an arbitrary unary mth-differentiable function.

In Transformational Method 1 $v = v(x_1, \cdots x_n)$ may be an unknown function completely or has a determinate form with unknown parameters, the solution of f may be an arbitrary or a certain unary mth-differentiable function, the solution of v and f may not be single and so on. Here we use Transformational Method 1 to solve three important PDEs.

Example 1.1.

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{10}$$

Eq. (10) is Laplace equation in Cartesian coordinate system. According to Transformational Method 1, set

$$u(x, y, z) = f(v) = f(k_1x + k_2y + k_3z + k_4), \tag{11}$$

where $v=k_1x+k_2y+k_3z+k_4$, k_1-k_4 are unascertained parameters and f is an undetermined unary 2th-differentiable function, by (7)

$$u_{xx}+u_{yy}+u_{zz}=(k_1^2+k_2^2+k_3^2)f_v^{(2)}=0.$$

The first case is

$$f_v^{(2)}=0, \tag{12}$$

according to Transformational Method 1 the solution of Eq. (10) is

$$u=f(v)=k_1x+k_2y+k_3z+k_4, \tag{13}$$

where k_1-k_4 are all arbitrary constants. The second case is

$$k_1^2+k_2^2+k_3^2=0 \Rightarrow k_1=\pm\sqrt{-k_2^2-k_3^2}, \tag{14}$$

where k_2 and k_3 are all arbitrary constants. By Transformational Method 1 the solution of Eq. (10) is

$$\begin{aligned} u = & f_1(x\sqrt{-k_2^2-k_3^2}+k_2y+k_3z+k_4) \\ & +f_2(-x\sqrt{-k_2^2-k_3^2}+k_2y+k_3z+k_4) \end{aligned} \tag{15}$$

where k_2, k_3 and k_4 are all arbitrary constants. Note the k_2, k_3 and k_4 in $f_1(x\sqrt{-k_2^2-k_3^2}+k_2y+k_3z+k_4)$ may be different with them in $f_2(-\sqrt{-k_2^2-k_3^2}+k_2y+k_3z+k_4)$, the k_1-k_4 in Eq. (13) may be different with them in Eq. (15), and Eq. (10) is a linear equation, in order to facilitate writing, the general solution of Laplace equation may be written as

$$\begin{aligned} u = & f_1(x\sqrt{-k_2^2-k_3^2}+k_1y+k_2z+k_3) \\ & +f_2(-x\sqrt{-k_4^2-k_5^2}+k_4y+k_5z+k_6)+k_7x+k_8y+k_9z+k_{10}, \end{aligned} \tag{16}$$

where $f_1$and f_2 are arbitrary unary second differentiable functions, k_1-k_{10} are arbitrary parameters.

Since k_1-k_{10} are arbitrary constants and Eq. (10) is a linear equation, (16) can also be written as the form of a function series

$$\begin{aligned} u = & \textstyle\sum_{i=1}^{s}(f_{1_i}(x\sqrt{-k_{1_i}^2-k_{2_i}^2}+k_{1_i}y+k_{2_i}z+k_{3_i})+f_{2_i}(-x \\ & \sqrt{-k_{4_i}^2-k_{5_i}^2}+k_{4_i}y+k_{5_i}z+k_{6_i})+k_{7_i}x+k_{8_i}y+k_{9_i}z+k_{10_i}), \end{aligned} \tag{17}$$

where f_{1_i} and f_{2_i} are arbitrary unary second differentiable functions, $(1\le s\le\infty)$, $k_{1_i}-k_{10_i}$ are arbitrary determined parameters.

We call (16) the **concise general solution** (CGS) which has the most simple form and call (17) the **series general solution** (SGS) which could have infinite arbitrary functions in it. Theoretically all specific series solutions of Eq. (10) can be obtained by SGS (17).

Example 1.2.

$$\begin{aligned} (u_x^{(m)})^r & +\tfrac{a(x,y,z)(k_1y+k_2z+k_4)^{mr}}{(k_1x+k_3z+k_5)^{mr}}(u_y^{(m)})^r \\ & -\tfrac{(1+a(x,y,z))(k_1y+k_2z+k_4)^{mr}}{(k_2x+k_3y+k_6)^{mr}}(u_z^{(m)})^r=0, \end{aligned} \tag{18}$$

where k_1-k_6 are arbitrary known constant, $a(x,y,z)$ is an arbitrary known function with 3 variables, by Transformational Method 1, set

$$u(x,y,z)=f(v)=f(k_1xy+k_2xz+k_3yz+k_4x+k_5y+k_6z+k_7), \tag{19}$$

where $v = k_1xy + k_2xz + k_3yz + k_4x + k_5y + k_6z + k_7, k_7$ is an undetermined parameter and f is an unary mth-differentiable function to be determined, then

$$u_x = f'_v v_x = (k_1y + k_2z + k_4)f'_v, \tag{20}$$

$$u_y = f'_v v_y = (k_1x + k_3z + k_5)f'_v, \tag{21}$$

$$u_z = f'_v v_z = (k_2x + k_3y + k_6)f'_v. \tag{22}$$

According to (20)-(22) and mathematical induction we get

$$u_x^{(m)} = (k_1y + k_2z + k_4)^m f_v^{(m)}, \tag{23}$$

$$u_y^{(m)} = (k_1x + k_3z + k_5)^m f_v^{(m)}, \tag{24}$$

$$u_z^{(m)} = (k_2x + k_3y + k_6)^m f_v^{(m)}. \tag{25}$$

Then

$$\begin{aligned}
&(u_x^{(m)})^r + \tfrac{a(x,y,z)(k_1y+k_2z+k_4)^{mr}}{(k_1x+k_3z+k_5)^{mr}}(u_y^{(m)})^r - \tfrac{(1+a(x,y,z))(k_1y+k_2z+k_4)^{mr}}{(k_2x+k_3y+k_6)^{mr}}(u_z^{(m)})^r \\
&= (k_1y + k_2z + k_4)^{mr}(f_v^{(m)})^r + \tfrac{a(x,y,z)(k_1y+k_2z+k_4)^{mr}}{(k_1x+k_3z+k_5)^{mr}}(k_1x + k_3z + k_5)^{mr}(f_v^{(m)})^r \\
&-\tfrac{(1+a(x,y,z))(k_1y+k_2z+k_4)^{mr}}{(k_2x+k_3y+k_6)^{mr}}(k_2x + k_3y + k_6)^{mr}(f_v^{(m)})^r = 0 \\
&\Rightarrow (k_1y + k_2z + k_4)^{mr}(f_v^{(m)})^r - (k_1y + k_2z + k_4)^{mr}(f_v^{(m)})^r = 0.
\end{aligned}$$

According to Transformational Method 1, the solution of Eq. (18) is

$$u = f(k_1xy + k_2xz + k_3yz + k_4x + k_5y + k_6z + k_7), \tag{26}$$

where $f(v)$ is an arbitrary unary mth-differentiable function and k_7 is an arbitrary constants.If $m = 1$, (26) is the general solution of Eq. (18).

Example 1.3.

$$a_1\left(u_{x_1}^{(m)}\right)^r + a_2\left(u_{x_2}^{(m)}\right)^r + \ldots + a_n\left(u_{x_n}^{(m)}\right)^r + a_{n+1}\left(u_{x_2x_3}^{(pq)}\right)^r = 0, \tag{27}$$

where $a_i, (i = 1, 2, \cdots n+1)$ are arbitrary known constants, $r \geq 1, 1 \leq p+q = m$, the left of Eq. (27) could be added any number and types of $\left(u_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)}\right)^r$,$(i_1 + i_2 + \cdots + i_n = m)$ with any constant coefficient, since the similar calculation method, for facilitating writing there is only the $a_{n+1}\left(u_{x_2x_3}^{(pq)}\right)^r$ in Eq. (27).

By Transformational Method 1, set $u(x_1, \cdots x_n) = f(v), v(x_1, \cdots x_n) = k_1x_1 + k_2x_2 + \cdots + k_nx_n + k_{n+1}$, where $k_1, k_2, \cdots k_{n+1}$ are unascertained parameters and f is an undetermined unary mth-differentiable function, by (7)

$$\begin{aligned}
a_1\left(u_{x_1}^{(m)}\right)^r &+a_2\left(u_{x_2}^{(m)}\right)^r + \cdots + a_n\left(u_{x_n}^{(m)}\right)^r + a_{n+1}\left(u_{x_2x_3}^{(pq)}\right)^r \\
&= (a_1k_1^{mr} + a_2k_2^{mr} + \cdots + a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr})\left(f_v^{(m)}\right)^r = 0.
\end{aligned}$$

The first case is

$$\left(f_v^{(m)}\right)^r = 0, \tag{28}$$

according to Transformational Method 1 the solution of Eq. (27) is

$$u = f\left(v\right) = c_{m-1}v^{m-1} + c_{m-2}v^{m-2} + \ldots + c_1v, \tag{29}$$

where $v(x_1, \cdots x_n) = k_1x_1 + k_2x_2 + \cdots + k_nx_n + k_{n+1}$, $k_1 - k_{n+1}$ and $c_1 - c_{m-1}$ are all arbitrary constants.
Since v contains arbitrary constants k_{n+1}, so there is no arbitrary constants c_0 in (29).
The second case is

$$a_1k_1^{mr} + a_2k_2^{mr} + \ldots a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr} = 0, \tag{30}$$

if m, r are both odd, then

$$k_1 = \left(-\frac{a_2k_2^{mr} + a_3k_3^{mr} + \ldots + a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr}}{a_1}\right)^{\frac{1}{mr}}, \tag{31}$$

where $k_2 - k_{n+1}$ are all arbitrary constants. By Transformational Method 1 the solution of Eq. (27) is

$$\begin{aligned} u =& f((-\frac{a_2k_2^{mr} + \ldots + a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr}}{a_1})^{\frac{1}{mr}} x_1 + k_2x_2 + \ldots \\ &+ k_nx_n + k_{n+1}), \end{aligned} \tag{32}$$

where f is an arbitrary unary mth-differentiable function.
If there is at least one even number among m and r in Eq. (27), then

$$k_1 = \pm\left(-\frac{a_2k_2^{mr} + a_3k_3^{mr} + \ldots + a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr}}{a_1}\right)^{\frac{1}{mr}}. \tag{33}$$

By Transformational Method 1, except (29) and (32) another solution of Eq. (27) is

$$\begin{aligned} u =& f(-(-\frac{a_2k_2^{mr} + \ldots + a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr}}{a_1})^{\frac{1}{mr}} x_1 + k_2x_2 + \ldots \\ &+ k_nx_n + k_{n+1}). \end{aligned} \tag{34}$$

In the case of $r = 1$, Eq. (27) becomes linear equation

$$a_1u_{x_1}^{(m)} + a_2u_{x_2}^{(m)} + \ldots + a_nu_{x_n}^{(m)} + a_{n+1}u_{x_2x_3}^{(pq)} = 0. \tag{35}$$

If m is odd, by (29) and (32) the solution of Eq. (35) is

$$\begin{aligned} u = & f\left(\left(-\frac{a_2k_2^{m} + \ldots + a_nk_n^{m} + a_{n+1}k_2^{p}k_3^{q}}{a_1}\right)^{\frac{1}{m}} x_1 + k_2x_2 + \ldots + k_nx_n + k_{n+1}\right) \\ &+ c_{m-1}v^{m-1} + c_{m-2}v^{m-2} + \ldots + c_1v, \end{aligned} \tag{36}$$

where $v = C_1x_1 + C_2x_2 + \cdots + C_nx_n + C_{n+1}, C_1 - C_{n+1}$ are arbitrary constants. If m is even, by (29), (32) and (34) the solution of Eq. (35) is

$$\begin{aligned} u = & f_1\left(\left(-\frac{a_2k_2^{m} + \ldots + a_nk_n^{m} + a_{n+1}k_2^{p}k_3^{q}}{a_1}\right)^{\frac{1}{m}} x_1 + k_2x_2 + \ldots + k_nx_n + k_{n+1}\right) \\ &+ f_2\left(-\left(-\frac{a_2l_2^{m} + \ldots + a_nl_n^{m} + a_{n+1}l_2^{p}l_3^{q}}{a_1}\right)^{\frac{1}{m}} x_1 + l_2x_2 + \ldots + l_nx_n + l_{n+1}\right) \\ &+ c_{m-1}v^{m-1} + c_{m-2}v^{m-2} + \ldots + c_1v \end{aligned} \tag{37}$$

where f_1 and f_2 are arbitrary unary mth-differentiable functions, $k_2 - k_{n+1}$ and $l_2 - l_{n+1}$ are arbitrary parameters. In Appendix A we proved that if $k_1, l_1 \neq 0$ and $k_1, l_1 \nrightarrow 0$ in (37), c_1v can be described by f_1 and f_2. If $m = 2, r = p = q = 1$, Eq. (27) becomes

$$a_1u_{x_1}^{(2)} + a_2u_{x_2}^{(2)} + \ldots + a_nu_{x_n}^{(2)} + a_{n+1}u_{x_2x_3} = 0. \tag{38}$$

According to (37) and (16), the CGS and SGS of Eq. (38) can be get respectively

$$\begin{aligned} u = f_1 &\left(\left(-\frac{a_2k_2^2+\ldots a_nk_n^2+a_{n+1}k_2k_3}{a_1}\right)^{\frac{1}{2}} x_1+k_2x_2+\ldots+k_nx_n+k_{n+1}\right) \\ &+ f_2\left(-\left(-\frac{a_2l_2^2+\ldots a_nl_n^2+a_{n+1}l_2l_3}{a_1}\right)^{\frac{1}{2}} x_1+l_2x_2+\ldots+l_nx_n+l_{n+1}\right)+c_1v. \end{aligned} \tag{39}$$

$$\begin{aligned} u = \sum_{i=1}^{s}(f_{1_i}((-&\frac{a_2k_{i_2}^2+\ldots+a_nk_{i_n}^2+a_{n+1}k_{i_2}k_{i_3}}{a_1})^{\frac{1}{2}} x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}) \\ + f_{2_i}(-(-&\frac{a_2l_{i_2}^2+\ldots+a_nl_{i_n}^2+a_{n+1}l_{i_2}l_{i_3}}{a_1})^{\frac{1}{2}} x_1+l_{i_2}x_2+\ldots+l_{i_n}x_n+l_{i_{n+1}}))+c_1v \end{aligned} \tag{40}$$

Consider the following Cauchy problem of Eq. (38)

$$u\left(0,x_2,\ldots x_n\right)=\sum_{i=1}^{s}\varphi_i\left(k_{i_2}x_2+k_{i_3}x_3+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right), \tag{41}$$

$$u_{x_1}\left(0,x_2,\ldots x_n\right)=\sum_{i=1}^{s}\psi_i\left(k_{i_2}x_2+k_{i_3}x_3+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right), \tag{42}$$

where $1\le s\le\infty$, x_1 sometimes equal to time t. In (40), set $c_1=0, k_{i_j}=l_{i_j}, (i=1,2,\cdots s, j=2,3,\cdots n+1)$, by further calculation which is in Appendix B, the exact solution of Eq. (38) in the conditions of (41) and (42) is

$$\begin{aligned} u = \ & \tfrac{1}{2}\textstyle\sum_{i=1}^{s}(\varphi_i(k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}) \\ & +\varphi_i(-k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}) \\ & +\tfrac{1}{k_{i_1}}\int_{-k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}}^{k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}}\psi(\xi_i)d\xi_i) \end{aligned} \tag{43}$$

where

$$k_{i_1}=\left(-\left(a_2k_{i_2}^2+\ldots+a_nk_{i_n}^2+a_{n+1}k_{i_2}k_{i_3}\right)/a_1\right)^{\frac{1}{2}}. \tag{44}$$

According to the above three typical cases, we know that the solutions of some linear or nonlinear PDEs can be obtained by using Transformational Method 1.

1.2. 1D wave equation

1D wave equation

$$u_{tt}-a^2u_{xx}=0, \tag{45}$$

is the first PDE studied deeply. Almost all current textbooks and professional books have pointed out that the general solution of Eq. (45) is

$$u=f_1\left(x+at\right)+f_2\left(x-at\right). \tag{46}$$

Fourier series solution of Eq. (45) is

$$u=\sum_{i=1}^{s}(A_n\cos(\frac{n\pi at}{l})+B_n\sin(\frac{n\pi at}{l}))\sin(\frac{n\pi x}{l}). \tag{47}$$

By (46) we cannot get (47) obviously, there is no answer why the particular solution (47) cannot be got by the general solution (46).

Eq. (45) is a special case of Eq. (38), according to (39) and (40), its CGS and SGS can be get respectively

$$u = f_1\left(k_1x + k_1at + k_2\right) + f_2\left(k_3x - k_3at + k_4\right) + k_5x + k_6t + k_7, \tag{48}$$

$$\begin{array}{l} u = \sum_{i=1}^{s}\left(f_{1_i}\left(k_{1_i}x + k_{1_i}at + k_{2_i}\right) + f_{2_i}\left(k_{3_i}x - k_{3_i}at + k_{4_i}\right)\right) \\ +k_5x + k_6t + k_7, \end{array} \tag{49}$$

where f_1, f_{1_i}, f_2 and f_{2_i} are arbitrary unary second differentiable functions, $k_1 - k_7$ are arbitrary parameters, $k_{1_i} - k_{4_i}$ are arbitrary determined parameters, $(1 \leq s \leq \infty)$. Of course, the general solution of Eq. (45) can also be written as

$$u = f_1\left(k_1x + k_1at + k_2\right) + \sum_{i=1}^{s} f_{2_i}\left(k_{3_i}x - k_{3_i}at + k_{4_i}\right) + k_5x + k_6t + k_7 \tag{50}$$

and so on, (50) is also a SGS, but in this paper we will not discuss the general solutions in special forms.

By the above results we can see that (46) is a special case of (48) and (49), and is not a general solution of Eq. (45) in fact, so using (46) we cannot get the Fourier series solution.

Theoretically every specific series solution of Eq. (45) can be obtained by the SGS (49), as a case, we will obtain the Fourier series solution (47).

Set

$$f_{1_n}\left(k_{1_n}x + k_{1_n}at + k_{2_n}\right) = C_n \sin(k_{1_n}x + k_{1_n}at + k_{2_n}), \tag{51}$$

$$f_{2_n}\left(k_{3_n}x - k_{3_n}at + k_{4_n}\right) = D_n \cos(k_{3_n}x - k_{3_n}at + k_{4_n}), \tag{52}$$

So

$$\begin{array}{l} u = \sum_{i=1}^{s}(f_{1_n}\left(k_{1_n}x + k_{1_n}at + k_{2_n}\right) + f_{2_n}\left(k_{3_n}x - k_{3_n}at + k_{4_n}\right)) \\ = \sum_{i=1}^{s}(C_n \sin(k_{1_n}x)\cos(k_{1_n}at)\cos k_{2_n} + \cos(k_{1_n}x)\sin(k_{1_n}at)\cos k_{2_n} \\ + \cos(k_{1_n}x)\cos(k_{1_n}at)\sin k_{2_n} - \sin(k_{1_n}x)\sin(k_{1_n}at)\sin k_{2_n}) \\ +D_n(\cos(k_{3_n}x)\cos(k_{3_n}at)\cos k_{4_n} + \sin(k_{3_n}x)\sin(k_{3_n}at)\cos k_{4_n} \\ - \sin(k_{3_n}x)\cos(k_{3_n}at)\sin k_{4_n} + \cos(k_{3_n}x)\sin(k_{3_n}at)\sin k_{4_n})). \end{array}$$

Set $k_{1_n} = k_{3_n} = k_n$, then

$$\begin{array}{l} u = \sum_{i=1}^{s}((C_n \cos k_{2_n} - D_n \sin k_{4_n})\sin\left(k_nx\right)\cos\left(k_nat\right) \\ + \left(C_n \cos k_{2_n} + D_n \sin k_{4_n}\right)\cos\left(k_nx\right)\sin\left(k_nat\right) \\ + \left(C_n \sin k_{2_n} + D_n \cos k_{4_n}\right)\cos\left(k_nx\right)\cos\left(k_nat\right) \\ + \left(-C_n \sin k_{2_n} + D_n \cos k_{4_n}\right)\sin\left(k_nx\right)\sin\left(k_nat\right)) \end{array} \tag{53}$$

Set

$$C_n \cos k_{2_n} + D_n \sin k_{4_n} = C_n \sin k_{2_n} + D_n \cos k_{4_n} = 0.$$

We have

$$k_{4_n} = \frac{(2m+1)\,\pi}{2} - k_{2_n}.$$

Set $k_{4_n} = \pi/2 - k_{2_n}$, we can get $C_n = -D_n$. Substituting the above results into (53)

$$\begin{aligned} u &= \textstyle\sum_{i=1}^{s}((C_n \cos k_{2_n} - D_n \sin k_{4_n}) \sin(k_n x) \cos(k_n at) \\ &\quad +(-C_n \sin k_{2_n} + D_n \cos k_{4_n}) \sin(k_n x) \sin(k_n at)) \\ &= \textstyle\sum_{i=1}^{s}(2C_n \cos k_{2_n} \sin(k_n x) \cos(k_n at) - 2C_n \sin k_{2_n} \sin(k_n x) \sin(k_n at)). \end{aligned}$$

Namely

$$u = \sum_{i=1}^{s} 2C_n(\cos k_{2_n} \cos(k_n at) - \sin k_{2_n} \sin(k_n at)) \sin(k_n x). \tag{54}$$

Since C_n, k_n and k_{2_n} are all arbitrary parameters, set

$$k_n = \frac{n\pi}{l}, 2C_n \cos k_{2_n} = A_n, -2C_n \sin k_{2_n} = B_n. \tag{55}$$

Then (54) may be translated into (47). (46) was first discovered by d' Alembert, then Daniel Bernoulli discovered an infinite series solution

$$u = \sum_{i=1}^{\infty} a_n \sin(\frac{n\pi x}{l}) \cos(\frac{n\pi at}{l}). \tag{56}$$

The relationship between (46) and (56) led to a well-known controversy in the history of mathematics [21], many famous mathematicians have been involved in this drastic and lengthy debate, even after discovering the Fourier series solution (47), the relationship between (46) and (47) is still unclear, now the problem is finally solved successfully, (46) and (47) are two different closed solutions, not the general solution, only the form of (46) is relatively close the general solution.

1.3. 2D wave equation

The form of 2D wave equation in Cartesian coordinate system is

$$u_{tt} - a^2 u_{xx} - a^2 u_{yy} = 0. \tag{57}$$

Eq. (57) is an especial case of Eq. (38), by (39) its CGS can be obtained

$$\begin{aligned} u &= f_1\left(k_1 x + k_2 y + at\sqrt{k_1^2 + k_2^2} + k_3\right) \\ &\quad +f_2\left(k_4 x + k_5 y - at\sqrt{k_4^2 + k_5^2} + k_6\right) + k_7 x + k_8 y + k_9 t + k_{10} \\ &= g\left(\tfrac{k_1 x}{\sqrt{k_1^2+k_2^2}} + \tfrac{k_2 y}{\sqrt{k_1^2+k_2^2}} + at + g_0\right) \\ &\quad +h\left(\tfrac{k_4 x}{\sqrt{k_4^2+k_5^2}} + \tfrac{k_5 y}{\sqrt{k_4^2+k_5^2}} - at + h_0\right) + k_7 x + k_8 y + k_9 t + k_{10} \\ &= g\left(x \cos\theta + y \sin\theta + at + g_0\right) \\ &\quad +h\left(x \cos\varphi + y \sin\varphi - at + h_0\right) + k_7 x + k_8 y + k_9 t + k_{10}, \end{aligned} \tag{58}$$

where f_1, f_2, g and h are arbitrary unary second differentiable functions, $k_1 - k_{10}, \theta, \varphi, g_0$ and h_0 are arbitrary parameters. $g(x \cos\theta + y \sin\theta + at + g_0)$ is a parallel wave with the speed a, the angle between x axis and spread direction of g is θ which is arbitrary. The SGS of Eq. (57) is

$$\begin{aligned} u(x, y, t) = \textstyle\sum_i(&g_i(x \cos\theta_i + y \sin\theta_i + at + g_{i0})+ \\ &h_i(x \cos\varphi_i + y \sin\varphi_i - at + h_{i0})) + k_7 x + k_8 y + k_9 t + k_{10}, \end{aligned} \tag{59}$$

where g_i and h_i are arbitrary unary second differentiable functions, $\theta_i, \varphi_i, g_{i0}$ and h_{i0} are arbitrary determined parameters.

A research hotspot is using numerical methods to study the 2D wave equation [21]. Consider the following initial value problem of Eq. (57)

$$\begin{array}{l} u\left(x,y,0\right)=\sum_i \varphi_i\left(k_{i_1}x+k_{i_2}y+k_{i_3}\right),\\ u_t\left(x,y,0\right)=\sum_i \psi_i\left(k_{i_1}x+k_{i_2}y+k_{i_3}\right). \end{array} \tag{60}$$

Similar to the solving method of (43), the exact solution of Eq. (57) on the conditions of (60) can be got

$$\begin{array}{l} u=\frac{1}{2}\sum_i(\varphi_i(k_{i_1}x+k_{i_2}y-at\sqrt{k_{i_1}^2+k_{i_2}^2}+k_{i_3})\\ \quad+\varphi_i(k_{i_1}x+k_{i_2}y+at\sqrt{k_{i_1}^2+k_{i_2}^2}+k_{i_3})\\ \quad+\frac{1}{a\sqrt{k_{i_1}^2+k_{i_2}^2}}\int_{k_{i_1}x+k_{i_2}y-at\sqrt{k_{i_1}^2+k_{i_2}^2}+k_{i_3}}^{k_{i_1}x+k_{i_2}y+at\sqrt{k_{i_1}^2+k_{i_2}^2}+k_{i_3}}\psi\left(\xi_i\right)d\xi_i). \end{array} \tag{61}$$

1.4. Acoustic wave equation

The form of acoustic wave equation is

$$p_{tt}-c_0^2\bigtriangleup p=0 \tag{62}$$

where Δ is the Laplace operator, p is the sound pressure, and c_0 is the sound speed. Eq. (62) is a special case of Eq. (38), according to (39) its CGS in Cartesian coordinate system is

$$\begin{array}{l} p=f_1\left(k_1x+k_2y+k_3z+c_0t\sqrt{k_1^2+k_2^2+k_3^2}+k_4\right)\\ \quad+f_2\left(k_5x+k_6y+k_7z-c_0t\sqrt{k_5^2+k_6^2+k_7^2}+k_8\right)+k_9x+k_{10}y+k_{11}z+k_{12}t+k_{13}\\ \quad=g\left(\frac{k_1x}{\sqrt{k_1^2+k_2^2+k_3^2}}+\frac{k_2y}{\sqrt{k_1^2+k_2^2+k_3^2}}+\frac{k_3z}{\sqrt{k_1^2+k_2^2+k_3^2}}+c_0t+\frac{k_4}{\sqrt{k_1^2+k_2^2+k_3^2}}\right)\\ \quad+h\left(\frac{k_5x}{\sqrt{k_5^2+k_6^2+k_7^2}}+\frac{k_6y}{\sqrt{k_5^2+k_6^2+k_7^2}}+\frac{k_7z}{\sqrt{k_5^2+k_6^2+k_7^2}}-c_0t+\frac{k_8}{\sqrt{k_5^2+k_6^2+k_7^2}}\right)\\ \quad+k_9x+k_{10}y+k_{11}z+k_{12}t+k_{13}\\ \quad=g(x\sin\theta\cos\varphi+y\sin\theta\sin\varphi+z\cos\theta+c_0t+g_0)\\ \quad+h(x\sin\phi\cos\psi+y\sin\phi\sin\psi+z\cos\phi+c_0t+h_0)\\ \quad+k_9x+k_{10}y+k_{11}z+k_{12}t+k_{13}, \end{array} \tag{63}$$

where f_1, f_2, g and h are arbitrary unary second differentiable functions, $k_1-k_{13}, \theta, \varphi, \phi, \psi, g_0$ and h_0 are arbitrary parameters.

$g(x\sin\theta\cos\varphi+y\sin\theta\sin\varphi+z\cos\theta+c_0t+g_0)$ is a parallel wave with the speed c_0, θ is the angle between z axis and spread direction of g, φ is the angle between x axis and the projection in xy plane of spread direction of g. The SGS of Eq. (62) is

$$\begin{array}{l} p=\sum_i\ g_i(x\sin\theta_i\cos\varphi_i+y\sin\theta_i\sin\varphi_i+z\cos\theta_i+c_0t+g_{i0})\\ \quad+\sum_i h_i(x\sin\phi_i\cos\psi_i+y\sin\phi_i\sin\psi_i+z\cos\phi_i-c_0t+h_{i0})\\ \quad+k_9x+k_{10}y+k_{11}z+k_{12}t+k_{13}, \end{array} \tag{64}$$

where g_i and h_i are arbitrary unary second differentiable functions, $\theta_i, \varphi_i, \phi_i, \psi_i, g_{i0}, h_{i0}$ are arbitrary determined parameters.

Consider the following initial value problem of Eq. (62)

$$p\left(x,y,z,0\right)=\sum_{i}\varphi_{i}\left(k_{i_1}x+k_{i_2}y+k_{i_3}z+k_{i_4}\right), \tag{65}$$

$$p_t\left(x,y,z,0\right)=\sum_{i}\psi_{i}\left(k_{i_1}x+k_{i_2}y+k_{i_3}z+k_{i_4}\right). \tag{66}$$

Similar to the solving method of (43), the exact solution of Eq. (62) on the conditions of (65) and (66) is

$$\begin{array}{l} p=\frac{1}{2}\sum_{i}(\varphi_i(k_{i_1}x+k_{i_2}y+k_{i_3}z+c_0t\sqrt{k_{i_1}^2+k_{i_2}^2+k_{i_3}^2}+k_{i_4}) \\ \quad +\varphi_i(k_{i_1}x+k_{i_2}y+k_{i_3}z-c_0t\sqrt{k_{i_1}^2+k_{i_2}^2+k_{i_3}^2}+k_{i_4}) \\ \quad +\frac{1}{c_0\sqrt{k_{i_1}^2+k_{i_2}^2+k_{i_3}^2}}\int_{k_{i_1}x+k_{i_2}y+k_{i_3}z-c_0t\sqrt{k_{i_1}^2+k_{i_2}^2+k_{i_3}^2}+k_{i_4}}^{k_{i_1}x+k_{i_2}y+k_{i_3}z+c_0t\sqrt{k_{i_1}^2+k_{i_2}^2+k_{i_3}^2}+k_{i_4}}\psi_i(\xi)d\xi) \end{array} \tag{67}$$

Nonlinear acoustic wave equation is a hot area of current research [22, 23], the solving method of nonlinear PDEs will be studied in our other papers.

1.5. Laplace equation

Laplace equation is importantly used not only in classical electrodynamics, thermodynamics and fluid dynamics etc., but also in the modern theory of the invisible [25, 26]. In recent decades a research hotspot is using many numerical methods for solving Laplace's equation in various geometries and boundary conditions, such as the moment methods [27], quasi-reversibility methods [28, 29], finite difference methods [30] and so on.

According to the previous calculation results, the CGS and SGS of $u_{xx}+u_{yy}+u_{zz}=0$ are (16) and (17) respectively. Assuming Eq. (10) satisfies the following boundary conditions

$$\begin{array}{l} u\left(0,y,z\right)=\sum_{i=1}^{s}\varphi_i\left(k_{i_1}y+k_{i_2}z+k_{i_3}\right) \\ u_x\left(0,y,z\right)=\sum_{i=1}^{s}\psi_i\left(k_{i_1}y+k_{i_2}z+k_{i_3}\right). \end{array} \tag{68}$$

According to (43) and (44), the exact solution of Eq. (10) on the conditions of (68) is

$$\begin{array}{l} u=\frac{1}{2}\sum_{i=1}^{s}(\varphi_i(x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3}) \\ \quad +\varphi_i(-x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3}) \\ \quad +\frac{1}{\sqrt{-k_{i_1}^2-k_{i_2}^2}}\int_{-x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3}}^{x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3}}\psi(\xi_i)d\xi_i) \end{array} \tag{69}$$

1.6. Basic principles and methods II

$v(x_1,\cdots x_n)$ and f are both undetermined in Transformational Method 1, to solve some PDEs we may be required to set f pending and $v(x_1,\cdots x_n)$ known, so put forward Transformational Method 2.

Transformational Method 2. *In the domain D, $(D\subset\mathbb{R}^n)$, any established mth-order PDE with n space variables $F(x_1,\cdots,x_n,u,u_{x_1},\cdots,u_{x_n},u_{x_1x_2},\cdots)=0$, set $v=v(x_1,\cdots,x_n)$ known and $u=f(v)$ undetermined $(u,v\in C^m(D))$, then substitute $u=f(v)$ and its partial derivatives into $F=0$*

1. In case of working out f, then $u = f(v)$ is the solution of $F = 0$,

2. In case of dividing out f and its partial derivative, also getting $0 = 0$, then $u = f(v)$ is the solution of $F = 0$, and f is an arbitrary unary mth-differentiable function,

3. In case of dividing out f and its partial derivative, also getting $k = 0$, but in fact $k \neq 0$, then $u = f(v)$ is not the solution of $F = 0$, and f is an arbitrary unary mth-differentiable function.

We will research the application of Transformational Method 2 later in this book. Through the comparison, we can find that the traveling wave method and the solitary wave method are the concrete applications of Transformation Method 1, 2.

Now we study another important compound law of multivariate functions. In $\mathbb{R}^n$ space ($n \geq 2$), assuming u, v and g are smooth functions, set

$$u(x_1, \cdots, x_n) = g(x_1, \cdots, x_n) f(v), (f'_v \neq 0), \tag{70}$$

where $v = v(x_1, \cdots x_n)$, f is an unary smooth function, then

$$\begin{aligned} du =& u_{x_1} dx_1 + u_{x_2} dx_2 + \ldots + u_{x_n} dx_n = f dg + g df = f dg + g f_v^{'} dv \\ =& \left(f g_{x_1} + g f_v^{'} v_{x_1} \right) dx_1 + \left(f g_{x_2} + g f_v^{'} v_{x_2} \right) dx_2 + \ldots + \left(f g_{x_n} + g f_v^{'} v_{x_n} \right) dx_n. \end{aligned}$$

So

$$u_{x_i} = f g_{x_i} + g f_v^{'} v_{x_i}. \tag{71}$$

By (71) we could obtain

$$u_{x_i x_i} = f g_{x_i x_i} + 2 g_{x_i} f_v^{'} v_{x_i} + g f_v^{''} v_{x_i}^2 + g f_v^{'} v_{x_i x_i}, \tag{72}$$

$$u_{x_i x_j} = f g_{x_i x_j} + g_{x_i} f_v^{'} v_{x_j} + g_{x_j} f_v^{'} v_{x_i} + g f_v^{''} v_{x_i} v_{x_j} + g f_v^{'} v_{x_i x_j}. \tag{73}$$

Higher order law may be deduced analogously.

According to the above laws we present Transformational Method 3.

Transformational Method 3. *In the domain D, ($D \subset \mathbb{R}^n$), any established mth-order PDE with n space variables $F(x_1, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1 x_2}, \cdots) = 0$, setting $f(v)$,$g(x_1, \cdots x_n)$ and $v(x_1, \cdots, x_n)$ are all undetermined function, $g, v \in C^m(D)$, then substitute $u = gf(v)$ and its partial derivatives into $F = 0$*

1. In case of working out f, g and v, then $u = gf(v)$ is the solution of $F = 0$,

2. In case of dividing out f and its partial derivative, also working out g and v, then $u = gf(v)$ is the solution of $F = 0$, and f is an arbitrary unary mth-differentiable function,

3. In case of getting $k = 0$, but in fact $k \neq 0$, then $u = gf(v)$ is not the solution of $F = 0$.

In Transformational Method 3 $v(x_1, \cdots x_n)$ and $g(x_1, \cdots x_n)$ may be unknown completely or have definite forms with unknown parameters, the solution of f, v and g may not be single. To solve some PDEs we may be required to set $f(v), v(x_1, \cdots x_n)$ pending and $g(x_1, \cdots x_n)$ known, so put forward Transformational Method 4.

Transformational Method 4. *In the domain D, ($D \subset \mathbb{R}^n$), any established mth-order PDE* *with n space variables* $F(x_1, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots) = 0$, *setting* $g(x_1, \cdots x_n)$ *is known and* $f(v), v(x_1, \cdots x_n)$ *are undetermined,* $g, v \in C^m(D)$, *then substitute* $u = gf(v)$ *and its partial derivatives into* $F = 0$

1. In case of working out f *and* v, *then* $u = gf(v)$ *is the solution of* $F = 0$,

2. In case of dividing out f *and its partial derivative, also working out* $v(x_1, \cdots x_n)$, *then* $u = gf(v)$ *is the solution of* $F = 0$, *and* f *is an arbitrary unary* m*th-differentiable function,*

3. In case of getting $k = 0$, *but in fact* $k \neq 0$, *then* $u = gf(v)$ *is not the solution of* $F = 0$.

To solve some PDEs we may be required to set $g(x_1, \cdots x_n)$, $f(v)$ undetermined and $v(x_1, \cdots x_n)$ known and so on. The forms of these laws are similar to Transformational Method 3-4, we will not present here.

1.7. Poisson equation

Consider the following Poisson equation

$$\triangle u = c\,(x, y, z)\,. \tag{74}$$

Supposing

$$c\,(x, y, z) = r\,(v)\,, v\,(x, y, z) = k_1x + k_2y + k_3z + k_4, \tag{75}$$

$c(x, y, z)$ is known in practical problems, so $v(x, y, z) = k_1x + k_2y + k_3z + k_4$ is known too. According to Transformational Method 2, set $u = f(v)$, f is an undetermined unary function, by (7), then

$$\begin{aligned}
&\triangle u = c\,(x, y, z)\\
&\quad \Longrightarrow \left(k_1^2 + k_2^2 + k_3^2\right) f_v'' = r\,(v)\\
&\quad \Longrightarrow u\,(x, y, z) = f\,(v) = \frac{\iint r\,(v)\,dvdv}{k_1^2 + k_2^2 + k_3^2} + C_1v + C_2,
\end{aligned}$$

where C_1 and C_2 are arbitrary constants. So particular solution of Eq. (74) is

$$u\,(x, y, z) = \frac{\iint r\,(v)\,dvdv}{k_1^2 + k_2^2 + k_3^2} + C_1v + C_2. \tag{76}$$

According to the general solution of Laplace equation, the CGS of Eq. (74) may be get

$$\begin{aligned}
u\,(x, y, z) = {} & f_1\left(x\sqrt{-l_1^2 - l_2^2} + l_1y + l_2z + l_3\right) + f_2\left(-x\sqrt{-l_4^2 - l_5^2} + l_4y + l_5z + l_6\right)\\
& + \frac{\iint r\,(v)\,dvdv}{k_1^2 + k_2^2 + k_3^2} + l_7x + l_8y + l_9z + l_{10},
\end{aligned} \tag{77}$$

where f_1 and f_2 are arbitrary unary second differentiable functions, since $l_1 - l_{10}$ are arbitrary constants, (77) can be written as

$$\begin{aligned}
u\,(x, y, z) = {} & \sum_{i=1}^{s}\left(f_{i_1}\left(x\sqrt{-l_{i_1}^2 - l_{i_2}^2} + l_{i_1}y + l_{i_2}z + l_{i_3}\right) + f_{i_2}\left(-x\sqrt{-l_{i_4}^2 - l_{i_5}^2} + l_{i_4}y + l_{i_5}z + l_{i_6}\right)\right)\\
& + \frac{\iint r\,(v)\,dvdv}{k_1^2 + k_2^2 + k_3^2} + l_7x + l_8y + l_9z + l_{10} \quad, (1 \leq s < \infty)\,,
\end{aligned} \tag{78}$$

where f_{i_1} and f_{i_2} are arbitrary unary second differentiable functions, $l_{i_1} - l_{i_6}$ are arbitrary determined constants.

Currently, using numerical methods to analyse Poisson equation is a hot research area [31], under the condition of (75) we set

$$u\left(0,y,z\right)=p\left(k_2y+k_3z+k_4\right)+\sum_{i=1}^{s}\varphi_i\left(l_{i_1}y+l_{i_2}z+l_{i_3}\right), \tag{79}$$

$$u_x\left(0,y,z\right)=k_1p'\left(k_2y+k_3z+k_4\right)+\sum_{i=1}^{s}\psi_i\left(l_{i_1}y+l_{i_2}z+l_{i_3}\right), \tag{80}$$

where φ_i, ψ_i and p are known functions, and

$$p\left(k_1x+k_2y+k_3z+k_4\right)=\frac{\iint r\left(v\right)dvdv}{k_1^2+k_2^2+k_3^2}+C_1v+C_2. \tag{81}$$

In (78) set $l_{i_1}=l_{i_4}$,$l_{i_2}=l_{i_5}$,$l_{i_3}=l_{i_6}$ and $l_7x+l_8y+l_9z+l_{10}=C_1v+C_2$, similar to the calculation of (43) we get

$$f_{i_1}\left(l_{i_1}y+l_{i_2}z+l_{i_3}\right)+f_{i_2}\left(l_{i_1}y+l_{i_2}z+l_{i_3}\right)=\varphi_i\left(l_{i_1}y+l_{i_2}z+l_{i_3}\right),$$

$$\begin{aligned}&f_{i_1}\left(l_{i_1}y+l_{i_2}z+l_{i_3}\right)-f_{i_2}\left(l_{i_1}y+l_{i_2}z+l_{i_3}\right)\\&=\frac{1}{\sqrt{-l_{i_1}^2-l_{i_2}^2}}\int_{l_{i_1}y_0+l_{i_2}z_0+l_{i_3}}^{l_{i_1}y+l_{i_2}z+l_{i_3}}\psi_i(\xi_i)d\xi_i+f_{i_1}(l_{i_1}y_0+l_{i_2}z_0+l_{i_3)}-f_{i_2}(l_{i_1}y_0+l_{i_2}z_0+l_{i_3}).\end{aligned}$$

By the further calculation, the exact solution of Eq. (74) on the conditions of (75), (79) and (80) is

$$\begin{aligned}u\left(x,y,z\right)&=p\left(k_1x+k_2y+k_3z+k_4\right)\\&+\sum_{i=1}^{s}(\frac{1}{2}\varphi_i\left(x\sqrt{-l_{i_1}^2-l_{i_2}^2}+l_{i_1}y+l_{i_2}z+l_{i_3}\right)+\frac{1}{2}\varphi_i\left(-x\sqrt{-l_{i_1}^2-l_{i_2}^2}+l_{i_1}y+l_{i_2}z+l_{i_3}\right)\\&+\frac{1}{2\sqrt{-l_{i_1}^2-l_{i_2}^2}}\int_{-x\sqrt{-l_{i_1}^2-l_{i_2}^2}+l_{i_1}y+l_{i_2}z+l_{i_3}}^{x\sqrt{-l_{i_1}^2-l_{i_2}^2}+l_{i_1}y+l_{i_2}z+l_{i_3}\int}\psi_i\left(\xi_i\right)d\xi_i)\end{aligned} \tag{82}$$

2D wave equation and acoustic wave equation which we studied previously are homogeneous equations, the general solution and the exact solution of the Cauchy problem for their non-homogeneous equation can be obtained similar to the Poisson equation.

1.8. Helmholtz equation

Before research Helmholtz equation, we first consider a PDE as follows

$$a_1u_{xx}+a_2u_{yy}+a_3u_{zz}+a_4u_{xy}+a_5u_{yz}+a_6u_{zx}=a_7, \tag{83}$$

where $a_i=a_i(x,y,z,u),(i=1,2,\cdots 7)$, according to Transformational Method 1, set

$$u\left(x,y,z\right)=f\left(v\right)=f\left(k_1x+k_2y+k_3z+k_4\right), \tag{84}$$

where $k_1 - k_4$ are parameters to be determined, f is an undetermined unary function, then

$$\begin{aligned}
&a_1u_{xx} + a_2u_{yy} + a_3u_{zz} + a_4u_{xy} + a_5u_{yz} + a_6u_{zx} \\
&= k_1^2a_1f_v^{''} + k_2^2a_2f_v^{''} + k_3^2a_3f_v^{''} + k_1k_2a_4f_v^{''} + k_2k_3a_5f_v^{''} + k_1k_3a_6f_v^{''} \\
&= a_7.
\end{aligned}$$

Namely

$$f_v^{''} = \frac{a_7}{k_1^2a_1 + k_2^2a_2 + k_3^2a_3 + k_1k_2a_4 + k_2k_3a_5 + k_1k_3a_6}. \tag{85}$$

If $\frac{a_7}{k_1^2a_1+k_2^2a_2+k_3^2a_3+k_1k_2a_4+k_2k_3a_5+k_1k_3a_6}$ can be converted into $g(v)$ or $h(f)$, it can be further computed,set

$$a_i\left(x,y,z,u\right) = a_i\left(v\right), \left(i = 1,2,\ldots 7\right)\ , \tag{86}$$

So the particular solution of Eq. (83) on the condition of (86) is

$$u\left(x,y,z\right) = \iint \frac{a_7dvdv}{k_1^2a_1 + k_2^2a_2 + k_3^2a_3 + k_1k_2a_4 + k_2k_3a_5 + k_1k_3a_6} + C_1v + C_2, \tag{87}$$

where C_1 and C_2 are arbitrary constant, $k_1 - k_4$ are determinate parameters. For instance

$$u_{xx} + \left(k_1x + k_2y + k_3z + k_4\right)^m u_{yy} + \left(k_1x + k_2y + k_3z + k_4\right)^n u_{zz} = \sin\left(k_1x + k_2y + k_3z + k_4\right). \tag{88}$$

According to (87) its particular solution is

$$u\left(x,y,z\right) = \iint \frac{\sin v dvdv}{k_1^2 + k_2^2v^m + k_3^2v^n} + C_1v + C_2,$$

where $v(x,y,z) = k_1x + k_2y + k_3z + k_4$. Set

$$a_i\left(x,y,z,u\right) = a_i\left(u\right), \left(i = 1,2,\ldots 7\right). \tag{89}$$

From (84)-(85) we have

$$\begin{aligned}
&f_v^{''} = \frac{a_7}{k_1^2a_1 + k_2^2a_2 + k_3^2a_3 + k_1k_2a_4 + k_2k_3a_5 + k_1k_3a_6} \\
&\Longrightarrow v = C_1 \pm \int \left(C_2 + 2\int \frac{a_7df}{k_1^2a_1 + k_2^2a_2 + k_3^2a_3 + k_1k_2a_4 + k_2k_3a_5 + k_1k_3a_6}\right)^{-\frac{1}{2}} df,
\end{aligned}$$

where $k_1 - k_4, C_1$ and C_2 are arbitrary constant. Namely

$$a_1\left(u\right)u_{xx} + a_2\left(u\right)u_{yy} + a_3\left(u\right)u_{zz} + a_4\left(u\right)u_{xy} + a_5\left(u\right)u_{yz} + a_6\left(u\right)u_{zx} = a_7\left(u\right). \tag{90}$$

The particular solution of Eq. (90) is

$$v = C_1 \pm \int \left(C_2 + 2\int \frac{a_7du}{k_1^2a_1 + k_2^2a_2 + k_3^2a_3 + k_1k_2a_4 + k_2k_3a_5 + k_1k_3a_6}\right)^{-\frac{1}{2}} du. \tag{91}$$

The solving method of Eq. (83) can be extended to any similar PDEs with n space variables. Emden-Fowler equation [32, 33], Klein-Gordon equation [34, 35] and sine-Gordon equation [36] are special cases of Eq. (90), which are the hotspots of current research.

Consider the following PDE

$$a_1u_{xx} + a_2u_{yy} + a_3u_{zz} + k^2u = 0 \tag{92}$$

It's a special case of Eq. (90), according to (91)

$$\begin{aligned} v &= C_1 \pm \int \left(C_2 - 2\int \frac{k^2udu}{k_1^2a_1 + k_2^2a_2 + k_3^2a_3} \right)^{-\frac{1}{2}} du \\ &= C_1 \pm \frac{\sqrt{k_1^2a_1 + k_2^2a_2 + k_3^2a_3}\text{arcsin}\left(C_3u\right)}{k} \\ \Rightarrow u &= \frac{1}{C_3}\text{sin}\left(\frac{\pm k\left(v - C_1\right)}{\sqrt{k_1^2a_1 + k_2^2a_2 + k_3^2a_3}} \right) \\ &= \pm C_4\text{sin}\left(\frac{C_5 + k\left(k_1x + k_2y + k_3z\right)}{\sqrt{k_1^2a_1 + k_2^2a_2 + k_3^2a_3}} \right). \end{aligned}$$

Since C_4 is an arbitrary constant, so the particular solution of Eq. (92) can be written as

$$u\left(x, y, z\right) = C_4\text{sin}\left(\frac{C_5 + k\left(k_1x + k_2y + k_3z\right)}{\sqrt{k_1^2a_1 + k_2^2a_2 + k_3^2a_3}} \right), \tag{93}$$

where $k_1 - k_3$,C_4 and C_5 are arbitrary constant. We use Transformational Method 3 to obtain the general solution of Eq. (92), set

$$u\left(x, y, z\right) = g\left(x, y, z\right)h\left(w\right) = g\left(x, y, z\right)h\left(l_1x + l_2y + l_3z + l_4\right), \tag{94}$$

where $w(x, y, z) = l_1x + l_2y + l_3z + l_4$,$l_1 - l_4$ are undetermined parameters, $h(w)$ and $g(x, y, z)$ are undetermined second differentiable functions, according to (72) and (94) we get

$$\begin{aligned} u_{xx} &= hg_{xx} + 2l_1g_xh_w^{'} + l_1^2gh_w^{''}, \\ u_{yy} &= hg_{yy} + 2l_2g_yh_w^{'} + l_2^2gh_w^{''}, \\ u_{zz} &= hg_{zz} + 2l_3g_zh_w^{'} + l_3^2gh_w^{''}. \end{aligned}$$

So

$$\begin{aligned} & a_1u_{xx} + a_2u_{yy} + a_3u_{zz} + k^2u \\ & = a_1hg_{xx} + 2a_1l_1g_xh_w^{'} + a_1l_1^2gh_w^{''} + a_2hg_{yy} + 2a_2l_2g_yh_w^{'} \\ & + a_2l_2^2gh_w^{''} + a_3hg_{zz} + 2a_3l_3g_zh_w^{'} + a_3l_3^2gh_w^{''} + k^2gh. \end{aligned}$$

Namely

$$\left(a_1l_1^2 + a_2l_2^2 + a_3l_3^2\right)gh_w^{''} + 2\left(a_1l_1g_x + a_2l_2g_y + a_3l_3g_z\right)h_w^{'} + \left(a_1g_{xx} + a_2g_{yy} + a_3g_{zz} + k^2g\right)h = 0. \tag{95}$$

Set $h(w)$ an arbitrary unary second differentiable function, according to (95) we obtain

$$a_1l_1^2 + a_2l_2^2 + a_3l_3^2 = 0 \Rightarrow l_1 = \pm\sqrt{\frac{-a_2l_2^2 - a_3l_3^2}{a_1}}, \tag{96}$$

$$a_1l_1g_x + a_2l_2g_y + a_3l_3g_z = 0, \tag{97}$$

$$a_1g_{xx} + a_2g_{yy} + a_3g_{zz} + k^2g = 0. \tag{98}$$

By (93) the particular solution of Eq. (98) is

$$g\left(x,y,z\right)=C_4\sin\left(\frac{C_5+k\left(k_1x+k_2y+k_3z\right)}{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}}\right) \tag{99}$$

Substituting from (99) into (97) we get

$$\begin{aligned}&a_1l_1g_x+a_2l_2g_y+a_3l_3g_z\\&=\frac{a_1l_1C_4kk_1+a_2l_2C_4kk_2+a_3l_3C_4kk_3}{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}}\cos\left(\frac{C_5+k\left(k_1x+k_2y+k_3z\right)}{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}}\right)=0\\&\Rightarrow a_1l_1C_4kk_1+a_2l_2C_4kk_2+a_3l_3C_4kk_3=0.\end{aligned}$$

Namely

$$k_1=\frac{-a_2k_2l_2-a_3k_3l_3}{a_1l_1}. \tag{100}$$

Then

$$\begin{aligned}&u\left(x,y,z\right)=g\left(x,y,z\right)h\left(w\right)\\&=\sin\left(\frac{C_5+k\left(k_1x+k_2y+k_3z\right)}{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}}\right)h\left(l_1x+l_2y+l_3z+l_4\right)\\&=\sin\left(\frac{C_5a_1l_1-k\left(a_2k_2l_2+a_3k_3l_3\right)x+ka_1l_1\left(k_2y+k_3z\right)}{\sqrt{\left(a_2k_2l_2+a_3k_3l_3\right)^2+\left(a_2k_2^2+a_3k_3^2\right)a_1^2l_1^2}}\right)h\left(l_1x+l_2y+l_3z+l_4\right).\end{aligned}$$

So the general solution of Eq. (92) is

$$\begin{aligned}u=&\sin\left(\frac{l_5-k\left(a_2k_2l_2+a_3k_3l_3\right)x+k\sqrt{-a_1a_2l_2^2-a_1a_3l_3^2}\left(k_2y+k_3z\right)}{\sqrt{\left(a_2k_2l_2+a_3k_3l_3\right)^2-a_1\left(a_2k_2^2+a_3k_3^2\right)\left(a_2l_2^2+a_3l_3^2\right)}}\right)\\&h_1\left(\sqrt{\frac{-a_2l_2^2-a_3l_3^2}{a_1}}x+l_2y+l_3z+l_4\right)\\&+\sin\left(\frac{l_{15}-k\left(a_2k_{12}l_{12}+a_3k_{13}l_{13}\right)x-k\sqrt{-a_1a_2l_{12}^2-a_1a_3l_{13}^2}\left(k_{12}y+k_{13}z\right)}{\sqrt{\left(a_2k_{12}l_{12}+a_3k_{13}l_{13}\right)^2-a_1\left(a_2k_{12}^2+a_3k_{13}^2\right)\left(a_2l_{12}^2+a_3l_{13}^2\right)}}\right)\\&h_2\left(-\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2}{a_1}}x+l_{12}y+l_{13}z+l_{14}\right),\end{aligned} \tag{101}$$

where h_1 and h_2 are arbitrary unary second differentiable functions, k_2,k_3,k_{12},k_{13},l_2-l_5 and $l_{12}-l_{15}$ are arbitrary constants.

Consider the following 3D Helmholtz equation

$$u_{xx}+u_{yy}+u_{zz}+k^2u=0. \tag{102}$$

According to (101) we can get the general solution of Eq. (102) is

$$u = \sin\left(\frac{l_5 - k\left(k_2 l_2 + k_3 l_3\right)x + k\sqrt{-l_2^2 - l_3^2}\left(k_2 y + k_3 z\right)}{\sqrt{\left(k_2 l_2 + k_3 l_3\right)^2 - \left(k_2^2 + k_3^2\right)\left(l_2^2 + l_3^2\right)}}\right) h_1\left(\sqrt{-l_2^2 - l_3^2}x + l_2 y + l_3 z + l_4\right)$$
$$+ \sin\left(\frac{l_{15} - k\left(k_{12} l_{12} + k_{13} l_{13}\right)x - k\sqrt{-l_{12}^2 - l_{13}^2}\left(k_{12} y + k_{13} z\right)}{\sqrt{\left(k_{12} l_{12} + k_{13} l_{13}\right)^2 - \left(k_{12}^2 + k_{13}^2\right)\left(l_{12}^2 + l_{13}^2\right)}}\right)$$
$$h_2\left(-\sqrt{-l_{12}^2 - l_{13}^2}x + l_{12} y + l_{13} z + l_{14}\right), \tag{103}$$

Consider the following 2D Helmholtz equation

$$u_{xx} + u_{yy} + k^2 u = 0. \tag{104}$$

By (101) the general solution of Eq. (102) could be got

$$u = \sin\left(\frac{C_6 - k\left(a_2 k_2 l_2 + a_3 k_3 l_3\right)x + k\sqrt{-a_1 a_2 l_2^2 - a_1 a_3 l_3^2}\left(k_2 y + k_3 z\right)}{\sqrt{\left(a_2 k_2 l_2\right)^2 - \left(a_2 k_2^2\right)\left(a_2 l_2^2\right)}}\right)$$
$$h_1\left(\sqrt{\frac{-a_2 l_2^2}{a_1}}x + l_2 y + l_4\right)$$
$$+ \sin\left(\frac{C_8 - k\left(a_2 k_{12} l_{12} + a_3 k_{13} l_{13}\right)x - k\sqrt{-a_1 a_2 l_{12}^2 - a_1 a_3 l_{13}^2}\left(k_{12} y + k_{13} z\right)}{\sqrt{\left(a_2 k_{12} l_{12}\right)^2 - \left(a_2 k_{12}^2\right)\left(a_2 l_{12}^2\right)}}\right)$$
$$h_2\left(-\sqrt{\frac{-a_2 l_{12}^2 - a_3 l_{13}^2}{a_1}}x + l_{12} y + l_{13} z + l_{14}\right).$$

The denominator of the above equation is equal to zero, so we can preliminarily judge that Eq. (104) has no general solution.

For the 1D Helmholtz equation $u_x x + k^2 u = 0$, according to (101) we can get that the denominator is equal to zero, so it can be judged preliminarily that 1D Helmholtz equation does not have any general solution.

Currently analysing the Helmholtz equation is mainly used numerical methods [37-40]. Here we consider the following boundary value problem of Eq. (102)

$$u\left(0, y, z\right) = sin\left(\sqrt{2}k\left(y + 2z\right)\right)\varphi\left(y + z\right), \tag{105}$$

$$u_x\left(0, y, z\right) = \sqrt{-2}sin\left(\sqrt{2}k\left(y + 2z\right)\right)\phi'\left(x + y\right) + 3kicos\left(\sqrt{2}k\left(y + 2z\right)\right)\phi\left(x + y\right), \tag{106}$$

where φ,ϕ are known function, comparing (103) with (105) we obtain

$$k_2 = k_{12} = l_2 = l_3 = l_{12} = l_{13} = 1\ , k_3 = k_{13} = 2, l_4 = l_{14} = C_6 = C_8 = 0.$$

Namely

$$\begin{aligned} u =& sin\left(3kix + \sqrt{2}k\left(y + 2z\right)\right) h_1\left(\sqrt{-2}x + y + z\right) \\ &+ sin\left(3kix - \sqrt{2}k\left(y + 2z\right)\right) h_2\left(-\sqrt{-2}x + y + z\right). \end{aligned} \tag{107}$$

Then

$$\begin{aligned}u(0,y,z) =& sin\left(\sqrt{2}k(y+2z)\right)h_1(y+z) - sin\left(\sqrt{2}k(y+2z)\right)h_2(y+z)\\ =& sin\left(\sqrt{2}k(y+2z)\right)\varphi(y+z) \Rightarrow h_1(y+z) - h_2(y+z) = \varphi(y+z),\end{aligned}$$

$$\begin{aligned}u_x(0,y,z) =& \sqrt{-2}sin\left(\sqrt{2}k(y+2z)\right)\left(h_1'(y+z) + h_2'(y+z)\right)\\ &+ 3kicos\left(\sqrt{2}k(y+2z)\right)(h_1(y+z) + h_2(y+z))\\ &= \sqrt{-2}sin\left(\sqrt{2}k(y+2z)\right)\phi'(x+y) + 3kicos\left(\sqrt{2}k(y+2z)\right)\phi(x+y)\\ &\Rightarrow h_1(y+z) + h_2(y+z) = \phi(x+y).\end{aligned}$$

Namely

$$h_1(y+z) - h_2(y+z) = \varphi(y+z) \tag{108}$$

$$h_1(y+z) + h_2(y+z) = \phi(x+y) \tag{109}$$

Then

$$\begin{aligned}h_1(y+z) =& \frac{1}{2}\left(\phi(y+z) + \varphi(y+z)\right)\\ \Rightarrow h_1\left(\sqrt{-2}x+y+z\right) =& \frac{1}{2}\left(\phi\left(\sqrt{-2}x+y+z\right) + \varphi\left(\sqrt{-2}x+y+z\right)\right),\end{aligned}$$

$$\begin{aligned}h_2(y+z) =& \frac{1}{2}\left(\phi(y+z) - \varphi(y+z)\right)\\ \Rightarrow h_2\left(-\sqrt{-2}x+y+z\right) =& \frac{1}{2}\left(\phi\left(-\sqrt{-2}x+y+z\right) - \varphi\left(-\sqrt{-2}x+y+z\right)\right).\end{aligned}$$

So the exact solution of Eq. (102) on the conditions of (105) and (106) can be get

$$\begin{aligned}u =& \frac{1}{2}sin\left(3kix + \sqrt{2}k(y+2z)\right)\left(\phi\left(\sqrt{-2}x+y+z\right) + \varphi\left(\sqrt{-2}x+y+z\right)\right)\\ &+ \frac{1}{2}sin\left(3kix - \sqrt{2}k(y+2z)\right)\left(\phi\left(\sqrt{-2}x+y+z\right) - \varphi\left(\sqrt{-2}x+y+z\right)\right).\end{aligned} \tag{110}$$

1.9. heat equation and diffusion equation

Consider the following PDE

$$a_0u_t + a_1u_{xx} + a_2u_{yy} + a_3u_{zz} = 0, \tag{111}$$

where a_i are known constants. For solving its particular solution, by Transformational Method 1 we set

$$u(t,x,y,z) = f(v) = f(k_0t + k_1x + k_2y + k_3z + k_4), \tag{112}$$

where $v(t,x,y,z) = k_0t + k_1x + k_2y + k_3z + k_4$,$k_0 - k_4$ are parameters to be determined, f is an undetermined unary second differentiable function. By (7)

$$a_0u_t + a_1u_{xx} + a_2u_{yy} + a_3u_{zz} = a_0k_0f_v' + \left(a_1k_1^2 + a_2k_2^2 + a_3k_3^2\right)f_v'' = 0.$$

Set $w = f_v'$,then

$$\begin{aligned}&a_0k_0f_v' + \left(a_1k_1^2 + a_2k_2^2 + a_3k_3^2\right)f_v'' = 0 \Rightarrow \left(a_1k_1^2 + a_2k_2^2 + a_3k_3^2\right)w_v' = -a_0k_0w\\ &\Rightarrow w = k_7e^{\frac{-a_0k_0v}{a_1k_1^2+a_2k_2^2+a_3k_3^2}} \Rightarrow f(v) = -k_7\frac{a_1k_1^2 + a_2k_2^2 + a_3k_3^2}{a_0k_0}e^{\frac{-a_0k_0v}{a_1k_1^2+a_2k_2^2+a_3k_3^2}} + k_6.\end{aligned}$$

So the particular solution of Eq. (111) is

$$u(t,x,y,z) = k_5 e^{\frac{-a_0k_0(k_0t+k_1x+k_2y+k_3z)}{a_1k_1^2+a_2k_2^2+a_3k_3^2}} + k_6, \tag{113}$$

where $k_0 - k_6$ are arbitrary constants.

In order to obtain the general solution of Eq. (111), according to Transformational Method 3 we set

$$u(t,x,y,z) = gh(w) = g(t,x,y,z)\,h(l_0t + l_1x + l_2y + l_3z + l_4), \tag{114}$$

where $w(t,x,y,z) = l_0t + l_1x + l_2y + l_3z + l_4, l_0 - l_4$ are parameters to be determined, h and g are undetermined second differentiable functions. By (71) and (72) we get

$$u_t = l_0 g h_w^{'} + h g_t,$$

$$\begin{aligned} u_{xx} &= l_1^2 g h_w^{''} + 2l_1 g_x h_w^{'} + h g_{xx}, \\ u_{yy} &= l_2^2 g h_w^{''} + 2l_2 g_y h_w^{'} + h g_{yy}, \\ u_{zz} &= l_3^2 g h_w^{''} + 2l_3 g_z h_w^{'} + h g_{zz}. \end{aligned}$$

Then

$$\begin{aligned} &a_0u_t + a_1u_{xx} + a_2u_{yy} + a_3u_{zz} \\ &= a_0l_0gh_w^{'} + a_0hg_t + a_1l_1^2gh_w^{''} + 2a_1l_1g_xh_w^{'} + a_1hg_{xx} + a_2l_2^2gh_w^{''} + 2a_2l_2g_yh_w^{'} + a_2hg_{yy} \\ &+ a_3l_3^2gh_w^{''} + 2a_3l_3g_zh_w^{'} + a_3hg_{zz}. \end{aligned}$$

Namely

$$\begin{aligned} &\left(a_1l_1^2 + a_2l_2^2 + a_3l_3^2\right) gh_w^{''} + \left(a_0l_0g + 2a_1l_1g_x + 2a_2l_2g_y + 2a_3l_3g_z\right) h_w^{'} \\ &+ \left(a_0g_t + a_1g_{xx} + a_2g_{yy} + a_3g_{zz}\right) h = 0. \end{aligned} \tag{115}$$

Set $h(w)$ an arbitrary unary second differentiable function, according to (115) we get

$$a_1l_1^2 + a_2l_2^2 + a_3l_3^2 = 0 \Rightarrow l_1 = \pm\sqrt{\frac{-a_2l_2^2 - a_3l_3^2}{a_1}}, \tag{116}$$

$$a_0l_0g + 2a_1l_1g_x + 2a_2l_2g_y + 2a_3l_3g_z = 0, \tag{117}$$

$$a_0g_t + a_1g_{xx} + a_2g_{yy} + a_3g_{zz} = 0. \tag{118}$$

By (113) the particular solution of Eq. (118) is

$$g(t,x,y,z) = k_5 e^{\frac{-a_0k_0(k_0t+k_1x+k_2y+k_3z)}{a_1k_1^2+a_2k_2^2+a_3k_3^2}} + k_6, \tag{119}$$

Set $k_6 = 0$, and substituting from (119) into (117), then

$$\begin{aligned} &a_0l_0g + 2a_1l_1g_x + 2a_2l_2g_y + 2a_3l_3g_z \\ &= a_0l_0k_5e^{\frac{-a_0k_0(k_0t+k_1x+k_2y+k_3z)}{a_1k_1^2+a_2k_2^2+a_3k_3^2}} - 2a_0k_0k_5e^{\frac{-a_0k_0(k_0t+k_1x+k_2y+k_3z)}{a_1k_1^2+a_2k_2^2+a_3k_3^2}} \frac{a_1l_1k_1 + a_2l_2k_2 + a_3l_3k_3}{a_1k_1^2 + a_2k_2^2 + a_3k_3^2} = 0 \\ &\Rightarrow l_0 = 2k_0\frac{a_1l_1k_1 + a_2l_2k_2 + a_3l_3k_3}{a_1k_1^2 + a_2k_2^2 + a_3k_3^2} \end{aligned}$$

We have

$$l_0 = 2k_0 \frac{a_1 l_1 k_1 + a_2 l_2 k_2 + a_3 l_3 k_3}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2}. \tag{120}$$

Therefore

$$\begin{aligned} u(x,y,z,t) &= g(x,y,z,t)\,h(w) = k_5 e^{\frac{-a_0 k_0 (k_0 t + k_1 x + k_2 y + k_3 z)}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2}} h(l_0 t + l_1 x + l_2 y + l_3 z + l_4) \\ &= e^{\frac{-a_0 k_0 (k_0 t + k_1 x + k_2 y + k_3 z)}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2}} h\left(\frac{2k_0 (a_1 l_1 k_1 + a_2 l_2 k_2 + a_3 l_3 k_3)\, t}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2} + l_1 x + l_2 y + l_3 z + l_4 \right). \end{aligned}$$

So the general solution of Eq. (111) is

$$\begin{aligned} u &= e^{\frac{-a_0 k_0 (k_0 t + k_1 x + k_2 y + k_3 z)}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2}} \\ & h_1 \left(\frac{2k_0 \left(k_1 \sqrt{-a_1 \left(a_2 l_2^2 + a_3 l_3^2 \right)} + a_2 l_2 k_2 + a_3 l_3 k_3 \right) t}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2} + \sqrt{\frac{-a_2 l_2^2 - a_3 l_3^2}{a_1}} x + l_2 y + l_3 z + l_4 \right) \\ &+ e^{\frac{-a_0 k_{10} (k_{10} t + k_{11} x + k_{12} y + k_{13} z)}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2}} h_2 (\frac{2k_{10} (-k_{11} \sqrt{-a_1 (a_2 l_{12}^2 + a_3 l_{13}^2)} + a_2 l_{12} k_{12} + a_3 l_{13} k_{13}) t}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2} \\ &- \sqrt{\frac{-a_2 l_{12}^2 - a_3 l_{13}^2}{a_1}} x + l_{12} y + l_{13} z + l_{14}) \end{aligned} \tag{121}$$

where h_1 and h_2 are arbitrary unary second differentiable functions, $k_0 - k_3$,$k_{10} - k_{13}$,$l_2 - l_4$ and $l_{12} - l_{14}$ are arbitrary constants.

The form of 3D heat equation and diffusion equation is

$$u_t - a^2 (u_{xx} + u_{yy} + u_{zz}) = 0, \tag{122}$$

According to (24) we can get the general solution of Eq. (122) is

$$\begin{aligned} u =& e^{\frac{k_0 (k_0 t + k_1 x + k_2 y + k_3 z)}{\left(k_1^2 + k_2^2 + k_3^2 \right) a^2}} h_1 \left(\frac{2k_0 \left(\sqrt{-l_2^2 - l_3^2} k_1 + l_2 k_2 + l_3 k_3 \right)}{k_1^2 + k_2^2 + k_3^2} t + \sqrt{-l_2^2 - l_3^2} x + l_2 y + l_3 z + l_4 \right) \\ &+ e^{\frac{k_{10} (k_{10} t + k_{11} x + k_{12} y + k_{13} z)}{\left(k_{11}^2 + k_{12}^2 + k_{13}^2 \right) a^2}} \\ & h_2 \left(\frac{2k_{10} \left(-\sqrt{-l_{12}^2 - l_{13}^2} k_{11} + l_{12} k_{12} + l_{13} k_{13} \right)}{k_{11}^2 + k_{12}^2 + k_{13}^2} t - \sqrt{-l_{12}^2 - l_{13}^2} x + l_{12} y + l_{13} z + l_{14} \right) \end{aligned} \tag{123}$$

The form of 2D heat equation and diffusion equation is

$$u_t - a^2 (u_{xx} + u_{yy}) = 0 \tag{124}$$

By (121) the general solution of Eq. (124) could be got

$$\begin{aligned} u =& e^{\frac{k_0 (k_0 t + k_1 x + k_2 y)}{\left(k_1^2 + k_2^2 \right) a^2}} h_1 \left(\frac{2k_0 (i l_2 k_1 + l_2 k_2)}{k_1^2 + k_2^2} t + i l_2 x + l_2 y + l_4 \right) \\ &+ e^{\frac{k_{10} (k_{10} t + k_{11} x + k_{12} y)}{\left(k_{11}^2 + k_{12}^2 \right) a^2}} h_2 \left(\frac{2k_{10} (-i l_{12} k_{11} + l_{12} k_{12})}{k_{11}^2 + k_{12}^2} t - i l_{12} x + l_{12} y + l_{14} \right) \end{aligned} \tag{125}$$

The form of 1D heat equation and diffusion equation is

$$u_t - a^2 u_{xx} = 0 \tag{126}$$

According to (121) we have

$$u = Ce^{\frac{k_0(k_0 t + k_1 x)}{a^2 k_1^2}} \tag{127}$$

Therefore, it can be preliminarily determined that Eq. (126) has no general solution.

Nonlinear problem [41-44] and numerical methods [45-47] are the research hotspots of the heat equation, here we consider the following initial value problem of Eq. (122)

$$u(x,y,z,0) = e^{\frac{x+y+z}{a^2}} \left(\varphi_1\left(\sqrt{-2}x + y + z\right) + \varphi_2\left(-\sqrt{-2}x + y + z\right)\right) \tag{128}$$

Comparing (123) with (128) we get

$$k_1 = k_2 = k_3 = \frac{k_4}{3}, k_{11} = k_{12} = k_{13} = \frac{k_{14}}{3}, l_2 = l_3 = l_{12} = l_{13} = 1, l_5 = l_{15} = 0$$

So

$$\begin{aligned} &u(x,y,z,t) = e^{\frac{x+y+z+3t}{a^2}} \\ &\left(h_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right) + h_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right)\right) \end{aligned} \tag{129}$$

Then

$$\begin{aligned} &u(x,y,z,0) \\ &= e^{\frac{x+y+z}{a^2}} \left(\varphi_1\left(\sqrt{-2}x + y + z\right) + \varphi_2\left(-\sqrt{-2}x + y + z\right)\right) \\ &= e^{\frac{x+y+z}{a^2}} \left(h_1\left(\sqrt{-2}x + y + z\right) + h_2\left(-\sqrt{-2}x + y + z\right)\right) \\ &\Rightarrow \varphi_1\left(\sqrt{-2}x + y + z\right) + \varphi_2\left(-\sqrt{-2}x + y + z\right) = h_1\left(\sqrt{-2}x + y + z\right) + h_2\left(-\sqrt{-2}x + y + z\right) \\ &\Rightarrow \varphi_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right) = h_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right) \\ &\varphi_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right) = h_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right) \end{aligned}$$

Namely

$$h_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right) = \varphi_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right)$$

$$h_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right) = \varphi_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right)$$

So the exact solution of Eq. (122) on the conditions of (128) can be get

$$\begin{aligned} &u(x,y,z,t) \\ &= e^{\frac{x+y+z+3t}{a^2}} \left(\varphi_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right) + \varphi_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right)\right) \end{aligned} \tag{130}$$

1.10. Schrödinger Equation

Linear [48-50] and nonlinear [51, 52] stationary state Schrödinger equation are the focus of current research, Consider the following linear equation

$$\frac{\hbar^2}{2m} \triangle u - (V(x,y,z) - E)u = 0, \tag{131}$$

where m is the mass of the described particle and $\hbar$ is the reduced Plank constant, by Transformational Method 2, set

$$u\left(x,y,z\right)=f\left(v\right)=f\left(k_1x+k_2y+k_3z+k_4\right), \tag{132}$$

$$V\left(x,y,z\right)-E=a\left(v\right)=a\left(k_1x+k_2y+k_3z+k_4\right), \tag{133}$$

where $v(x,y,z)=k_1x+k_2y+k_3z+k_4, k_1-k_4$ are known parameters, $V(x,y,z)-E=a(v)$ is a known function, f is an undetermined unary second differentiable function, then

$$\begin{aligned}&\frac{\hbar^2}{2m}\triangle u-\left(V\left(x,y,z\right)-E\right)u=\frac{\hbar^2}{2m}\left(k_1^2+k_2^2+k_3^2\right)f_v^{''}-a\left(v\right)f=0\\&\Rightarrow f_v^{''}+b\left(v\right)f=0.\end{aligned}$$

Namely

$$f_v^{''}+b\left(v\right)f=0. \tag{134}$$

where

$$b\left(v\right)=\frac{-2ma\left(v\right)}{\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}=\frac{-2m\left(V\left(x,y,z\right)-E\right)}{\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}. \tag{135}$$

If $b(v)$ is some special function [53], Eq. (134) has a particular solution and its general solution may be obtained by the law of second-order linear ODEs (LODEs), such as

$$b\left(v\right)=-c\left(cv^{2n}+nv^{n-1}\right), \tag{136}$$

$$V\left(x,y,z\right)=a\left(v\right)+E=\frac{c\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}{2m}\left(cv^{2n}+nv^{n-1}\right)+E, \tag{137}$$

where c is an arbitrary constant, the particular solution of Eq. (134) under the condition of (137) is

$$f\left(v\right)=\exp\left(\frac{cv^{n+1}}{n+1}\right)=\exp\left(\frac{c(k_1x+k_2y+k_3z+k_4)^{n+1}}{n+1}\right).$$

So the particular solution of Eq. (131) under the condition of (137) is

$$u\left(x,y,z\right)=\exp\left(\frac{c(k_1x+k_2y+k_3z+k_4)^{n+1}}{n+1}\right). \tag{138}$$

For getting the general solution of Eq. (131) under the condition of (137), according to Transformational Method 3, we set

$$u\left(x,y,z\right)=g\left(x,y,z\right)h\left(w\right)=g\left(x,y,z\right)h\left(l_1x+l_2y+l_3z+l_4\right), \tag{139}$$

where $w(x,y,z)=l_1x+l_2y+l_3z+l_4, l_1-l_4$ are parameters to be determined, $h(w)$ and $g(x,y,z)$ are undetermined second differentiable function, by (129) and (72) we obtain

$$u_{xx}=hg_{xx}+2l_1g_xh_w^{'}+l_1^2gh_w^{''}, \tag{140}$$

$$u_{yy}=hg_{yy}+2l_2g_yh_w^{'}+l_2^2gh_w^{''}, \tag{141}$$

$$u_{zz} = hg_{zz} + 2l_3 g_z h_w^{'} + l_3^2 g h_w^{''}. \tag{142}$$

Then

$$\begin{aligned}
&\frac{\hbar^2}{2m}\triangle u - V((x,y,z) - E)u \\
&= \frac{\hbar^2}{2m}hg_{xx} + \frac{\hbar^2}{m}l_1 g_x h_w^{'} + \frac{\hbar^2}{2m}l_1^2 g h_w^{''} + \frac{\hbar^2}{2m}hg_{yy} + \frac{\hbar^2}{m}l_2 g_y h_w^{'} + \frac{\hbar^2}{2m}l_2^2 g h_w^{''} + \frac{\hbar^2}{2m}hg_{zz} \\
&+ \frac{\hbar^2}{m}l_3 g_z h_w^{'} + \frac{\hbar^2}{2m}l_3^2 g h_w^{''} + (V(x,y,z) - E)gh = 0.
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{\hbar^2}{2m}\left(l_1^2 + l_2^2 + l_3^2\right) g h_w^{''} + \frac{\hbar^2}{m}\left(l_1 g_x + l_2 g_y + l_3 g_z\right) h_w^{'} \\
&+ \left(\frac{\hbar^2}{2m}g_{xx} + \frac{\hbar^2}{2m}g_{yy} + \frac{\hbar^2}{2m}g_{zz} + \left(V\left(x,y,z\right) - E\right) g\right) h = 0.
\end{aligned} \tag{143}$$

Set $h(w)$ an arbitrary second differentiable function, by (143) we get

$$l_1^2 + l_2^2 + l_3^2 = 0 \Longrightarrow l_1 = \pm\sqrt{-l_2^2 - l_3^2}, \tag{144}$$

$$l_1 g_x + l_2 g_y + l_3 g_z = 0, \tag{145}$$

$$\frac{\hbar^2}{2m}g_{xx} + \frac{\hbar^2}{2m}g_{yy} + \frac{\hbar^2}{2m}g_{zz} + \left(V\left(x,y,z\right) - E\right) g = 0. \tag{146}$$

By (138), the particular solution of Eq. (146) on the condition of (137) is

$$g\left(x,y,z\right) = \exp\left(\frac{c(k_1 x + k_2 y + k_3 z + k_4)^{n+1}}{n+1}\right). \tag{147}$$

Substituting from (147) into (145) we get

$$\begin{aligned}
&l_1 c k_1 (k_1 x + k_2 y + k_3 z + k_4)^n \exp\left(\frac{c(k_1 x + k_2 y + k_3 z + k_4)^{n+1}}{n+1}\right) \\
&+ l_2 c k_2 (k_1 x + k_2 y + k_3 z + k_4)^n \exp\left(\frac{c(k_1 x + k_2 y + k_3 z + k_4)^{n+1}}{n+1}\right) \\
&+ l_3 c k_3 (k_1 x + k_2 y + k_3 z + k_4)^n \exp\left(\frac{c(k_1 x + k_2 y + k_3 z + k_4)^{n+1}}{n+1}\right) = 0 \\
&\Longrightarrow l_1 = \frac{-k_2 l_2 - k_3 l_3}{k_1} = \pm\sqrt{-l_2^2 - l_3^2} \\
&\Longrightarrow l_2 = \frac{-k_1 k_2 l_3 \pm \sqrt{-k_1^4 l_3^2 - k_1^2 k_3^2 l_3^2 - k_2^2 k_3^2 l_3^2}}{k_1^2 + k_2^2}.
\end{aligned}$$

Namely

$$l_2 = \frac{-k_1 k_2 l_3 \pm \sqrt{-k_1^4 l_3^2 - k_1^2 k_3^2 l_3^2 - k_2^2 k_3^2 l_3^2}}{k_1^2 + k_2^2}. \tag{148}$$

Then

$$u\left(x,y,z\right) = g\left(x,y,z\right) h\left(w\right) = \exp\left(\frac{c(k_1 x + k_2 y + k_3 z + k_4)^{n+1}}{n+1}\right) h\left(l_1 x + l_2 y + l_3 z + l_4\right).$$

So the general solution of Eq. (131) on the condition of (137) is

$$u = \exp\left(\frac{c(k_1x+k_2y+k_3z+k_4)^{n+1}}{n+1}\right)$$
$$\left(h_1\left(\sqrt{-l_2^2-l_3^2}x+l_2y+l_3z+l_4\right)+h_2\left(-\sqrt{-l_{12}^2-l_{13}^2}x+l_{12}y+l_{13}z+l_{14}\right)\right), \tag{149}$$

where h_1 and h_2 are arbitrary second differentiable unary function, k_1-k_4 and c are determinate parameters, l_3, l_4, l_{13} and l_{14} are arbitrary constants,$l_{12} = \frac{-k_1k_2l_{13}\pm l_{13}\sqrt{-k_1^4-k_1^2k_3^2-k_2^2k_3^2}}{k_1^2+k_2^2}$.

Time dependent Schrödinger equation is always the focus of research [54-59], in addition, the related nonlinear equation [60, 61] and the time fractional Schrödinger equations (TFSEs) [62, 63] are the deeply researched field. Consider the following linear equation

$$i\hbar u_t + \frac{\hbar^2}{2m}\triangle u - V(x,y,z,t)\,u = 0. \tag{150}$$

According to Method 2, set

$$u(x,y,z,t) = f(v) = f(k_1x+k_2y+k_3z+k_4t+k_5), \tag{151}$$

$$V(x,y,z,t) = a(v) = a(k_1x+k_2y+k_3z+k_4t+k_5), \tag{152}$$

where $v = k_1x+k_2y+k_3z+k_4t+k_5, k_1-k_5$ are known parameters, $V(x,y,z,t) = a(v)$ is a known function, f is an undetermined unary second differentiable function, then

$$i\hbar u_t + \frac{\hbar^2}{2m}\triangle u - V(x,y,z,t)\,u = i\hbar k_4 f_v' + \frac{\hbar^2}{2m}\left(k_1^2+k_2^2+k_3^2\right)f_v'' - a(v)\,f = 0.$$

Namely

$$f_v'' + kf_v' + b(v)\,f = 0, \tag{153}$$

where

$$k = \frac{i2mk_4}{\hbar\left(k_1^2+k_2^2+k_3^2\right)}\ ,\ b(v) = \frac{-2ma(v)}{\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}. \tag{154}$$

If $b(v)$ is some special function [53], Eq. (153) has a particular solution and its general solution may be obtained by the law of second-order LODEs, such as

$$b(v) = c\left(-cv^{2n}+kv^n+nv^{n-1}\right).$$

The particular solution of Eq. (153) is

$$f(v) = \exp\left(-\frac{cv^{n+1}}{n+1}\right).$$

Namely

$$V(x,y,z,t) = a(v) = \frac{-\,c\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}{2m}\left(-cv^{2n}+kv^n+nv^{n-1}\right). \tag{155}$$

The particular solution of Eq. (150) on the condition of (155) is

$$u(x,y,z,t) = \exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right). \tag{156}$$

By

$$f_v^{''} + (g(v) + h(v)) f_v^{'} + \left(g(v) h(v) + g_v^{'}\right) f = 0. \tag{157}$$

The particular solution of Eq. (157) is

$$f(v) = \exp\left(-\int g(v)\, dv\right). \tag{158}$$

Set $h(v) = -g(v) + k$, where $g(v)$ is an arbitrary unary first differentiable function, then

$$b(v) = -g^2(v) + kg(v) + g_v^{'}. \tag{159}$$

Namely

$$V(x,y,z,t) = a(v) = \frac{\hbar^2\left(k_1^2 + k_2^2 + k_3^2\right)}{2m}\left(g^2(v) - kg(v) - g_v^{'}\right). \tag{160}$$

The particular solution of Eq. (150) on the condition of (160) is

$$u(x,y,z,t) = f(v) = \exp\left(-\int g(v)\, dv\right). \tag{161}$$

For getting the general solution of Eq. (150) on the condition of (155), according to Transformational Method 3, we set

$$u(x,y,z,t) = g(x,y,z,t)\, h(w) = g(x,y,z,t)\, h(l_1x + l_2y + l_3z + l_4t + l_5), \tag{162}$$

where $w = l_1x + l_2y + l_3z + l_4t + l_5, l_1 - l_5$ are parameters to be determined, h and g are undetermined second differentiable functions, by (71)-(72) and (162)

$$\begin{aligned}
&i\hbar u_t + \frac{\hbar^2}{2m} \triangle u - V(x,y,z,t)\, u \\
&= i\hbar l_4 g h_w^{'} + i\hbar h g_t + \frac{\hbar^2}{2m} l_1^2 g h_w^{''} + \frac{\hbar^2}{m} l_1 g_x h_w^{'} + \frac{\hbar^2}{2m} h g_{xx} + \frac{\hbar^2}{2m} l_2^2 g h_w^{''} + \frac{\hbar^2}{m} l_2 g_y h_w^{'} \\
&+ \frac{\hbar^2}{2m} h g_{yy} + \frac{\hbar^2}{2m} l_3^2 g h_w^{''} + \frac{\hbar^2}{m} l_3 g_z h_w^{'} + \frac{\hbar^2}{2m} h g_{zz} - Vgh \frac{\hbar^2}{2m}\left(l_1^2 + l_2^2 + l_3^2\right) g h_w^{''} \\
&+ \hbar\left(il_4 g + \frac{\hbar}{m} l_1 g_x + \frac{\hbar}{m} l_2 g_y + \frac{\hbar}{m} l_3 g_z\right) h_w^{'} + \left(i\hbar g_t + \frac{\hbar^2}{2m} g_{xx} + \frac{\hbar^2}{2m} g_{yy} + \frac{\hbar^2}{2m} g_{zz} - Vg\right) h = 0.
\end{aligned}$$

Namely

$$\begin{aligned}
&\frac{\hbar^2}{2m}\left(l_1^2 + l_2^2 + l_3^2\right) g h_w^{''} + \hbar\left(il_4 g + \frac{\hbar}{m} l_1 g_x + \frac{\hbar}{m} l_2 g_y + \frac{\hbar}{m} l_3 g_z\right) h_w^{'} \\
&+ \left(i\hbar g_t + \frac{\hbar^2}{2m} g_{xx} + \frac{\hbar^2}{2m} g_{yy} + \frac{\hbar^2}{2m} g_{zz} - Vg\right) h = 0.
\end{aligned} \tag{163}$$

Set $h(w)$ an arbitrary second differentiable function, by (163) we get

$$l_1^2 + l_2^2 + l_3^2 = 0 \Longrightarrow l_1 = \pm\sqrt{-l_2^2 - l_3^2}, \tag{164}$$

$$il_4 g + \frac{\hbar}{m} l_1 g_x + \frac{\hbar}{m} l_2 g_y + \frac{\hbar}{m} l_3 g_z = 0, \tag{165}$$

$$i\hbar g_t + \frac{\hbar^2}{2m} g_{xx} + \frac{\hbar^2}{2m} g_{yy} + \frac{\hbar^2}{2m} g_{zz} - Vg = 0. \tag{166}$$

By (156) the particular solution of Eq. (166) on the condition of (155) is

$$g\left(x,y,z,t\right)=\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right). \tag{167}$$

Substituting from (167) into (165) we get

$$\begin{aligned}&il_4\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right)\\&-\frac{\hbar}{m}l_1ck_1(k_1x+k_2y+k_3z+k_4t+k_5)^n\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right)\\&-\frac{\hbar}{m}l_2ck_2(k_1x+k_2y+k_3z+k_4t+k_5)^n\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right)\\&-\frac{\hbar}{m}l_3ck_3(k_1x+k_2y+k_3z+k_4t+k_5)^n\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right)=0\\&\Longrightarrow l_4=-\frac{\hbar c}{m}\left(l_1k_1+l_2k_2+l_3k_3\right)\left(k_1x+k_2y+k_3z+k_4t+k_5\right)^n.\end{aligned}$$

Namely

$$l_4=-\frac{\hbar c}{m}\left(l_1k_1+l_2k_2+l_3k_3\right)\left(k_1x+k_2y+k_3z+k_4t+k_5\right)^n. \tag{168}$$

Since l_4 is a constant and is not a function of x,y,z and t, if (167) is the particular solution of Eq. (166), by (168) n must equal 0, then

$$V=\frac{-\,c\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}{2m}\left(-cv^{2n}+kv^n+nv^{n-1}\right)=\frac{-\,c\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}{2m}\left(-c+k\right). \tag{169}$$

Since k_1-k_5,k and c are determinate constants, so $V(x,y,z,t)$ is an determinate constants too, namely

$$l_4=-\frac{\hbar c}{m}\left(l_1k_1+l_2k_2+l_3k_3\right). \tag{170}$$

Then

$$u=gh=\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right)h\left(l_1x+l_2y+l_3z+l_4t+l_5\right).$$

So the general solution of Eq. (150) in the condition of (169) is

$$\begin{aligned}&u=e^{-c(k_1x+k_2y+k_3z+k_4t+k_5)}\\&\left(h_1\left(\sqrt{-l_2^2-l_3^2}x+l_2y+l_3z+l_4t+l_5\right)+h_2\left(-\sqrt{-l_{12}^2-l_{13}^2}x+l_{12}y+l_{13}z+l_{14}t+l_{15}\right)\right)\end{aligned} \tag{171}$$

where h_1 and h_2 are arbitrary unary second differentiable functions,$l_{14}=-\frac{\hbar c}{m}\left(l_{11}k_1+l_{12}k_2+l_{13}k_3\right)$, $l_2,l_3,l_5,l_{12},l_{13}$ and l_{15} are arbitrary parameters.

Consider the following initial value problem of Eq. (150) on the condition of (169)

$$u\left(x,y,z,0\right)=e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right)+\varphi_2\left(-\sqrt{-2}x+y+z\right)\right), \tag{172}$$

$$
\begin{aligned}
&u_t(x,y,z,0)\\
&=e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right)+\varphi_2\left(-\sqrt{-2}x+y+z\right)\right)\\
&+\frac{\hbar}{m}e^{x+y+z}\left(\left(2+\sqrt{-2}\right)\varphi_1'\left(\sqrt{-2}x+y+z\right)+\left(2-\sqrt{-2}\right)\varphi_2'\left(-\sqrt{-2}x+y+z\right)\right).
\end{aligned}
\tag{173}
$$

Comparing (171) with (172) we have

$$k_1=k_2=k_3=-\frac{1}{c}\ ,l_2=l_3=l_{12}=l_{13}=1\ ,k_5=l_5=l_{15}=0.$$

By further calculation which is in Appendix C, the exact solutions of the initial value problem is

$$
\begin{aligned}
u=&e^{x+y+z+t}(\varphi_1(\sqrt{-2}x+y+z+\frac{\hbar}{m}(2+\sqrt{-2})t)\\
&+\varphi_2(-\sqrt{-2}x+y+z+\frac{\hbar}{m}(2-\sqrt{-2})t)).
\end{aligned}
\tag{174}
$$

When $V(x,y,z,t)$ is a constant, Eq. (150) is also an important case of the diffusion equation with a source [64], its general solution and the exact solutions of the Cauchy problem are applicable to the diffusion equation.

1.11. Singular general solution of the Helmholtz equation

For the 3D Helmholtz equation

$$\triangle u+k^2u=0. \tag{175}$$

By (93), its particular solution is

$$u(x,y,z)=C_4\sin\left(\frac{C_5+k\left(k_1x+k_2y+k_3z\right)}{\sqrt{k_1^2+k_2^2+k_3^2}}\right), \tag{176}$$

where k_1-k_3,C_4 and C_5 are arbitrary constant. Similar to the solving method of (101), by (176) and Transformational Method 3 the SGS of Eq. (175) is

$$
\begin{aligned}
u=&\sin\left(\frac{C_6-k\left(k_2l_2+k_3l_3\right)x+k\sqrt{-l_2^2-l_3^2}\left(k_2y+k_3z\right)}{\sqrt{\left(k_2l_2+k_3l_3\right)^2-\left(k_2^2+k_3^2\right)\left(l_2^2+l_3^2\right)}}\right)h_1\left(\sqrt{-l_2^2-l_3^2}x+l_2y+l_3z+l_4\right)\\
&+\sin\left(\frac{C_8-k\left(k_{12}l_{12}+k_{13}l_{13}\right)x-k\sqrt{-l_{12}^2-l_{13}^2}\left(k_{12}y+k_{13}z\right)}{\sqrt{\left(k_{12}l_{12}+k_{13}l_{13}\right)^2-\left(k_{12}^2+k_{13}^2\right)\left(l_{12}^2+l_{13}^2\right)}}\right)\\
&h_2\left(-\sqrt{-l_{12}^2-l_{13}^2}x+l_{12}y+l_{13}z+l_{14}\right),
\end{aligned}
\tag{177}
$$

where h_1 and h_2 are arbitrary unary second differentiable functions, $k_2,k_3,k_{12},k_{13},l_2-l_4,l_{12}-l_{14},C_6$ and C_8 are arbitrary constants.

According to [65]

$$u=\left(k_1\cosh\alpha x+k_2\sinh\alpha x\right)\left(k_3\cosh\beta y+k_4\sinh\beta y\right)\left(k_5\cosh\gamma z+k_6\sinh\gamma z\right),\ k^2=-\alpha^2-\beta^2-\gamma^2, \tag{178}$$

where k_1-k_6 are arbitrary constants, (178) is a particular solution of Eq. (175). Other particular solutions of the Helmholtz equation can be referred to [65, 66].

Here we shall use (178) and Transformational Method 3 to obtain a new solution of Eq. (175), and shall analyze the relationship between it and (177).

According to Transformational Method 3, we set

$$u(x,y,z)=g(x,y,z)\,h(w)=g(x,y,z)\,h(l_1x+l_2y+l_3z+l_4)\,, \tag{179}$$

where $h(w)$ and $g(x,y,z)$ are undetermined second differentiable functions,$w(x,y,z)=l_1x+l_2y+l_3z+l_4$,$l_1-l_4$ are undetermined parameters, by (179) we have

$$u_{xx}=hg_{xx}+2l_1g_xh_w^{'}+l_1^2gh_w^{''}, \tag{180}$$

$$u_{yy}=hg_{yy}+2l_2g_yh_w^{'}+l_2^2gh_w^{''}, \tag{181}$$

$$u_{zz}=hg_{zz}+2l_3g_zh_w^{'}+l_3^2gh_w^{''}. \tag{182}$$

So

$$\begin{aligned}&\triangle u+k^2u\\&=hg_{xx}+2l_1g_xh_w^{'}+l_1^2gh_w^{''}+hg_{yy}+2l_2g_yh_w^{'}+l_2^2gh_w^{''}+hg_{zz}+2l_3g_zh_w^{'}+l_3^2gh_w^{''}+k^2gh=0.\end{aligned}$$

Namely

$$\left(l_1^2+l_2^2+l_3^2\right)gh_w^{''}+2\left(l_1g_x+l_2g_y+l_3g_z\right)h_w^{'}+\left(g_{xx}+g_{yy}+g_{zz}+k^2g\right)h=0. \tag{183}$$

Set $h(w)$ an arbitrary unary second differentiable function, according to (183) we obtain

$$l_1^2+l_2^2+l_3^2=0\Longrightarrow l_1=\pm\sqrt{-l_2^2-l_3^2}, \tag{184}$$

$$l_1g_x+l_2g_y+l_3g_z=0, \tag{185}$$

$$g_{xx}+g_{yy}+g_{zz}+k^2g=0. \tag{186}$$

By (178), a particular solution of Eq. (186) is

$$g=(k_1\cosh\alpha x+k_2\sinh\alpha x)\,(k_3\cosh\beta y+k_4\sinh\beta y)\,(k_5\cosh\gamma z+k_6\sinh\gamma z)\,,\ k^2=-\alpha^2-\beta^2-\gamma^2, \tag{187}$$

Substituting (187) into (185)

$$\begin{aligned}&l_1g_x+l_2g_y+l_3g_z\\&=\alpha l_1\,(k_2\cosh\alpha x+k_1\sinh\alpha x)\,(k_3\cosh\beta y+k_4\sinh\beta y)\,(k_5\cosh\gamma z+k_6\sinh\gamma z)\\&+\beta l_2\,(k_1\cosh\alpha x+k_2\sinh\alpha x)\,(k_4\cosh\beta y+k_3\sinh\beta y)\,(k_5\cosh\gamma z+k_6\sinh\gamma z)\\&+\gamma l_3\,(k_1\cosh\alpha x+k_2\sinh\alpha x)\,(k_3\cosh\beta y+k_4\sinh\beta y)\,(k_6\cosh\gamma z+k_5\sinh\gamma z)=0,\end{aligned}$$

we get

$$k_1=k_2,k_3=k_4,k_5=k_6,\ l_1=\frac{-\beta l_2-\gamma l_3}{\alpha}. \tag{188}$$

Since

$$\begin{aligned}&l_1=\pm\sqrt{-l_2^2-l_3^2}=\frac{-\beta l_2-\gamma l_3}{\alpha}\\&\Longrightarrow\left(\sqrt{k^2+\gamma^2}l_2-\frac{\beta\gamma l_3}{\sqrt{k^2+\gamma^2}}\right)^2+\left(k^2+\beta^2-\frac{\beta^2\gamma^2}{k^2+\gamma^2}\right)l_3^2=0\\&\Longrightarrow\sqrt{k^2+\gamma^2}l_2-\frac{\beta\gamma l_3}{\sqrt{k^2+\gamma^2}}=0,k^2+\beta^2-\frac{\beta^2\gamma^2}{k^2+\gamma^2}=0\\&\Longrightarrow l_2=\frac{\beta\gamma l_3}{k^2+\gamma^2},\beta=\pm\sqrt{-k^2-\gamma^2},\alpha=0.\end{aligned}$$

For $\alpha = 0$, we have

$$k_3 = k_4, k_5 = k_6,\ l_3 = \frac{-\beta l_2}{\gamma}, k^2 = -\beta^2 - \gamma^2, \tag{189}$$

and

$$k^2 = -\beta^2 - \gamma^2 \Longrightarrow \beta = \pm\sqrt{-k^2 - \gamma^2} \Longrightarrow l_1 = \pm\sqrt{-l_2^2 - l_3^2} = \pm\sqrt{-l_2^2 - \frac{\beta^2 l_2^2}{\gamma^2}} = \pm\frac{kl_2}{\gamma}$$

Namely

$$\beta = \pm\sqrt{-k^2 - \gamma^2}, \alpha = 0, l_1 = \pm\frac{kl_2}{\gamma} \tag{190}$$

We set $\omega = \sqrt{-k^2 - \gamma^2}$,then

$$\begin{aligned}
&u(x,y,z) = g(x,y,z)\,h(w)\\
&= (k_2\cosh\alpha x + k_1\sinh\alpha x)(k_3\cosh\beta y + k_4\sinh\beta y)(k_5\cosh\gamma z + k_6\sinh\gamma z)\,h(l_1x + l_2y + l_3z + l_4)\\
&= (\cosh\beta y + \sinh\beta y)(\cosh\gamma z + \sinh\gamma z)\,h(l_1x + l_2y + l_3z + l_4) = e^{\pm\omega y + \gamma z}h(l_1x + l_2y + l_3z + l_4).
\end{aligned}$$

So

$$\begin{aligned}
&u(x,y,z)\\
&= e^{\omega y+\gamma z}(h_1(kx + \gamma y + \omega z + C_1) + h_2(-kx + \gamma y + \omega z + C_2) + h_3(kx + \gamma y - \omega z + C_3)\\
&+ h_4(-kx + \gamma y - \omega z + C_4)) + e^{-\omega y+\gamma z}(h_5(kx + \gamma y + \omega z + C_5) + h_6(-kx + \gamma y + \omega z + C_6)\\
&+ h_7(kx + \gamma y - \omega z + C_7) + h_8(-kx + \gamma y - \omega z + C_8))
\end{aligned} \tag{191}$$

where $h_1 - h_8$ are arbitrary unary second differentiable functions, $C_1 - C_8$ are arbitrary constants. Substituting (191) into Eq. (175), we find the necessary to set $h_1 + h_2 = C_9$ and $h_7 + h_8 = C_{10}$, where C_9 and C_{10} are arbitrary constants, so the verified new general solution of Eq. (175) is:

$$\begin{aligned}
u(x,y,z) =& e^{\omega y+\gamma z}(h_3(kx + \gamma y - \omega z + C_3) + h_4(-kx + \gamma y - \omega z + C_4) + C_9)\\
&+ e^{-\omega y+\gamma z}(h_5(kx + \gamma y + \omega z + C_5) + h_6(-kx + \gamma y + \omega z + C_6) + C_{10}).
\end{aligned}$$

In order to facilitate the writing and application, we rewrite the above equation

$$\begin{aligned}
u(x,y,z) =& e^{\omega y+\gamma z}(h_1(kx + \gamma y - \omega z + C_1) + h_2(-kx + \gamma y - \omega z + C_2) + C_5)\\
&+ e^{-\omega y+\gamma z}(h_3(kx + \gamma y + \omega z + C_3) + h_4(-kx + \gamma y + \omega z + C_4) + C_6).
\end{aligned} \tag{192}$$

where $h_1 - h_4$ are arbitrary unary second differentiable functions, $C_1 - C_6$ are arbitrary constants.

Note that the number of arbitrary functions in (192) is four! Namely the number of arbitrary functions in the SGS of 2th-order linear PDE (LPDE) is more than 2!

For the modified Helmholtz equation

$$\triangle u - k^2 u = 0. \tag{193}$$

Similar to the calculation of (101), the SGS of Eq. (193) is

$$\begin{aligned}
u = &\sin\left(\frac{C_1 - ik\left(k_2 l_2 + k_3 l_3\right)x + ik\sqrt{-l_2^2 - l_3^2}\left(k_2 y + k_3 z\right)}{\sqrt{\left(k_2 l_2 + k_3 l_3\right)^2 - \left(k_2^2 + k_3^2\right)\left(l_2^2 + l_3^2\right)}}\right) \\
&h_1\left(\sqrt{-l_2^2 - l_3^2}x + l_2 y + l_3 z + l_4\right) \\
&+\sin\left(\frac{C_2 + ik\left(k_{12} l_{12} + k_{13} l_{13}\right)x - ik\sqrt{-l_{12}^2 - l_{13}^2}\left(k_{12} y + k_{13} z\right)}{\sqrt{\left(k_{12} l_{12} + k_{13} l_{13}\right)^2 - \left(k_{12}^2 + k_{13}^2\right)\left(l_{12}^2 + l_{13}^2\right)}}\right) \\
&h_2\left(\sqrt{-l_{12}^2 - l_{13}^2}x + l_{12} y + l_{13} z + l_{14}\right) \\
&+\sin\left(\frac{C_3 - ik\left(k_{22} l_{22} + k_{23} l_{23}\right)x - ik\sqrt{-l_{22}^2 - l_{23}^2}\left(k_{22} y + k_{23} z\right)}{\sqrt{\left(k_{22} l_{22} + k_{23} l_{23}\right)^2 - \left(k_{22}^2 + k_{23}^2\right)\left(l_{22}^2 + l_{23}^2\right)}}\right) \\
&h_3\left(-\sqrt{-l_{22}^2 - l_{23}^2}x + l_{22} y + l_{23} z + l_{24}\right) \\
&+\sin\left(\frac{C_4 + ik\left(k_{32} l_{32} + k_{33} l_{33}\right)x + ik\sqrt{-l_{32}^2 - l_{33}^2}\left(k_{32} y + k_{33} z\right)}{\sqrt{\left(k_{32} l_{32} + k_{33} l_{33}\right)^2 - \left(k_{32}^2 + k_{33}^2\right)\left(l_{32}^2 + l_{33}^2\right)}}\right) \\
&h_4\left(-\sqrt{-l_{32}^2 - l_{33}^2}x + l_{32} y + l_{33} z + l_{34}\right),
\end{aligned} \tag{193}$$

where $h_1 - h_4$ are arbitrary unary second differentiable functions, $k_2, k_3, k_{12}, k_{13}, k_{22}, k_{23}, k_{32}, k_{33}, l_2 - l_4, l_{12} - l_{14}, l_{22} - l_{24}, l_{32} - l_{34}$ and $C_1 - C_4$ are arbitrary constants. Note that the number of arbitrary functions in (194) is four, the SGS of Eq. (193) which similar to (192) is more complex.

According to the above calculation, we propose a new problem: Comparing (192) with (177), whether they are independent of each other? How to prove?

In the theory of ordinary differential equations (ODEs), it has been proven that the number of the arbitrary constants in the general solution of mth-order ODEs is m [67, 68]. In the theory of PDEs, almost all of the textbooks and professional books directly or indirectly declare that the number of the arbitrary functions in the general solution of mth-order PDEs is m [20, 64, 69, 70], but no related rigorous proof up to now, so the above problem is very important, it relates to how many arbitrary functions in the SGS of mth-order PDEs. By the law of superposition, if (192) and (177) are independent, the number of arbitrary functions in the SGS of Eq. (175) is six.

Because the concise general solution of a PDE may not be the only, we can use symbols G_n^m express these different solution, where n is the number of arbitrary functions and m is the number of discretionary parameters in the corresponding concise general solution.

2. Principles of transformational equations

We presents five new concepts and five new theorems in this chapter, using Theorem 1 if the solution of a PDE is known, the solutions of its various independent variable transformational equations (IVTEs) can be obtained directly. According to Theorem 2 we can use two solvable equations to obtain a new solvable PDE. By Theorem 4 and 5, the general solutions of the vector wave equation in cartesian, cylindrical and spherical coordinate systems have been solved for the first time. We point out that the general solutions or particular solutions of various symmetric vector partial differential equations can be obtained similarly in any orthogonal coordinate system, such as vector Helmholtz equation, vector Laplace equation and the magnetic vector potential equation and so on.

2.1. The principle of independent variable transformational equations

Many MPEs' solutions under non-Cartesian coordinate system have been investigated deeply [71-73]. One PDE in different coordinate systems has different forms, such as the Laplace equation in spherical coordinates and cylindrical coordinates are written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}=0, \tag{195}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}=0. \tag{196}$$

Here we begin to study the relationship law of the PDEs' solutions in different coordinate systems. First, we define independent variable transformational equations (IVTEs).

Definition. In the domain $D,(D\subset\mathbb{R}^n),(n\geq 2)$, set x_i are independent and y_i are independent too, $(i=1,2,\cdots n)$, by $x_i=x_i(y_1,y_2,\cdots y_n)$,$y_i=y_i(x_1,x_2\cdots x_n)$, which are known,$m$th-order PDE $F\left(x_1,\ldots x_n,u,u_{x_1},\ldots u_{x_n},u_{x_1x_2},\ldots\right)=0$ is converted into another mth-order PDE $G\left(y_1,\ldots y_n,u,u_{y_1},\ldots\ u_{y_n},u_{y_1y_2},\ldots\right)=0$, then $G=0$ is called a IVTE of $F=0$.

Since the concrete forms of $x_i=x_i\left(y_1,y_2,\ldots y_n\right),y_i=y_i\left(x_1,x_2\ldots x_n\right)$ are infinite, so every PDE has infinite IVTEs. In the orthogonal coordinate system theory the forms of IVTEs are always obtained by the metric and exterior calculus method, in fact, they could also be get by known $x_i=x_i(y_1,\ldots y_n)$ and $y_i=y_i(x_1,\ldots x_n)$,such as the Laplace equation in cylindrical coordinates, using

$$x=r\cos\theta\ ,y=r\sin\theta\ ,z=z. \tag{197}$$

We obtain

$$r=\sqrt{x^2+y^2}\ ,\theta=\arctan\frac{y}{x}\ ,z=z, \tag{198}$$

and further

$$r_x=\frac{x}{\sqrt{x^2+y^2}}=\cos\theta\ ,r_y=\frac{y}{\sqrt{x^2+y^2}}=\sin\theta\ ,r_z=0$$

$$\theta_x=\frac{-y}{x^2+y^2}=\frac{-\sin\theta}{r}\ ,\theta_y=\frac{x}{x^2+y^2}=\frac{\cos\theta}{r}\ \ ,\theta_z=0\ ,z_x=z_y=0$$

$$u_x=u_r r_x+u_\theta\theta_x+u_z z_x=u_r\cos\theta-u_\theta\frac{\sin\theta}{r}$$

$$\begin{aligned}u_{xx}&=u_{rr}r_x\cos\theta-u_r\sin\theta\theta_x-u_{\theta\theta}\theta_x\frac{\sin\theta}{r}-u_\theta\frac{\cos\theta}{r}\theta_x+u_\theta\frac{\sin\theta}{r^2}r_x\\&=u_{rr}\cos^2\theta+u_r\frac{\sin^2\theta}{r}+u_{\theta\theta}\frac{\sin^2\theta}{r^2}+2u_\theta\frac{\sin\theta\cos\theta}{r^2}\end{aligned}$$

$$u_y=u_r r_y+u_\theta\theta_y+u_z z_y=u_r\sin\theta+u_\theta\frac{\cos\theta}{r}$$

$$\begin{aligned}u_{yy}&=u_{rr}r_y\sin\theta+u_r\cos\theta\theta_y+u_{\theta\theta}\theta_y\frac{\cos\theta}{r}-u_\theta\frac{\sin\theta}{r}\theta_y-u_\theta\frac{\cos\theta}{r^2}r_y\\&=u_{rr}\sin^2\theta+u_r\frac{\cos^2\theta}{r}+u_{\theta\theta}\frac{\cos^2\theta}{r^2}-2u_\theta\frac{\sin\theta\cos\theta}{r^2}.\end{aligned}$$

Then

$$
\begin{aligned}
&u_{xx}+u_{yy}+u_{zz}\\
&=u_{rr}\cos^2\theta+u_r\frac{\sin^2\theta}{r}+u_{\theta\theta}\frac{\sin^2\theta}{r^2}+2u_\theta\frac{\sin\theta\cos\theta}{r^2}+u_{rr}\sin^2\theta+u_r\frac{\cos^2\theta}{r}+u_{\theta\theta}\frac{\cos^2\theta}{r^2}\\
&-2u_\theta\frac{\sin\theta\cos\theta}{r^2}+u_{zz}=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+u_{zz}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}.
\end{aligned}
$$

Namely

$$u_{xx}+u_{yy}+u_{zz}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}.$$

Using the definition of IVTEs we present Theorem 1.

Theorem 1. *In the domain $D,(D\subset\mathbb{R}^n)$,if the solution $u=f(x_1,\cdots x_n)$ of a mth-order PDE $F(x_1,\cdots x_n,u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)=0$ is known, then the solution of its IVTE $G(y_1,\cdots y_n,u,$ $u_{y_1},\cdots u_{y_n},u_{y_1y_2},\cdots)=0$ is $u=f(x_1,\cdots x_n)=g(y_1,\cdots y_n)$.*

Proof. Using $x_i=x_i(y_1,\cdots y_n)$ and $y_i=y_i(x_1,\cdots x_n)$, $G(y_1,\cdots y_n,u,u_{y_1},\cdots u_{y_n},u_{y_1y_2},\cdots)=0$ could be converted into $F(x_1,\cdots x_n,u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)=0$, because the solution $u=f(x_1,\cdots x_n)$ of $F=0$ is known, so $u=f(x_1,\cdots x_n)$ is the solution of $G=0$ too, by $x_i=x_i(y_1,\cdots y_n)$, $u=f(x_1,\cdots x_n)$ may be varied into $u=g(y_1,\cdots y_n)$.

According to Theorem 1, if the solution of a PDE is known, the solutions of its various IVTEs can be obtained directly, as in spherical coordinates

$$x=r\sin\theta\cos\varphi\ ,y=r\sin\theta\sin\varphi\ ,z=r\cos\theta. \tag{199}$$

By the general solution (16) of Laplace equation in the Cartesian coordinate system, its general solution in the spherical coordinate system can be got directly

$$
\begin{aligned}
u\left(r,\theta,\varphi\right)=&f_1\left(r\sin\theta\cos\varphi\sqrt{-k_1^2-k_2^2}+k_1r\sin\theta\sin\varphi+k_2r\cos\theta+k_3\right)\\
&+f_2\left(-r\sin\theta\cos\varphi\sqrt{-k_4^2-k_5^2}+k_4r\sin\theta\sin\varphi+k_5r\cos\theta+k_6\right)\\
&+k_7r\sin\theta\cos\varphi+k_8r\sin\theta\sin\varphi+k_9r\cos\theta+k_{10}.
\end{aligned}\tag{200}
$$

Using (197) the general solution of Laplace equation in the cylindrical coordinate system is

$$
\begin{aligned}
u\left(r,\theta,z\right)=&f_1\left(r\cos\theta\sqrt{-k_1^2-k_2^2}+k_1r\sin\theta+k_2z+k_3\right)\\
&+f_2\left(-r\cos\theta\sqrt{-k_4^2-k_5^2}+k_4r\sin\theta+k_5z+k_6\right)\\
&+k_7r\cos\theta+k_8r\sin\theta+k_9z+k_{10}.
\end{aligned}\tag{201}
$$

All the solutions of PDEs obtained in this paper, we can use them to obtain the solutions in all orthogonal coordinate system by Theorem 1, exact solutions of Cauchy problems can be similarly analysed yet.

2.2. The principle of dependent variable transformational equations

First, we define the dependent variable transformational equations (DVTEs).

Definition. In the domain$D,(D\subset\mathbb{R}^n)$,set $v=h(x_1,x_n,u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)$, so a PDE

$F(x_1,\cdots x_n,v,v_{x_1},\cdots v_{x_n},v_{x_1x_2},\cdots)=0$ may be converted into $G(x_1,\cdots x_n,u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)=0$, then $G=0$ is called a DVTE of $F=0$.

Using the definition of DVTE we present Theorem 2.

Theorem 2. *In the domain* $D,(D\subset\mathbb{R}^n)$ *,if the solution* $v=f(x_1,\cdots x_n)$ *of a PDE* $F(x_1,\cdots x_n,v,v_{x_1},\cdots v_{x_n},v_{x_1x_2},\cdots)=0$ *is known, set* $v=h(x_1,\cdots x_n,u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)$*, then the solution of its DVTE* $G(x_1,\cdots x_n,u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)=0$ *is the solution of* $h(x_1,\cdots x_n,u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)=f(x_1,\cdots x_n)$.

Proof. Set $v=h(x_1,\cdots x_n,u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)$, then $G(y_1,\cdots y_n,u,u_{y_1},\cdots u_{y_n},u_{y_1y_2},\cdots)=0$ could be converted into $F(x_1,\cdots x_n,v,v_{x_1},\cdots v_{x_n},v_{x_1x_2},\cdots)=0$, because the solution $v=f(x_1,\cdots x_n)$ of $F=0$ is known, so if the solution $u=g(x_1,\cdots x_n) of h(x_1,\cdots x_n,u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)=f(x_1,\cdots x_n)$ can be solved, then $u=g(x_1,\cdots x_n)$ is the solution of $G=0$ yet.

Essentially, Theorem 2 uses two solvable equations to obtain a new solvable PDE, the two solvable equations are

$$F\left(x_1,\ldots x_n,v,v_{x_1},\ldots v_{x_n},v_{x_1x_2},\ldots\right)=0, \tag{202}$$

$$h\left(x_1,\ldots x_n,u,u_{x_1},\ldots u_{x_n},u_{x_1x_2},\ldots\right)=f\left(x_1,x_2\ldots x_n\right)=v. \tag{203}$$

The new solvable PDE which is the DVTE is

$$G\left(x_1,\ldots x_n,u,u_{x_1},\ldots u_{x_n},u_{x_1x_2},\ldots\right)=0. \tag{204}$$

Eq. (202) may be a solvable PDE or ODE, Eq. (203) may be a solvable PDE, ODE or function equation, so the concretely using methods of Theorem 2 may be infinite. Actually, Transformational Method 1-4 are specific applications of Theorem 2. Here we use Theorem 2 to get the general solution of two linear PDEs first, in Chapter 4, we'll use it to solve the general solution of two nonlinear PDEs.

Example 2.1

$$u_{xy}+a\left(x,y\right)u_x+b\left(x,y\right)u_y+\left(a_x\left(x,y\right)+b\left(x,y\right)a\left(x,y\right)\right)u=c\left(x,y\right), \tag{205}$$

where a,b,c are any known binary functions, Eq. (205) is a second order linear hyperbolic PDE, due to

$$v_x+b\left(x,y\right)v=c\left(x,y\right), \tag{206}$$

the general solution of Eq. (206) is [74]:

$$v\left(x,y\right)=e^{-\int b(x,y)dx}\left(\varphi\left(y\right)+\int c\left(x,y\right)e^{\int b(x,y)dx}dx\right), \tag{207}$$

where $\varphi(y)$ is an arbitrary unary function, by Theorem 2 we set

$$v\left(x,y\right)=u_y+a\left(x,y\right)u. \tag{208}$$

According to (207) we get

$$\begin{aligned}u\left(x,y\right)&=e^{-\int a(x,y)dy}\left(\phi\left(x\right)+\int v\left(x,y\right)e^{\int a(x,y)dy}dy\right)\\&=e^{-\int a(x,y)dy}\left(\phi\left(x\right)+\int\left(e^{-\int b(x,y)dx}\left(\varphi\left(y\right)+\int c\left(x,y\right)e^{\int b(x,y)dx}dx\right)\right)e^{\int a(x,y)dy}dy\right),\end{aligned}$$

where $\phi(x)$ is an arbitrary unary function, by (208) we could get that Eq. (205) is a DVTE of Eq. (206), according to Theorem 2, the general solution of Eq. (205) is

$$u = e^{-\int a(x,y)dy} \left(\phi(x) + \int \left(e^{-\int b(x,y)dx} \left(\varphi(y) + \int c(x,y) e^{\int b(x,y)dx} dx \right) \right) e^{\int a(x,y)dy} dy \right). \tag{209}$$

Example 2.2

$$a_y u_{xy} + a_x u_{yy} + a_y b u_x + a_x b u_y + (a_y b_x + a_x b_y) u = 0, \tag{210}$$

where a,b are any known binary functions, due to

$$a_y v_x + a_x v_y = 0, \tag{211}$$

the general solution of Eq. (211) is [74]:

$$v = g(a(x,y)), \tag{212}$$

where g is an arbitrary unary first differentiable function, by Theorem 2 we set

$$v(x,y) = u_y + b(x,y) u. \tag{213}$$

According to (207) and (212) we get

$$\begin{aligned} u(x,y) &= e^{-\int b(x,y)dy} \left(\phi(x) + \int v(x,y) e^{\int a(x,y)dy} dy \right) \\ &= e^{-\int b(x,y)dy} \left(\phi(x) + \int g(a(x,y)) e^{\int b(x,y)dy} dy \right), \end{aligned}$$

where $\phi(x)$ is an arbitrary unary function, by (213) we could get that Eq. (210) is a DVTE of Eq. (211), according to Theorem 2, the general solution of Eq. (210) is

$$u(x,y) = e^{-\int b(x,y)dy} \left(\phi(x) + \int g(a(x,y)) e^{\int b(x,y)dy} dy \right). \tag{214}$$

The definition and rule of DVTEs can be extended to ODEs.

Definition. In the domain $D, (D \subset \mathbb{R}^1)$,set $w = h\left(x, y, y', y'', \dots y^{(m)}\right)$,so an nth-order ODE $F\left(x, w, w', w'', \dots w^{(n)}\right)$ may be converted into an n+mth-order ODE $G\left(x, y, y', y'', \dots y^{(m+n)}\right) = 0$ then $G = 0$ is called a DVTE of $F = 0$.

Theorem 3. *In the domain $D, (D \subset \mathbb{R}^1)$,if the solution $w = f(x)$ of an ODE $F(x, w, w', w'', \dots w^{(n)}) = 0$ is known, set $w = h(x, y, y', y'', \dots y^{(m)})$,then the solution of its DVTE $G(x, y, y', y'', \dots y^{(m+n)}) = 0$, is the solution of $h\left(x, y, y', y'', \dots y^{(m)}\right) = f(x)$.*

For the theoretical system' completeness we present Theorem 3, the proof method of Theorem 3 is similar to Theorem 2. In fact, some ODEs have been solved by Theorem 3 [53], so here we will not study and example.

2.3. General Solutions of Vector Equation in Various Orthogonal Coordinate Systems

In $\mathbb{R}^n$ space ($n \geq 2$), the expression of a vector function **u** is

$$u = u_1e_1 + u_2e_2 + \ldots + u_ne_n, \tag{215}$$

where $u_i = u_i(x_1, x_2, \cdots x_n)$, $i = (1, 2, \cdots n)$, Suppose **u** satisfies the PDE

$$F\left(x_1, \ldots x_n, u, \frac{\partial u}{\partial x_1}, \ldots \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots\right) = 0. \tag{216}$$

If Eq. (216) satisfies

$$\begin{aligned} &F\left(x_1, \ldots x_n, u, \frac{\partial u}{\partial x_1}, \ldots \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots\right) \\ &= \sum_{i=1}^{n} F_i\left(x_1, \ldots x_n, u_i, \frac{\partial u_i}{\partial x_1}, \ldots \frac{\partial u_i}{\partial x_n}, \frac{\partial^2 u_i}{\partial x_1 \partial x_2}, \ldots\right) e_i = 0. \end{aligned} \tag{217}$$

That is, only the dependent variable u_i in the ith component equation, and the form of each component $F_i\left(x_1, \ldots x_n, u_i, \frac{\partial u_i}{\partial x_1}, \ldots \frac{\partial u_i}{\partial x_n}, \frac{\partial^2 u_i}{\partial x_1 \partial x_2}, \ldots\right)$ in Eq. (217) are the same, we call the vector PDEs as the **symmetric vector partial differential equations** (SVPDEs).

By Maxwell equations we can get the vector wave equation which represents the electromagnetic wave spreading in the vacuum [75]

$$\triangle u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{218}$$

where c is the speed of light, **u** is the electric-field strength **E** or magnetic induction **B**, in 3-dimensional space

$$u = u_xe_x + u_ye_y + u_ze_z, \tag{219}$$

$$\triangle u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \left(\triangle u_x - \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2}\right)e_x + \left(\triangle u_y - \frac{1}{c^2}\frac{\partial^2 u_y}{\partial t^2}\right)e_y + \left(\triangle u_z - \frac{1}{c^2}\frac{\partial^2 u_z}{\partial t^2}\right)e_z = 0, \tag{220}$$

the form of each component $F_i\left(x, y, z, u_i, \frac{\partial u_i}{\partial x}, \ldots \frac{\partial u_i}{\partial z}, \frac{\partial^2 u_i}{\partial x \partial x}, \ldots\right)$ in Eq. (220) are the same, so Eq. (218) is a SVPDEs. Another example is that time harmonic electromagnetic waves distribution in space satisfies the vector Helmholtz equation

$$\triangle u + k^2 u = 0, \tag{221}$$

where k is a constant related to the medium and the frequency of electromagnetic wave, **u** is the electric-field strength **E** or magnetic induction **B**, for

$$\triangle u + k^2 u = \left(\triangle u_x + k^2 u_x\right)e_x + \left(\triangle u_y + k^2 u_y\right)e_y + \left(\triangle u_z + k^2 u_z\right)e_z = 0. \tag{222}$$

the form of each component F_i in Eq. (222) are the same, so it is a SVPDEs too. A vector PDE may be a SVPDE in a certain orthogonal coordinate system and may not in other orthogonal coordinate system, such as the form of vector Helmholtz equation in the cylindrical coordinate system is [20]

$$\left(\triangle u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + k^2 u_r\right)e_r + \left(\triangle u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + k^2 u_\theta\right)e_\theta + \left(\triangle u_z + k^2 u_z\right)e_z = 0. \tag{223}$$

Obviously Eq. (223) is not a SVPDE.

Any a n-dimensional SVPDE $F = 0$ can be decomposed into n independent equations $F_i = 0, (i = 1, 2, \cdots n)$, we call $F_i = 0$ the **corresponding scalar equation** (CSE) of $F = 0$, due to each F_i in $F = 0$ has the same form, so every SVPDE has a unique CSE. The CSE of Eq. (218) is:

$$\triangle u_i - \frac{1}{c^2}\frac{\partial^2 u_i}{\partial t^2} = 0 \tag{224}$$

The CSE of Eq. (221) is

$$\triangle u_i + k^2 u_i = 0 \tag{225}$$

Here we propose a new theorem based on the above two new definitions.

Theorem 4. *If there are m arbitrary functions in the general solution of the CSE $F_i = 0$, then the number of arbitrary functions in the general solution of the n-dimensional SVPDE $F = 0$ is mn*

Proof. Because a n-dimensional SVPDE $F = 0$ can be decomposed into n independent e-quations $F_i = 0, (i = 1, 2, \cdots n)$, and each $F_i = 0$ has the same form, so the numbers of arbitrary functions in the general solution of every $F_i = 0$ are all m, according to Eq. (215) we can obtain the number of arbitrary functions in the general solution of $F = 0$ is mn.

Now we begin to analyze the solution law of vector PDEs in various orthogonal coordinate system. By exterior differential and the form in Cartesian coordinate system, the form of a vector PDE in various orthogonal coordinate systems can be calculated [20]. In this paper, we propose a new simple method which can not only easily calculate the form of a vector PDE in any orthogonal coordinate systems, and can directly get its solutions in all kinds of orthogonal coordinate systems by the solution of any an orthogonal coordinate system, therefore it has great advantages.

According to the definition of different orthogonal coordinate system and the magnitude of every unit vector is equal to 1, we can write out the mathematical relationship between the unit vectors of different orthogonal coordinate system, such as the relation between the unit vector e_r, e_θ, e_z of cylinder coordinate system and the unit vectors e_x, e_y, e_z of Cartesian coordinate system, for

$$e_r = \cos\theta e_x + \sin\theta e_y, e_\theta = -\sin\theta e_x + \cos\theta e_y, e_z = e_z. \tag{226}$$

By Eq. (226) we can get

$$e_x = \cos\theta e_r - \sin\theta e_\theta, e_y = \sin\theta e_r + \cos\theta e_\theta, e_z = e_z. \tag{227}$$

For the spherical coordinate system
From Figure 1.1 we get

$$e_r = \sin\theta\cos\varphi e_x + \sin\theta\sin\varphi e_y + \cos\theta e_z \tag{228}$$

$$e_\varphi = -\sin\varphi e_x + \cos\varphi e_y \tag{229}$$

According to Figure 1.2

$$e_\theta = \cos\theta\cos\varphi e_x + \cos\theta\sin\varphi e_y - \sin\theta e_z. \tag{230}$$

By further calculation we have

$$e_x = \cos\varphi\sin\theta e_r + \cos\varphi\cos\theta e_\theta - \sin\varphi e_\varphi \tag{231}$$

$$e_y = \sin\varphi\sin\theta e_r + \sin\varphi\cos\theta e_\theta + \cos\varphi e_\varphi \tag{232}$$

$$e_z = \cos\theta e_r - \sin\theta e_\theta \tag{233}$$

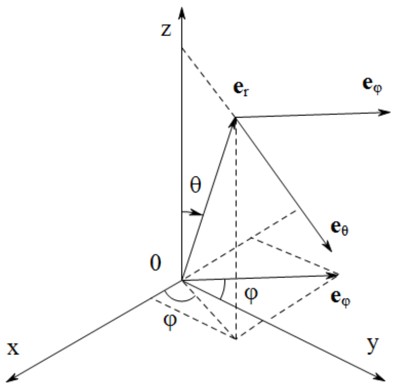

Figure 1.1:

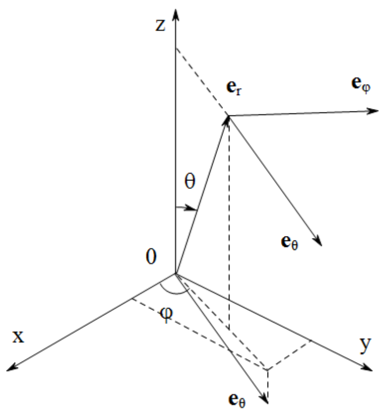

Figure 1.2:

By the functional relationship between the unit vector in different coordinate systems, the function relation of dependent variable components in diversified coordinate systems can be deduced, and the form of a vector PDE in various coordinate systems can be got too, such as the cylindrical coordinate system

$$\begin{aligned} u &= u_x e_x + u_y e_y + u_z e_z = u_x\left(\cos\theta e_r - \sin\theta e_\theta\right) + u_y\left(\sin\theta e_r + \cos\theta e_\theta\right) + u_z e_z \\ &= \left(u_x\cos\theta + u_y\sin\theta\right) e_r + \left(-u_x\sin\theta + u_y\cos\theta\right) e_\theta + u_z e_z = u_r e_r + u_\theta e_\theta + u_z e_z \end{aligned} \tag{234}$$

From Eq. (234) we get

$$u_r = u_x\cos\theta + u_y\sin\theta, u_\theta = -u_x\sin\theta + u_y\cos\theta \tag{235}$$

By Eq. (235) we can obtain

$$u_x = \cos\theta u_r - \sin\theta u_\theta, u_y = \sin\theta u_r + \cos\theta u_\theta \tag{236}$$

So

$$\begin{aligned} \triangle u &= \left(\triangle u_x\right) e_x + \left(\triangle u_y\right) e_y + \left(\triangle u_z\right) e_z \\ &= \left(\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right)\left(\cos\theta u_r - \sin\theta u_\theta\right)\right)\left(\cos\theta e_r - \sin\theta e_\theta\right) \\ &+ \left(\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right)\left(\sin\theta u_r + \cos\theta u_\theta\right)\right)\left(\sin\theta e_r + \cos\theta e_\theta\right) \\ &+ \left(\triangle u_z\right) e_z \\ &= (\cos^2\theta\triangle u_r - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{\cos^2\theta u_r}{r^2} - \sin\theta\cos\theta\triangle u_\theta - \frac{2\cos^2\theta}{r^2}\frac{\partial u_\theta}{\partial\theta} + \frac{\sin\theta\cos\theta u_\theta}{r^2} \\ &+ \sin^2\theta\triangle u_r + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{\sin^2\theta u_r}{r^2} + \sin\theta\cos\theta\triangle u_\theta - \frac{2\sin^2\theta}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{\sin\theta\cos\theta u_\theta}{r^2})e_r \\ &+ (-\sin\theta\cos\theta\triangle u_r + \frac{2\sin^2\theta}{r^2}\frac{\partial u_r}{\partial\theta} + \frac{\sin\theta\cos\theta u_r}{r^2} + \sin^2\theta\triangle u_\theta + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{\sin^2\theta u_\theta}{r^2} \\ &+ \sin\theta\cos\theta\triangle u_r + \frac{2\cos^2\theta}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{\sin\theta\cos\theta u_r}{r^2} + \cos^2\theta\triangle u_\theta - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{\cos^2\theta u_\theta}{r^2})e_\theta \\ &+ \left(\triangle u_z\right) e_z \end{aligned}$$

Namely

$$\triangle u = \left(\triangle u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right) e_r + \left(\triangle u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right) e_\theta + \left(\triangle u_z\right) e_z \tag{237}$$

Similar to the independent variable transformational equations (IVTE) presented in [10], essentially the vector PDE in different coordinate systems are the IVTE each other. Therefore, we propose the concept of an independent variable transformation vector equation (IVTVE).

Definition. In the domain $D,(D\subset\mathbb{R}^n)(n\geq 2)$,set x_i are independent and y_i are independent too,$(i=1,2,\cdots n)$,by $x_i=x_i\left(y_1,y_2,\ldots y_n\right)$,$y_i=y_i\left(x_1,x_2,\ldots x_n\right)$, $e_{x_i}=f_i(y_1,y_2,\ldots y_n,e_{y_1},e_{y_2},\ldots e_{y_n})$ and $e_{y_i}=g_i\left(x_1,x_2\ldots x_n,e_{x_1},e_{x_2},\ldots e_{x_n}\right)$ which are known, mth-order vector PDE $F\left(x_1,\ldots x_n,u,\frac{\partial u}{\partial x_1},\ldots\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_1\partial x_2},\ldots\right)=0$ can be converted into another mth-order vector PDE $G\left(y_1,\ldots y_n,u,\frac{\partial u}{\partial y_1},\ldots\frac{\partial u}{\partial y_n},\frac{\partial^2 u}{\partial y_1\partial y_2},\ldots\right)=0$,then $G=0$ is called a IVTVE of $F=0$.

According to the define of IVTVE we propose theorem 5.

Theorem 5. *In the domain* $D,(D\subset\mathbb{R}^n)$,*if the solution* $u=f\left(x_1,x_2\ldots x_n,e_{x_1},e_{x_2},\ldots e_{x_n}\right)$ *of a* m*th-order vector PDE* $F\left(x_1,\ldots x_n,u,\frac{\partial u}{\partial x_1},\ldots\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_1\partial x_2},\ldots\right)=0$ *is known, then the solution of its IVTVE* $G\left(y_1,\ldots y_n,u,\frac{\partial u}{\partial y_1},\ldots\frac{\partial u}{\partial y_n},\frac{\partial^2 u}{\partial y_1\partial y_2},\ldots\right)=0$, *is* $u=f\left(x_1,x_2\ldots x_n,e_{x_1},e_{x_2},\ldots e_{x_n}\right)=g\left(y_1,y_2\ldots y_n,e_{y_1},e_{y_2},\ldots e_{y_n}\right)$.

Proof. Using $x_i=x_i\left(y_1,y_2,\ldots y_n\right)$,$y_i=y_i\left(x_1,x_2,\ldots x_n\right)$, $e_{x_i}=f_i\left(y_1,y_2,\ldots y_n,e_{y_1},e_{y_2},\ldots e_{y_n}\right)$ and $e_{y_i}=g_i\left(x_1,x_2\ldots x_n,e_{x_1},e_{x_2},\ldots e_{x_n}\right)$,$G\left(y_1,\ldots y_n,u,\frac{\partial u}{\partial y_1},\ldots\frac{\partial u}{\partial y_n},\frac{\partial^2 u}{\partial y_1\partial y_2},\ldots\right)=0$, could be converted into $F\left(x_1,\ldots x_n,u,\frac{\partial u}{\partial x_1},\ldots\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_1\partial x_2},\ldots\right)=0$ because the solution $u=f\left(x_1,x_2\ldots x_n,e_{x_1},e_{x_2},\ldots e_{x_n}\right)$ of $F=0$ is known, so $u=f\left(x_1,x_2\ldots x_n,e_{x_1},e_{x_2},\ldots e_{x_n}\right)$ is the solution of $G=0$ too, by $x_i=x_i\left(y_1,y_2,\ldots y_n\right)$, and $e_{x_i}=f_i\left(y_1,y_2,\ldots y_n,e_{y_1},e_{y_2},\ldots e_{y_n}\right)$, $u=f\left(x_1,x_2\ldots x_n,e_{x_1},e_{x_2},\ldots e_{x_n}\right)$ may be varied into $u=g\left(y_1,y_2\ldots y_n,e_{y_1},e_{y_2},\ldots e_{y_n}\right)$.

Using theorem 5, we can solve the general solutions of vector wave equation and Helmholtz equation in Cartesian coordinate system, the CSE Eq. (224) of Eq. (218) is the acoustic equation, we had solved its general solution in Section 1.4

$$\begin{aligned}u_i=&f_{1i}\left(k_{1i}x+k_{2i}y+k_{3i}z+ct\sqrt{k_{1i}^2+k_{2i}^2+k_{3i}^2}+k_{4i}\right)\\&+f_{2i}\left(k_{5i}x+k_{6i}y+k_{7i}z-ct\sqrt{k_{5i}^2+k_{6i}^2+k_{7i}^2}+k_{8i}\right)+k_{9i}x+k_{10i}y+k_{11i}z+k_{12i}t+k_{13i}\end{aligned}\tag{238}$$

where f_{1i} andf_{2i} are arbitrary unary second differentiable functions, $k_{1i}-k_{13i}$ are arbitrary parameters, $(i=x,y,z)$. Using Eq. (219) and (238), we can obtain the general solution of Eq. (218) with six arbitrary second differentiable functions

$$\begin{aligned}u=\sum_{i=x,y,z}&(f_{1i}\left(k_{1i}x+k_{2i}y+k_{3i}z+ct\sqrt{k_{1i}^2+k_{2i}^2+k_{3i}^2}+k_{4i}\right)\\&+f_{2i}\left(k_{5i}x+k_{6i}y+k_{7i}z-ct\sqrt{k_{5i}^2+k_{6i}^2+k_{7i}^2}+k_{8i}\right)\\&+k_{9i}x+k_{10i}y+k_{11i}z+k_{12i}t+k_{13i})e_i\end{aligned}\tag{239}$$

It is necessary to pay an attention to that the wave velocity u_i are all the speed of light velocity c.

The general solution of the Helmholtz equation Eq. (225) is

$$\begin{aligned} u_i =&\sin\left(\frac{k_{3i}-k\left(k_{1i}l_{1i}+k_{2i}l_{2i}\right)x+k\sqrt{-l_{1i}^2-l_{2i}^2}\left(k_{1i}y+k_{2i}z\right)}{\sqrt{\left(k_{1i}l_{1i}+k_{2i}l_{2i}\right)^2-\left(k_{1i}^2+k_{2i}^2\right)\left(l_{1i}^2+l_{2i}^2\right)}}\right) \\ &h_{1i}\left(\sqrt{-l_{1i}^2-l_{2i}^2}x+l_{1i}y+l_{2i}z+l_{3i}\right) \\ &+\sin\left(\frac{k_{6i}-k\left(k_{4i}l_{4i}+k_{5i}l_{5i}\right)x-k\sqrt{-l_{4i}^2-l_{5i}^2}\left(k_{4i}y+k_{5i}z\right)}{\sqrt{\left(k_{4i}l_{4i}+k_{5i}l_{5i}\right)^2-\left(k_{4i}^2+k_{5i}^2\right)\left(l_{4i}^2+l_{5i}^2\right)}}\right) \\ &h_{2i}\left(-\sqrt{-l_{4i}^2-l_{5i}^2}x+l_{4i}y+l_{5i}z+l_{6i}\right), \end{aligned} \tag{240}$$

where h_{1i} and h_{2i} are arbitrary unary second differentiable functions, $k_{1i}-k_{6i}$and $l_{1i}-l_{6i}$are arbitrary constants. Using Eq. (219) and (240), we can obtain the general solution of the vector Helmholtz equation Eq. (221) with six arbitrary second differentiable functions

$$\begin{aligned} u=\sum_{i=x,y,z}&(\sin\left(\frac{k_{3i}-k\left(k_{1i}l_{1i}+k_{2i}l_{2i}\right)x+k\sqrt{-l_{1i}^2-l_{2i}^2}\left(k_{1i}y+k_{2i}z\right)}{\sqrt{\left(k_{1i}l_{1i}+k_{2i}l_{2i}\right)^2-\left(k_{1i}^2+k_{2i}^2\right)\left(l_{1i}^2+l_{2i}^2\right)}}\right) \\ &h_{1i}\left(\sqrt{-l_{1i}^2-l_{2i}^2}x+l_{1i}y+l_{2i}z+l_{3i}\right) \\ &+\sin\left(\frac{k_{6i}-k\left(k_{4i}l_{4i}+k_{5i}l_{5i}\right)x-k\sqrt{-l_{4i}^2-l_{5i}^2}\left(k_{4i}y+k_{5i}z\right)}{\sqrt{\left(k_{4i}l_{4i}+k_{5i}l_{5i}\right)^2-\left(k_{4i}^2+k_{5i}^2\right)\left(l_{4i}^2+l_{5i}^2\right)}}\right) \\ &h_{2i}\left(-\sqrt{-l_{4i}^2-l_{5i}^2}x+l_{4i}y+l_{5i}z+l_{6i}\right))e_i \end{aligned} \tag{241}$$

The solutions of PDEs in different orthogonal coordinate systems have been the focus of research [71-73]. According to Theorem 5, if a solution of a vector PDE is known, the solutions of its various IVTVEs can be obtained directly, as in the cylindrical coordinate system

$$x=r\cos\theta\ ,y=r\sin\theta\ ,z=z. \tag{242}$$

Using Eq. (227), (239), (242) and Theorem 5, the general solution of vector wave equation in cylindrical coordinate system can be obtained directly

$$\begin{aligned} u&=\left(f_{1x}+f_{2x}+g_x\right)\left(\cos\theta e_r-\sin\theta e_\theta\right)+\left(f_{1y}+f_{2y}+g_y\right)\left(\sin\theta e_r+\cos\theta e_\theta\right)+\left(f_{1z}+f_{2z}+g_z\right)e_z \\ &=\left(\cos\theta f_{1x}+\cos\theta f_{2x}+\cos\theta g_x+\sin\theta f_{1y}+\sin\theta f_{2y}+\sin\theta g_y\right)e_r \\ &+\left(-\sin\theta f_{1x}-\sin\theta f_{2x}-\sin\theta g_x+\cos\theta f_{1y}+\cos\theta f_{2y}+\cos\theta g_y\right)e_\theta+\left(f_{1z}+f_{2z}+g_z\right)e_z \end{aligned}$$

Namely

$$\begin{aligned} u=&\left(\cos\theta\left(f_{1x}+f_{2x}+g_x\right)+\sin\theta\left(f_{1y}+f_{2y}+g_y\right)\right)e_r \\ &+\left(-\sin\theta\left(f_{1x}+f_{2x}+g_x\right)+\cos\theta\left(f_{1y}+f_{2y}+g_y\right)\right)e_\theta+\left(f_{1z}+f_{2z}+g_z\right)e_z, \end{aligned} \tag{243}$$

where

$$f_{1i}=f_{1i}\left(k_{1i}r\cos\theta+k_{2i}r\sin\theta+k_{3i}z+ct\sqrt{k_{1i}^2+k_{2i}^2+k_{3i}^2}+k_{4i}\right) \tag{244}$$

$$f_{2i}=f_{2i}\left(k_{5i}r\cos\theta+k_{6i}r\sin\theta+k_{7i}z-ct\sqrt{k_{5i}^2+k_{6i}^2+k_{7i}^2}+k_{8i}\right) \tag{245}$$

$$g_i = k_{9i} r\cos\theta + k_{10i} r\sin\theta + k_{11i} z + k_{12i} t + k_{13i} \tag{246}$$

For the spherical coordinate system

$$x = r\sin\theta\cos\varphi \ , y = r\sin\theta\sin\varphi \ , z = r\cos\theta. \tag{247}$$

By Eq. (227), (231-233), (247) and Theorem 5, the general solution of vector wave equation in spherical coordinate system can be obtained straightway

$$\begin{aligned} u = & \left(f_{1x} + f_{2x} + g_x\right)\left(\cos\varphi\sin\theta e_r + \cos\varphi\cos\theta e_\theta - \sin\varphi e_\varphi\right) \\ & + \left(f_{1y} + f_{2y} + g_y\right)\left(\sin\varphi\sin\theta e_r + \sin\varphi\cos\theta e_\theta + \cos\varphi e_\varphi\right) \\ & + \left(f_{1z} + f_{2z} + g_z\right)\left(\cos\theta e_r - \sin\theta e_\theta\right) \end{aligned}$$

Namely

$$\begin{aligned} u = & \left(f_{1x} + f_{2x} + g_x\right)\left(\cos\varphi\sin\theta e_r + \cos\varphi\cos\theta e_\theta - \sin\varphi e_\varphi\right) \\ & + \left(f_{1y} + f_{2y} + g_y\right)\left(\sin\varphi\sin\theta e_r + \sin\varphi\cos\theta e_\theta + \cos\varphi e_\varphi\right) \\ & + \left(f_{1z} + f_{2z} + g_z\right)\left(\cos\theta e_r - \sin\theta e_\theta\right) \end{aligned}$$

Namely

$$\begin{aligned} u = & \left(\cos\varphi\sin\theta\left(f_{1x} + f_{2x} + g_x\right) + \sin\varphi\sin\theta\left(f_{1y} + f_{2y} + g_y\right) + \cos\theta\left(f_{1z} + f_{2z} + g_z\right)\right) e_r \\ & + \cos\varphi\cos\theta\left(f_{1x} + f_{2x} + g_x\right) + \sin\varphi\cos\theta\left(f_{1y} + f_{2y} + g_y\right) - \sin\theta\left(f_{1z} + f_{2z} + g_z\right) e_\theta \\ & + \left(-\sin\varphi\left(f_{1x} + f_{2x} + g_x\right) + \cos\varphi\left(f_{1y} + f_{2y} + g_y\right)\right) e_\varphi, \end{aligned} \tag{243}$$

where

$$f_{1i} = f_{1i}\left(k_{1i} r\sin\theta\cos\varphi + k_{2i} r\sin\theta\sin\varphi + k_{3i} r\cos\theta + ct\sqrt{k_{1i}^2 + k_{2i}^2 + k_{3i}^2} + k_{4i}\right), \tag{249}$$

$$f_{2i} = f_{2i}\left(k_{5i} r\sin\theta\cos\varphi + k_{6i} r\sin\theta\sin\varphi + k_{7i} r\cos\theta + ct\sqrt{k_{5i}^2 + k_{6i}^2 + k_{7i}^2} + k_{8i}\right), \tag{250}$$

$$g_i = k_{9i} r\sin\theta\cos\varphi + k_{10i} r\sin\theta\sin\varphi + k_{11i} r\cos\theta + k_{12i} t + k_{13i}, \left(i = x, y, z\right). \tag{251}$$

The general solutions of the vector Laplace equation and Helmholtz equation and so on in cylindrical coordinate system and spherical coordinate system can be obtained similarly.

3. General solutions laws of linear partial differential equations

In recent years, many numerical methods have been developed to solve LPDEs, such as Finite integration method [76, 77], Bernoulli matrix method [78], Chebyshev matrix method [79] and so on, the existence [80], uniqueness [81, 82] and stability [83] of the solution are also the focus of research. Using the principle of transformational equations, if the solution of a LPDE is known, the solutions of its infinite IVTEs and DVTEs can be obtained, which may be LPDEs or NLPDEs. In this chapter, we will research new general solutions laws of LPDEs.

3.1. General solutions laws of linear partial differential equations with variable coefficients

In this section, if there is no special interpretation,$a_i = a_i\left(x_1, \ldots x_n\right)$, $a_{j_i} = a_{j_i}\left(x_1, \ldots x_n\right)$, $a_{i_1 i_2 \ldots i_n} = a_{i_1 i_2 \ldots i_n}\left(x_1, \ldots x_n\right)$, $b_i = b_i\left(x_1, \ldots x_n\right)$, $b_{j_i} = b_{j_i}\left(x_1, \ldots x_n\right)$,$l_i$ and l_{j_i} are arbitrary constants,f and f_i are arbitrary unary smooth functions $(i, j = 1, 2, \cdots)$.

Proposition 1. *If $v(x_1, \cdots x_n)$ is the particular solution of $a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_n u_{x_n} = 0$,then* *its general solution is* $u = f(v(x_1, \cdots x_n))$.

Prove. By Transformational Method 1 set $u(x_1, \ldots x_n) = f(v) = f(v(x_1, \ldots x_n))$ then

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots a_n u_{x_n} = a_1 v_{x_1} f' + a_2 v_{x_2} f' + \ldots a_n v_{x_n} f' = 0.$$

Namely $a_1 v_{x_1} + a_2 v_{x_2} + \ldots a_n v_{x_n} = 0$, if its particular solution is known, the general solution of $a_1 u_{x_1} + a_2 u_{x_2} + \ldots a_n u_{x_n} = 0$ is $u(x_1, \cdots x_n) = f(v(x_1, \cdots x_n))$.

Proposition 2. *If* $a_1 = \frac{-a_2 a_{j_{x_2}} - a_3 a_{j_{x_3}} - \ldots - a_n a_{j_{x_n}}}{a_{j_{x_1}}}$ *then the general solution of* $a_1 u_{x_1} + a_2 u_{x_2} + \ldots a_n u_{x_n} = 0$, *is* $u = f(a_j(x_1, \cdots x_n))$.

Prove. According to Transformational Method 2, set $u(x_1, \cdots x_n) = f(a_j(x_1, \cdots x_n))$,$(j = 2, 3, \cdots n)$, then

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_n u_{x_n} = a_1 a_{j_{x_1}} f' + a_2 a_{j_{x_2}} f' + \ldots + a_n a_{j_{x_n}} f' = 0.$$

Namely

$$a_1 = \frac{-a_2 a_{j_{x_2}} - a_3 a_{j_{x_3}} - \ldots - a_n a_{j_{x_n}}}{a_{j_{x_1}}}.$$

So the general solution of
$a_1 u_{x_1} + a_2 u_{x_2} + \ldots a_n u_{x_n} = 0$ under the condition of $a_1 = \frac{-a_2 a_{j_{x_2}} - a_3 a_{j_{x_3}} - \ldots - a_n a_{j_{x_n}}}{a_{j_{x_1}}}$. is $u = f(a_j(x_1, \cdots x_n))$.

Proposition 3.

$$a_1 v_{x_1} + a_2 v_{x_2} + \ldots + a_n v_{x_n} = 0, \tag{252}$$

$$a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_n g_{x_n} + a_{n+1} g = 0. \tag{253}$$

If the particular solutions of Eq. (252) and Eq. (253) are known, the general solution of $a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_n u_{x_n} + a_{n+1} u = 0$ *is* $u = g(x_1, \cdots x_n) f(v(x_1, \cdots x_n))$.

Prove. According to Transformational Method 3, set $u(x_1, \cdots x_n) = g(x_1, \cdots x_n) f(v(x_1, \cdots x_n))$, the

$$\begin{aligned} & a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_n u_{x_n} + a_{n+1} u \\ & = (a_1 v_{x_1} + a_2 v_{x_2} + \ldots + a_n v_{x_n}) g f_v^{'} + (a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_n g_{x_n} + a_{n+1} g) f = 0. \end{aligned}$$

Setting f an arbitrary unary first differentiable function, we obtain

$$a_1 v_{x_1} + a_2 v_{x_2} + \ldots + a_n v_{x_n} = 0, \tag{252}$$

$$a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_n g_{x_n} + a_{n+1} g = 0. \tag{253}$$

So if the particular solutions of Eq. (252) and Eq. (253) are known, the general solution of $a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_n u_{x_n} + a_{n+1} u = 0$ is

$$u(x_1, \cdots x_n) = g(x_1, \cdots x_n) f(v(x_1, \cdots x_n)). \tag{254}$$

Proposition 4. *If the general solution* $u = f(v(x_1, \cdots x_n))$ *of* $a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_n u_{x_n} = 0$ *is known, then the general solution of* $a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_n u_{x_n} + a_{n+1} = 0$ *is* $u = f(v(x_1, \cdots x_n)) + g(x_1, \cdots x_n)$,*where* $u = g(x_1, \cdots x_n)$ *is the particular solution of* $a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_n u_{x_n} + a_{n+1} = 0$.

Proposition 4 is obvious, and almost needs no proof.

Proposition 5. *If the general solution $u = f(v(x, y, z))$ of $(b_1D_x + b_2D_y + b_3D_z)u = 0$ is known, where $D_{x_j} \equiv \frac{\partial}{\partial x_j}$, $(x_j = x, y, z)$ then the general solution of $(b_1D_x + b_2D_y + b_3D_z)^2u = 0$ is $u = f_1 (v (x, y, z)) + (l_1x + l_2y + l_3z + l_4) f_2 (v (x, y, z))$.*

Prove. Because the general solution $u = f(v(x, y, z))$ of $(b_1D_x + b_2D_y + b_3D_z)u = 0$ is known, apparently $u = f(v(x, y, z))$ is also the solution of $(b_1D_x + b_2D_y + b_3D_z)^2u = 0$, setting $u = g(x, y, z)f(v(x, y, z))$ is the solution of is the solution of $(b_1D_x + b_2D_y + b_3D_z)^2u = 0$ too, then

$$
\begin{aligned}
&(b_1D_x + b_2D_y + b_3D_z)^2u\\
=&\left(b_1^2D_x^2 + b_2^2D_y^2 + b_3^2D_z^2 + 2b_1b_2D_xD_y + 2b_1b_3D_xD_z + 2b_2b_3D_yD_z\right) gf\\
=&b_1^2 (gf_{xx} + 2f_xg_x + g_{xx}f) + b_2^2 (gf_{yy} + 2f_yg_y + g_{yy}f) + b_3^2 (gf_{zz} + 2f_zg_z + g_{zz}f)\\
&+ 2b_1b_2 (g_xf_y + gf_{xy} + g_{xy}f + g_yf_x) + 2b_1b_3 (g_xf_z + gf_{xz} + g_{xz}f + g_zf_x)\\
&+ 2b_2b_3 (g_yf_z + gf_{yz} + g_{yz}f + g_zf_y)\\
=&\left(b_1^2f_{xx} + b_2^2f_{yy} + b_3^2f_{zz} + 2b_1b_2f_{xy} + 2b_1b_3f_{xz} + 2b_2b_3f_{yz}\right) g\\
&+ \left(b_1^2g_{xx} + b_2^2g_{yy} + b_3^2g_{zz} + 2b_1b_2g_{xy} + 2b_1b_3g_{xz} + 2b_2b_3g_{yz}\right) f\\
&+ \left(2b_1^2g_x + 2b_1b_2g_y + 2b_1b_3g_z\right) f_x + \left(2b_2^2g_y + 2b_1b_2g_x + 2b_2b_3g_z\right) f_y\\
&+ \left(2b_3^2g_z + 2b_1b_3g_x + 2b_2b_3g_y\right) f_z\\
=&\left(b_1^2g_{xx} + 2b_1b_2g_{xy} + b_2^2g_{yy} + 2b_1b_3g_{xz} + 2b_2b_3g_{yz} + b_3^2g_{zz}\right) f\\
&+ 2b_1g_x (b_1f_x + b_2f_y + b_3f_z) + 2b_2g_y (b_1f_x + b_2f_y + b_3f_z) + 2b_3g_z (b_1f_x + b_2f_y + b_3f_z)\\
=&\left(b_1^2g_{xx} + 2b_1b_2g_{xy} + b_2^2g_{yy} + 2b_1b_3g_{xz} + 2b_2b_3g_{yz} + b_3^2g_{zz}\right) f = 0\\
&\Longrightarrow b_1^2g_{xx} + 2b_1b_2g_{xy} + b_2^2g_{yy} + 2b_1b_3g_{xz} + 2b_2b_3g_{yz} + b_3^2g_{zz} = 0.
\end{aligned}
$$

Namely

$$b_1^2g_{xx} + 2b_1b_2g_{xy} + b_2^2g_{yy} + 2b_1b_3g_{xz} + 2b_2b_3g_{yz} + b_3^2g_{zz} = 0, \tag{255}$$

$g(x, y, z)$ has to be a particular solution of Eq. (255), due to

$$g (x, y, z) = l_1x + l_2y + l_3z + l_4, \tag{256}$$

(256) is a particular solution of Eq. (255), so the general solution of $(b_1D_x + b_2D_y + b_3D_z)^2u = 0$ is $u = f_1 (v (x, y, z)) + (l_1x + l_2y + l_3z + l_4) f_2 (v (x, y, z))$.

Proposition 6. *If the general solution $u = f(v(x_1, \cdots x_n))$ of $(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_nD_{x_n})u = 0$ is known, then the general solution of $(b_1D_{x_1} + b_2D_{x_2} + \ldots + b_nD_{x_n})^2u = 0$ is $u = f_1(v(x_1, \ldots x_n)) + (l_1x_1 + l_2x_2 + \ldots + l_nx_n) f_2 (v (x_1, \ldots x_n))$.*

Prove. Because the general solution $u = f(v(x_1, \cdots x_n))$ of $(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_nD_{x_n})u = 0$ is known, apparently $u = f(v(x_1, \cdots x_n))$ is the solution of $(b_1D_{x_1} + b_2D_{x_2} + \ldots + b_nD_{x_n})^2u = 0$ yet, according to Proposition 5, we set that $u = g(x_1, \cdots x_n)f = l_sx_sf$ is the solution of $(b_1D_{x_1} + b_2D_{x_2} + \ldots + b_nD_{x_n})^2u = 0$ too, namely $g(x_1, \cdots x_n) = l_sx_s$, l_s is an arbitrary constant,$(s=1,2,\cdots n)$, then

$$(b_1D_{x_1}+b_2D_{x_2}+\dots b_nD_{x_n})^2u=\left(\sum_{i=1}^{n}b_i^2D_{x_i}^2+2\sum_{i<j}b_ib_jD_{x_i}D_{x_j}\right)(gf)$$

$$=\sum_{i=1}^{n}b_i^2\left(g_{x_ix_i}f+2g_{x_i}f_{x_i}+gf_{x_ix_i}\right)+2\sum_{i\neq j}b_ib_j\left(g_{x_ix_j}f+g_{x_i}f_{x_j}+g_{x_j}f_{x_i}+gf_{x_ix_j}\right)$$

$$=g\left(\sum_{i=1}^{n}b_i^2f_{x_ix_i}+2\sum_{i<j}b_ib_jf_{x_ix_j}\right)+f\left(\sum_{i=1}^{n}b_i^2g_{x_ix_i}+2\sum_{i<j}b_ib_jg_{x_ix_j}\right)+2\sum_{i=1}^{n}b_i^2g_{x_i}f_{x_i}$$

$$+2\sum_{i\neq j}b_ib_j\left(g_{x_i}f_{x_j}+g_{x_j}f_{x_i}\right)=2\sum_{i=1}^{n}b_i^2g_{x_i}f_{x_i}+2\sum_{i<j}b_ib_jg_{x_i}f_{x_j}+2\sum_{i<j}b_ib_jg_{x_j}f_{x_i}$$

$$=2b_s^2l_s^2f_{x_s}+2l_sb_s\sum_{s<j}b_jf_{x_j}+2l_sb_s\sum_{i<s}b_if_{x_i}=2l_sb_s\sum_{i=1}^{n}b_if_{x_i}=0.$$

That $u=l_sx_sf(v(x_1,\cdots x_n))$ is the solution of $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n})^2u=0$ is proved. So its general solution is

$$u=f_1\left(v\left(x_1,\dots x_n\right)\right)+\left(l_1x_1+l_2x_2+\ldots+l_nx_n\right)f_2\left(v\left(x_1,\dots x_n\right)\right). \tag{257}$$

Note (257) may be written as

$$u=f_1\left(v\left(x_1,\dots x_n\right)\right)+\left(l_1x_1+l_2x_2+\ldots+l_nx_n+l_{n+1}\right)f_2\left(v\left(x_1,\dots x_n\right)\right).$$

According to Proposition 6, we present a conjecture

Conjecture 1. *If the general solution $u=f(v(x_1,\cdots x_n))$ of $(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_nD_{x_n})u=0$ is known, then the general solution of $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n})^mu=0$ is* $u=\sum_{j=1}^{m}\left(l_{j_1}x_1+l_{j_2}x_2+\ldots+l_{j_n}x_n\right)^{j-1}f_j\left(v\left(x_1,\dots x_n\right)\right)$.

Theoretically Conjecture 1 can be proved by mathematical induction, we shall not analyse it further.

For the mth-order linear PDE with variable coefficients

$$\sum_{i_1+i_2+\ldots+i_n=m}a_{i_1i_2\ldots i_n}u_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)}=0, \tag{258}$$

where i_j are positive integers, $1\le j\le n$, If Eq. (258) can be translated into

$$\begin{aligned}&\left(b_{1_1}D_{x_1}+b_{1_2}D_{x_2}+\ldots+b_{1_n}D_{x_n}\right)\left(b_{2_1}D_{x_1}+b_{2_2}D_{x_2}+\ldots+b_{2_n}D_{x_n}\right)\dots\\&\left(b_{m_1}D_{x_1}+b_{m_2}D_{x_2}+\ldots+b_{m_n}D_{x_n}\right)u=0.\end{aligned} \tag{259}$$

For

$$\left(b_{j_1}D_{x_1}+b_{j_2}D_{x_2}+\ldots+b_{j_n}D_{x_n}\right)u=b_{j_1}u_{x_1}+b_{j_2}u_{x_2}+\ldots+b_{j_n}u_{x_n}=0\ \ ,(j=1,2,\dots m). \tag{260}$$

If the particular solutions $u=v_j(x_1,\cdots x_n),(j=1,2,\cdots m)$ of Eq. (260) are all known, by Proposition 1 the general solution of Eq. (258) is

$$u=\sum_{j=1}^{m}f_j\left(v_j\left(x_1,x_2,\dots x_n\right)\right). \tag{261}$$

If Eq. (258) can be translated into:

$$\prod_{j=1}^{q}(b_{j_1}D_{x_1}+b_{j_2}D_{x_2}+\ldots+b_{j_n}D_{x_n})^{p_j}u=0, \tag{262}$$

where $\sum_{j=1}^{q}p_j=m$,its general solution of conjecture may be written by Conjecture 1.

Proposition 7. *If the general solution* $u=g(x_1,\cdots x_n)f(v(x_1,\cdots x_n))$ *of* $(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_nD_{x_n}+b_{n+1})u=0$ *is known, then the general solution of* $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n}+b_{n+1})^2u=0$ *is* $u=g(x_1,\ldots x_n)\left(f_1(v(x_1,\ldots x_n))+(l_1x_1+l_2x_2+\ldots+l_nx_n)f_2(v(x_1,\ldots x_n))\right).$

Prove. If the general solution $u=g(x_1,\cdots x_n)f(v(x_1,\cdots x_n))$ of $(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_nD_{x_n}+b_{n+1})u=0$ is known,is known, apparently $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n}+b_{n+1})^2u=0$ is $u=g(x_1,\ldots x_n)f(v(x_1,\ldots x_n))$ is the solution of $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n}+b_{n+1})^2u=0$ yet, set

$$u=ht=l_sx_sg(x_1,x_2,\ldots x_n)f(v(x_1,x_2,\ldots x_n)), \tag{263}$$

where $h=l_sx_s$,$t=g(x_1,x_2,\ldots x_n)f(v(x_1,\ldots x_n))$ and l_s is an arbitrary constant($s=1,2,\cdots n$). Assuming (263) is a particular solution of $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n}+b_{n+1})^2u=0$, then

$$\begin{aligned}
&(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n}+b_{n+1})^2u\\
=&\sum_{i=1}^{n}b_i^2\left(h_{x_ix_i}t+2h_{x_i}t_{x_i}+ht_{x_ix_i}\right)+b_{n+1}^2ht\\
&+2\sum_{1\le i<j\le n}b_ib_j\left(h_{x_ix_j}t+h_{x_i}t_{x_j}+h_{x_j}t_{x_i}+ht_{x_ix_j}\right)+2b_{n+1}\sum_{i=1}^{n}b_i\left(h_{x_i}t+ht_{x_i}\right)\\
=&h\left(\sum_{i=1}^{n}b_i^2t_{x_ix_i}+b_{n+1}^2t+2\sum_{1\le i<j\le n}b_ib_jt_{x_ix_j}+2b_{n+1}\sum_{i=1}^{n}b_it_{x_i}\right)+t\sum_{i=1}^{n}b_i^2h_{x_ix_i}\\
&+2\sum_{i=1}^{n}b_i^2h_{x_i}t_{x_i}+2t\sum_{1\le i<j\le n}b_ib_jh_{x_ix_j}+2\sum_{1\le i<j\le n}b_ib_jh_{x_i}t_{x_j}+2\sum_{1\le i<j\le n}b_ib_jh_{x_j}t_{x_i}\\
&+2b_{n+1}t\sum_{i=1}^{n}b_ih_{x_i}\\
=&2\sum_{i=1}^{n}b_i^2h_{x_i}t_{x_i}+2\sum_{1\le i<j\le n}b_ib_jh_{x_i}t_{x_j}+2\sum_{1\le i<j\le n}b_ib_jh_{x_j}t_{x_i}+2b_{n+1}t\sum_{i=1}^{n}b_ih_{x_i}\\
=&2b_s^2h_{x_s}t_{x_s}+2b_sh_{x_s}\sum_{1\le s<j\le n}b_jt_{x_j}+2b_sh_{x_s}\sum_{1\le i<s\le n}b_it_{x_i}+2b_{n+1}b_sh_{x_s}t
\end{aligned}$$

That (263) is a solution of $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n}+b_{n+1})^2u=0$ is proved, so its general solution is $u=g(x_1,\ldots x_n)\left(f_1(v(x_1,\ldots x_n))+(l_1x_1+l_2x_2+\ldots+l_nx_n)f_2(v(x_1,\ldots x_n))\right).$

According to Proposition 7, we present Conjecture 2.

Conjecture 2. *If the general solution* $u=g(x_1,\ldots x_n)f(v(x_1,\ldots x_n))$ *of* $(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_nD_{x_n})u=0$ *is known, the general solution of* $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n}+b_{n+1})^mu=0$ *is* $u=g(x_1,\ldots x_n)\sum_{j=1}^{m}(l_{j_1}x_1+l_{j_2}x_2+\ldots+l_{j_n}x_n)^{j-1}f_j(v(x_1,\ldots x_n)),(m\ge 2).$

For the mth-order linear PDE with variable coefficients

$$\sum_{0\le i_1+i_2+\ldots+i_n\le m} a_{i_1 i_2\ldots i_n} u_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)} = 0, \tag{264}$$

where i_j are positive integers, $1 \le j \le n$. Suppose Eq. (264) can be translated into:

$$\begin{aligned}&\left(b_{1_1}D_{x_1} + b_{1_2}D_{x_2} + \ldots + b_{1_n}D_{x_n} + b_{1_{n+1}}\right)\left(b_{2_1}D_{x_1} + b_{2_2}D_{x_2} + \ldots + b_{2_n}D_{x_n} + b_{2_{n+1}}\right)\\ &\ldots\left(b_{m_1}D_{x_1} + b_{m_2}D_{x_2} + \ldots + b_{m_n}D_{x_n} + b_{m_{n+1}}\right)u = 0.\end{aligned} \tag{265}$$

If the particular solutions $g_j(x_1, \cdots x_n)$ of $(b_{j_1}D_{x_1} + b_{j_2}D_{x_2} + \cdots + b_{j_n}D_{x_n} + b_{j_{n+1}})u = 0$ and the particular solutions $v_j(x_1, \cdots x_n)$ of $(b_{j_1}D_{x_1} + b_{j_2}D_{x_2} + \cdots + b_{j_n}D_{x_n})u = 0$ re all known$(j = 1, 2, \cdots n)$, by Proposition 3 the general solution of Eq. (265) is

$$u\left(x_1, \ldots x_n\right) = \sum_{j=1}^{m}\left(g_j\left(x_1, \ldots x_n\right) f_j\left(v_j\left(x_1, \ldots x_n\right)\right)\right). \tag{266}$$

Suppose Eq. (264) can be translated into

$$\prod_{j=1}^{q}\left(b_{j_1}D_{x_1} + b_{j_2}D_{x_2} + \ldots + b_{j_n}D_{x_n} + b_{j_{n+1}}\right)^{p_j}u = 0, \tag{267}$$

where $\sum\limits_{j=1}^{q} p_j = m$,its general solution of conjecture may be written by Conjecture 2.

For the mth-order linear PDE with variable coefficients

$$\sum_{0\le i_1+i_2+\ldots+i_n\le m} a_{i_1 i_2,\ldots i_n} u_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)} = h\left(x_1, x_2, \ldots x_n\right), \tag{268}$$

where $h(x_1, x_2, \cdots x_n)$ is an arbitrary known function, we need first solve the particular solution of Eq. (268), by the general solution of its homogeneous equation, the general solution of Eq. (268) could be got.

3.2. General solutions laws of linear partial differential equations with constant coefficients

Here we will research the general solutions laws of LPDEs with constant coefficients, which are the special cases of LPDEs with variable coefficients. In this section, if there is no special interpretation $a_i, a_{j_i}, b_i, b_{j_i}, c_i, k_i, k_{j_i}, l_i, l_{j_i}$ and $a_{i_1i_2\ldots i_n}$ are arbitrary constants, f and f_i are arbitrary unary smooth functions $(i, j = 1, 2, \cdots)$.

Proposition 8. *The general solution of* $a_1u_{x_1}+a_2u_{x_2}+\ldots+a_nu_{x_n} = 0$ *is* $u = f(\frac{-1}{a_1}(a_2k_2+a_3k_3+\ldots a_nk_n)x_1 + k_2x_2 + \ldots + k_nx_n + k_{n+1})$. **Prove.** According to Transformational Method 1, set $u\left(x_1, \ldots x_n\right) = f\left(v\right), v\left(x_1, \ldots x_n\right) = k_1x_1 + k_2x_2 + \ldots + k_nx_n + k_{n+1}$,then

$$\begin{aligned}&a_1u_{x_1} + a_2u_{x_2} + \ldots + a_nu_{x_n} = a_1k_1f_v^{'} + a_2k_2f_v^{'} + \ldots + a_nk_nf_v^{'} = 0\\ &\Longrightarrow k_1 = \frac{-1}{a_1}\left(a_2k_2 + a_3k_3 + \ldots + a_nk_n\right).\end{aligned}$$

So the general solution of $a_1u_{x_1}+a_2u_{x_2}+\ldots+a_nu_{x_n} = 0$ is $u\left(x_1, x_2, \ldots x_n\right) = f(\frac{-1}{a_1}(a_2k_2+a_3k_3+\ldots a_nk_n)x_1 + k_2x_2 + \ldots + k_nx_n + k_{n+1})$.

Proposition 9. *The general solution of* $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n})^2u=0$ *is* $u=f_1(\frac{-1}{b_1}(b_2k_{1_2}+b_3k_{1_3}+\ldots+b_nk_{1_n})x_1+k_{1_2}x_2+\ldots+k_{1_n}x_n+k_{1_{n+1}})+(l_1x_1+l_2x_2+\ldots+l_nx_n)f_2(\frac{-1}{b_1}(b_2k_{2_2}+b_3k_{2_3}+\ldots+b_nk_{2_n})x_1+k_{2_2}x_2+\ldots+k_{2_n}x_n+k_{2_{n+1}})$.

Prove. According to Proposition 6 and Proposition 8, we can get Proposition 9 directly.

By Proposition 9 and Conjecture 1, we present Conjecture 3.

Conjecture 3. *The general solution of* $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n})^mu=0$ *is* $u=\sum\limits_{j=1}^{m}(l_{j_1}x_1+\ldots+l_{j_n}x_n)^{j-1}f_j\left(\frac{-1}{b_1}(b_2k_{j_2}+\ldots+b_nk_{j_n})x_1+k_{j_2}x_2+\ldots+k_{j_n}x_n+k_{j_{n+1}}\right)$.

For the mth-order LPDE with constant coefficients

$$\sum_{i_1+i_2+\ldots+i_n=m} a_{i_1i_2\ldots i_n}u_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)}=0, \tag{269}$$

where i_j are positive integers, $1\le j\le n$. If Eq. (269) can be translated into

$$\begin{aligned}&(b_{1_1}D_{x_1}+b_{1_2}D_{x_2}+\ldots+b_{1_n}D_{x_n})(b_{2_1}D_{x_1}+b_{2_2}D_{x_2}+\ldots+b_{2_n}D_{x_n})\\&\ldots(b_{m_1}D_{x_1}+b_{m_2}D_{x_2}+\ldots+b_{m_n}D_{x_n})u=0.\end{aligned} \tag{270}$$

By (261) and Proposition 8, the general solution of Eq. (270) is

$$u=\sum_{j=1}^{m}f_j\left(\frac{-1}{b_{j_1}}(b_{j_2}k_{j_2}+b_{j_3}k_{j_3}+\ldots+b_{j_n}k_{j_n})x_1+k_{j_2}x_2+\ldots+k_{j_n}x_n+k_{j_{n+1}}\right). \tag{271}$$

If Eq. (269) can be converted into

$$\prod_{j=1}^{q}(b_{j_1}D_{x_1}+b_{j_2}D_{x_2}+\ldots+b_{j_n}D_{x_n})^{p_j}u=0, \tag{272}$$

where $\sum\limits_{j=1}^{q}p_j=m$,its general solution of conjecture may be written by Conjecture 3.

Proposition 10. *The general solution of* $a_1u_{x_1}+a_2u_{x_2}+\ldots+a_nu_{x_n}+a_{n+1}u=0$ *is* $u=f\left(\frac{-1}{a_1}(a_2k_2+a_3k_3+\ldots+a_nk_n)x_1+k_2x_2+\ldots+k_nx_n+k_{n+1}\right)\sum\limits_{i=1}^{n}c_ie^{\frac{-a_{n+1}x_i}{a_i}}$.

Prove. We set $u=g(x_1,\cdots x_n)f(v)$, where $v(x_1,\ldots x_n)=k_1x_1+\ldots+k_nx_n+k_{n+1}$,then

$$\begin{aligned}&a_1u_{x_1}+a_2u_{x_2}+\ldots+a_nu_{x_n}+a_{n+1}u\\&=a_1g_{x_1}f+a_1k_1gf_v'+a_2g_{x_2}f+a_2k_2gf_v'+\ldots+a_ng_{x_n}f+a_nk_ngf_v'+a_{n+1}gf\\&=(a_1k_1+a_2k_2+\ldots+a_nk_n)gf_v'+(a_1g_{x_1}+a_2g_{x_2}+\ldots+a_ng_{x_n}+a_{n+1}g)f=0.\end{aligned}$$

Setting f an arbitrary unary first differentiable function, then we get

$$a_1k_1+a_2k_2+\ldots+a_nk_n=0\Longrightarrow k_1=\frac{-1}{a_1}(a_2k_2+a_3k_3+\ldots a_nk_n), \tag{273}$$

$$a_1g_{x_1}+a_2g_{x_2}+\ldots+a_ng_{x_n}+a_{n+1}g=0. \tag{274}$$

Set $g(x_1,\cdots x_n)=h(x_i),(i=1,2,\cdots n)$, then

$$a_1g_{x_1}+\ldots+a_ng_{x_n}+a_{n+1}g=a_ih_{x_i}+a_{n+1}h=0\Longrightarrow h(x_i)=g(x_1,\ldots x_n)=c_ie^{\frac{-a_{n+1}x_i}{a_i}},$$

so $u=\sum_{i=1}^{n} c_i e^{\frac{-a_{n+1}x_i}{a_i}}$ is the particular solution of $a_1u_{x_1}+a_2u_{x_2}+\ldots+a_nu_{x_n}+a_{n+1}u=0$,thus its general solution is

$$u=f\left(\frac{-1}{a_1}\left(a_2k_2+a_3k_3+\ldots+a_nk_n\right)x_1+k_2x_2+\ldots+k_nx_n+k_{n+1}\right)\sum_{i=1}^{n} c_i e^{\frac{-a_{n+1}x_i}{a_i}}. \quad (275)$$

Proposition 11. *The general solution of* $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n}+b_{n+1})^2u=0$ *is*

$$\begin{aligned} u=&(\sum_{j=1}^{2}(l_{j_1}x_1+l_{j_2}x_2+\ldots+l_{j_n}x_n)^{j-1}f_j(\frac{-1}{b_1}(b_2k_{j_2}+b_3k_{j_3}+\ldots+b_nk_{j_n})x_1+k_{j_2}x_2\\ &+\ldots+k_{j_n}x_n+k_{j_{n+1}}))\sum_{i=1}^{n} c_i e^{\frac{-b_{n+1}x_i}{b_i}}. \end{aligned}$$

Prove. According to Proposition 7 and Proposition 10, we can get Proposition 11 directly.

By Proposition 11, we present Conjecture 4.

Conjecture 4. *The general solution of* $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_nD_{x_n}+b_{n+1})^m u=0$ *is*

$$\begin{aligned} u=&(\sum_{j=1}^{m}(l_{j_1}x_1+l_{j_2}x_2+\ldots+l_{j_n}x_n)^{j-1}f_j(\frac{-1}{b_1}(b_2k_{j_2}+b_3k_{j_3}+\ldots+b_nk_{j_n})x_1+k_{j_2}x_2+\ldots\\ &+k_{j_n}x_n+k_{j_{n+1}}))\sum_{i=1}^{n} c_i e^{\frac{-b_{n+1}x_i}{b_i}},(m\geq 2) \end{aligned}$$

.

For the mth-order linear PDE with constant coefficients

$$\sum_{0\leq i_1+i_2+\ldots+i_n\leq m} a_{i_1i_2\ldots i_n}u_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)}=0, \quad (276)$$

where i_j are positive integers, $1\leq j\leq n$. Suppose Eq. (276) can be translated into

$$\begin{aligned} &\left(b_{1_1}D_{x_1}+b_{1_2}D_{x_2}+\ldots+b_{1_n}D_{x_n}+b_{1_{n+1}}\right)\left(b_{2_1}D_{x_1}+b_{2_2}D_{x_2}+\ldots+b_{2_n}D_{x_n}+b_{2_{n+1}}\right)\\ &\ldots\left(b_{m_1}D_{x_1}+b_{m_2}D_{x_2}+\ldots+b_{m_n}D_{x_n}+b_{m_{n+1}}\right)u=0. \end{aligned} \quad (277)$$

According to Proposition 10 the general solution of Eq. (277) is

$$u=\sum_{j=1}^{m}\left(f_j\left(\frac{-1}{b_{j_1}}\left(b_{j_2}k_{j_2}+\ldots+b_{j_n}k_{j_n}\right)x_1+k_{j_2}x_2+\ldots+k_{j_n}x_n+k_{j_{n+1}}\right)\sum_{i=1}^{n} c_{j_i} e^{\frac{-b_{j_{n+1}}x_i}{b_{j_i}}}\right). \quad (278)$$

Suppose Eq. (276) can be translated into

$$\prod_{j=1}^{q}\left(b_{j_1}D_{x_1}+b_{j_2}D_{x_2}+\ldots+b_{j_n}D_{x_n}+b_{j_{n+1}}\right)^{p_j}u=0, \quad (279)$$

where $\sum_{j=1}^{q} p_j=m$,its general solution of conjecture may be written by Conjecture 4.

For the mth-order linear PDE with constant coefficients

$$\sum_{0\le i_1+i_2+\ldots+i_n\le m} a_{i_1i_2\ldots i_n}u^{(i_1i_2\ldots i_n)}_{x_1x_2\ldots x_n}=g\left(x_1,x_2,\ldots x_n\right). \tag{280}$$

If $g(x_1,\cdots x_n)=h(v)$,$v=k_1x_1+\ldots+k_nx_n+k_{n+1}$, h is a known unary function, $k_1,k_2,\cdots k_{n+1}$ are known parameters, according to Transformational Method 2 set $u(x_1,x_2,\cdots x_n)=f(v)$, Eq. (280) could be converted into a mth-order nonhomogeneous LODE with constant coefficients, and its particular solution $f(v)$ could be obtained. Using (278) and so on, the general solution of the homogeneous equation of Eq. (280) can be had, thus the general solution of Eq. (280) may be gained further.

4. General solutions and particular solutions of nonlinear partial differential e-quations

We have studied Eq. (18), (27) and Eq. (83), which are NLPDEs, and have obtained their exact solutions or general solutions. Since the importance of NLPDEs, many effective methods have been proposed to obtain their exact solutions, such as F-expansion method [84], tanh-sech method [85C87], extended tanh method [88C90], multiple exp-function method [91, 92], hyperbolic function method [93], Jacobi elliptic function expansion method [94], homogeneous balance method [95C97], sineCcosine method [98C100] and so on. In this section we will solve some typical examples by the new laws and methods proposed in this paper.

According to (1), (2) we present Theorem 6 first.

Theorem 6. *In the domain $D,(D\subset\mathbb{R}^n)$, if $u(x_1,x_2,\cdots x_n)$ is a first differentiable function, then*

$$\frac{\partial\int f\left(u\right)du}{\partial x_i}=f\left(u\right)u_{x_i}. \tag{281}$$

Proof. Set

$$v\left(x_1,x_2,\ldots x_n\right)=g\left(u\right)=\int f\left(u\right)du. \tag{282}$$

By (2) and (282)

$$v_{x_i}=\frac{\partial\int f\left(u\right)du}{\partial x_i}=g'\left(u\right)u_{x_i}=f\left(u\right)u_{x_i}.$$

Thus the theorem is proved.

Example 4.1.

$$b\left(u\right)u_x=a\left(x,y\right), \tag{283}$$

where $a(x,y)$ is an any known binary function, $b(u)$ is an arbitrary known unary function, by Theorem 6

$$b\left(u\right)u_x=\frac{\partial\int b\left(u\right)du}{\partial x}=a\left(x,y\right)\int b\left(u\right)du=\phi\left(y\right)+\int a\left(x,y\right)dx,$$

where $\phi(y)$ is an arbitrary unary function, so the general solution of Eq. (283) is:

$$\int b\left(u\right)du=\phi\left(y\right)+\int a\left(x,y\right)dx. \tag{284}$$

Example 4.2.

$$a_yu_{xx}+a_xu_{xy}+a_yb\left(u\right)u_x^2+a_xb\left(u\right)u_xu_y=0, \tag{285}$$

where a is an any known binary function, $b(u)$ is an arbitrary known unary function, due to

$$a_y v_x + a_x v_y = 0, \tag{211}$$

the general solution of Eq. (211) is

$$v = g\left(a\left(x, y\right)\right), \tag{212}$$

where g is an arbitrary unary first differentiable function, by Theorem 2 we set

$$v\left(x, y\right) = f\left(u\right) u_x, \tag{286}$$

where f is an undetermined unary function, according to (284) and (212)

$$\int f\left(u\right) du = \phi\left(y\right) + \int v\left(x, y\right) dx = \phi\left(y\right) + \int g\left(a\left(x, y\right)\right) dx,$$

where $\phi(y)$ is an arbitrary unary function, by (286)

$$\begin{aligned}
& a_y v_x + a_x v_y = a_y f\left(u\right) u_{xx} + a_y f'\left(u\right) u_x^2 + a_x f\left(u\right) u_{xy} + a_x f'\left(u\right) u_x u_y = 0 \\
& \Longrightarrow a_y u_{xx} + a_x u_{xy} + a_y \frac{f'\left(u\right)}{f\left(u\right)} u_x^2 + a_x \frac{f'\left(u\right)}{f\left(u\right)} u_x u_y \\
& = a_y u_{xx} + a_x u_{xy} + a_y b\left(u\right) u_x^2 + a_x b\left(u\right) u_x u_y = 0 \\
& \Longrightarrow b\left(u\right) = \frac{f'\left(u\right)}{f\left(u\right)} \Longrightarrow f\left(u\right) = e^{\int b(u) du}.
\end{aligned}$$

So Eq. (285) is a DVTE of Eq. (211), according to Theorem 2, the general solution of Eq. (285) is

$$\int e^{\int b(u) du} du = \phi\left(y\right) + \int g\left(a\left(x, y\right)\right) dx. \tag{287}$$

Example 4.3.

$$u_{xy} + a\left(u\right) u_x u_y + b\left(x, y\right) u_y = 0, \tag{288}$$

where $a(u)$ is an any known unary function, $b(x, y)$ is an arbitrary known binary function, due to

$$v_x + b\left(x, y\right) v = 0, \tag{289}$$

the general solution of Eq. (289) is [74]

$$v = \varphi\left(y\right) e^{-\int b(x,y) dx}, \tag{290}$$

where $\varphi(y)$ is an arbitrary unary function, by Theorem 2 we set

$$v = f\left(u\right) u_y, \tag{291}$$

where f is an undetermined unary first differentiable function, according to (284) and (290)

$$\int f\left(u\right) du = \int v dy = \phi\left(x\right) + \int \varphi\left(y\right) e^{-\int b(x,y) dx} dy,$$

where $\phi(x)$ is an arbitrary unary function, by (291)

$$\begin{aligned}
& v_x + b\left(x, y\right) v = f\left(u\right) u_{xy} + f'\left(u\right) u_x u_y + b\left(x, y\right) f\left(u\right) u_y = 0 \\
& \Longrightarrow u_{xy} + \frac{f'\left(u\right)}{f\left(u\right)} u_x u_y + b\left(x, y\right) u_y = u_{xy} + a\left(u\right) u_x u_y + b\left(x, y\right) u_y = 0 \\
& \Longrightarrow \frac{f'\left(u\right)}{f\left(u\right)} = a\left(u\right) \Longrightarrow f\left(u\right) = e^{\int a(u) du}.
\end{aligned}$$

So Eq. (288) is a DVTE of Eq. (289), according to Theorem 2, the general solution of Eq. (288) is

$$\int e^{\int a(u)du} du = \phi(x) + \int \varphi(y) e^{-\int b(x,y)dx} dy. \tag{292}$$

Example 4.4.

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots a_n u_{x_n} + a_{n+1} h(u) = 0, \tag{293}$$

where $a_i = a_i(x_1, \cdots x_n), (i = 1, 2, \cdots n+1)$, and $h(u)$ is an arbitrary known unary function. By Proposition 3, supposing the general solution $w = g(x_1, \cdots x_n) f(v(x_1, \cdots x_n))$ of $a_1 w_{x_1} + a_2 w_{x_2} + \cdots a_n w_{x_n} + a_{n+1} w = 0$ is known, set

$$w = p(u). \tag{294}$$

Then

$$w_{x_i} = p_u' u_{x_i}, \tag{295}$$

$$a_1 w_{x_1} + a_2 w_{x_2} + \ldots a_n w_{x_n} + a_{n+1} w = a_1 p_u' u_{x_1} + a_2 p_u' u_{x_2} + \ldots a_n p_u' u_{x_n} + a_{n+1} p(u) = 0$$

$$\Longrightarrow a_1 u_{x_1} + a_2 u_{x_2} + \ldots a_n u_{x_n} + a_{n+1} \frac{p(u)}{p_u'} = a_1 u_{x_1} + a_2 u_{x_2} + \ldots a_n u_{x_n} + a_{n+1} h(u) = 0$$

$$\Longrightarrow \frac{p(u)}{p_u'} = h(u)$$

$$\Longrightarrow w = p(u) = e^{\int \frac{du}{h(u)}} = g(x_1, x_2, \ldots x_n) f(v(x_1, x_2, \ldots x_n)).$$

So Eq. (293) is a DVTE of $a_1 w_{x_1} + a_2 w_{x_2} + \cdots a_n w_{x_n} + a_{n+1} w = 0$,according to Theorem 2, the general solution of Eq. (293) is

$$e^{\int \frac{du}{h(u)}} = g(x_1, x_2, \ldots x_n) f(v(x_1, x_2, \ldots x_n)). \tag{296}$$

If a_i are constants, by Proposition 10 and (296), the general solution of Eq. (293) can be get

$$e^{\int \frac{du}{h(u)}} = f\left(\frac{-1}{a_1}(a_2 k_2 + a_3 k_3 + \ldots a_n k_n) x_1 + k_2 x_2 + \ldots + k_n x_n + k_{n+1}\right) \sum_{i=1}^{n} c_i e^{\frac{-a_{n+1} x_i}{a_i}}, \tag{297}$$

where $k_2 - k_{n+1}$ and $c_1 - c_n$ are arbitrary constants.

In the theory of PDEs, converting a PDE to a relatively simple ODE is a classical method, such as Laplace transform, Fourier transform, and so on integral transformation belong to this type of way. According to Eq. (7) and (8), two kinds of PDEs can be transformed into ODEs, and now we use two new theorems to express this idea:

Theorem 7. *In the domain $D, (D \subset \mathbb{R}^n)$, any established mth-order PDE with n space variables $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1 x_2}, \cdots) = 0$, If all the known functions satisfy $a_i(x_1, \ldots x_n) = a_i(k_1 x_1 + \ldots + k_n x_n + k_{n+1})$,where $k_1, k_2, \cdots k_{n+1}$ are known parameters, set $u(x_1, \ldots x_n) = f(k_1 x_1 + \ldots + k_n x_n + k_{n+1})$, then substitute $u = f(k_1 x_1 + \ldots + k_n x_n + k_{n+1})$ and its partial derivatives into $F = 0$*
1. If $F = 0$ is a linear PDE, then it can be converted to a linear ODE,

2. If $F = 0$ is a non-linear PDE, then it can be converted to a non-linear ODE.

Proof. Set $v(x_1, \cdots x_n) = k_1 x_1 + \cdots + k_n x_n + k_{n+1}$, then $a_i(x_1, \cdots x_n) = a_i v, u = f(v)$, by (7) and (8) $F = 0$ can be converted to an ODE whose dependent variable is f and independent variable is v, since each linear term in $F=0$ is transformed into a new linear term, each non-linear term is transformed into a new nonlinear term, so the theorem is proved.

Theorem 8. *In the domain* $D,(D\subset\mathbb{R}^n)$,*any established* m*th-order PDE with* n *space variables* $F(u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)=0$, *namely in the equation there is no known function* $a_i(x_1,\cdots x_n)$ *set* $u(x_1,\cdots x_n)=f(k_1x_1+\cdots+k_nx_n+k_{n+1})$, *where* $k_1,k_2,\cdots k_{n+1}$ *are unascertained parameters, then substitute* $u=f(k_1x_1+\cdots+k_nx_n+k_{n+1})$ *and its partial derivatives into* $F=0$
1. If $F=0$ *is a linear PDE, then it can be converted to a linear ODE,*

2. If $F=0$ *is a non-linear PDE, then it can be converted to a non-linear ODE.*

Proof. Set $v(x_1,\cdots x_n)=k_1x_1+\cdots+k_nx_n+k_{n+1}$, then $u=f(v)$, by (2.7) and (2.8) $F=0$ can be converted to an ODE whose dependent variable is f and independent variable is v, since each linear term in $F=0$ is transformed into a new linear term, each non-linear term is transformed into a new nonlinear term, so the theorem is proved.

Theorem 7 and 8 is a further application of Transformational Method 1 and 2. In the previous, the method of solving the particular solution of Poisson equation is in fact using theorem 7, the method of solving Eq. (27) is actually the application of theorem 8. Now we use theorem 8 to solve two typical nonlinear PDEs.

Example 6.4.

$$a\left(u\right)u_t+b\left(u\right)u_x+c\left(u\right)u_{xx}=0. \tag{298}$$

According to Theorem 8, set $u(x,t)=f(v)=f(k_1x+k_2t+k_3)$, k_1-k_3 are parameters to be determined, f is an undetermined unary second differentiable function, then

$$a\left(u\right)u_t+b\left(u\right)u_x+c\left(u\right)u_{xx}=k_2a\left(f\right)f_v^{'}+k_1b\left(f\right)f_v^{'}+k_1^2c\left(f\right)f_v^{''}=0.$$

Namely

$$f_v^{''}+\frac{k_2a\left(f\right)+k_1b\left(f\right)}{k_1^2c\left(f\right)}f_v^{'}=0 \tag{299}$$

Because

$$y''+b\left(y\right)\left(y'\right)^2+c\left(y\right)\left(y'\right)^m=0. \tag{300}$$

The general solution of Eq. (300) is [53]

$$x=C_2+\int\left(\frac{-e^{(2-m)\int b(y)dy}}{C_1+(2-m)\int c\left(y\right)e^{(2-m)\int b(y)dy}dy}\right)^{\frac{1}{2-m}}dy, \tag{301}$$

where C_1 and C_2 are arbitrary constants, so the exact solution of Eq. (298) is

$$v=k_1x+k_2t+k_3=\int\frac{du}{C_1-\int\frac{k_2a(u)+k_1b(u)}{k_1^2c(u)}du}. \tag{302}$$

where k_1-k_3 are arbitrary constants, Burgers equation

$$u_t+uu_x+\alpha u_{xx}=0, \tag{303}$$

is a special case of Eq. (298), according to (302) its exact solution is

$$k_1x+k_2t+k_3=-2k_1^2\alpha\int\frac{du}{C_1+2k_2u+k_1u^2}. \tag{304}$$

Example 6.5.

$$a\left(u\right)u_t+b\left(u\right)u_x+c\left(u\right)u_{xxx}=0. \tag{305}$$

According to Theorem 8, set $u(x,t)=f(v)=f(k_1x+k_2t+k_3)$, k_1-k_3 are parameters to be determined, f is an undetermined unary third differentiable function, then

$$a\left(u\right)u_t+b\left(u\right)u_x+c\left(u\right)u_{xxx}=k_2a\left(u\right)f_v^{'}+k_1b\left(u\right)f_v^{'}+k_1^3c\left(u\right)f_v^{'''}=0.$$

Namely

$$f_v^{'''}+\frac{k_2a\left(u\right)+k_1b\left(u\right)}{k_1^3c\left(u\right)}f_v^{'}=0. \tag{306}$$

Because

$$y'''+b\left(y\right)y'\left(y''\right)^m=0. \tag{307}$$

The general solution of Eq. (307) is [53]

$$x=C_1+\int\left(C_2-2\int\left(C_3+(1-m)\int b\left(y\right)dy\right)^{\frac{1}{1-m}}dy\right)^{\frac{-1}{2}}dy, \tag{308}$$

where C_1-C_3 are arbitrary constants, so the exact solution of Eq. (305) is

$$k_1x+k_2t+k_3=\int\left(C_1-C_2u-2\int\int\frac{k_2a\left(u\right)+k_1b\left(u\right)}{k_1^3c\left(u\right)}dudu\right)^{\frac{-1}{2}}du. \tag{309}$$

where k_1-k_3 are arbitrary constants, KdV equation, mKdV equation and KdV-mKdV equation

$$u_t+uu_x+\beta u_{xxx}=0, \tag{310}$$

$$u_t+\alpha u^2u_x+\beta u_{xxx}=0, \tag{311}$$

$$u_t+\gamma uu_x+\alpha u^2u_x+\beta u_{xxx}=0, \tag{312}$$

are special cases of Eq. (305), according to (309) their exact solutions are

$$k_1x+k_2t+k_3=\int\left(C_1-C_2u-\frac{k_2}{k_1^3\beta}u^2-\frac{k_1}{3k_1^3\beta}u^3\right)^{\frac{-1}{2}}du, \tag{313}$$

$$k_1x+k_2t+k_3=\int\left(C_1-C_2u-\frac{k_2}{k_1^3\beta}u^2-\frac{k_1\alpha}{6k_1^3\beta}u^4\right)^{\frac{-1}{2}}du, \tag{314}$$

$$k_1x+k_2t+k_3=\int\left(C_1-C_2u-\frac{k_2}{k_1^3\beta}u^2-\frac{k_1\gamma}{3k_1^3\beta}u^3-\frac{k_1\alpha}{6k_1^3\beta}u^4\right)^{\frac{-1}{2}}du, \tag{315}$$

respectively. Since k_1-k_3 are arbitrary constants, (304), (313)-(315) are all solitary wave solutions.

5. Extention and conclusion

5.1. Two axioms and a conjecture

For revealing the relationship between an arbitrary mth-order partial differential equation (PDE) of n variables and an arbitrary mth-differentiable function of n variables, we present Axiom 1.

Axiom 1. *In the domain $D, (D \subset \mathbb{R}^n)$, any established mth-order PDE with n variables $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1x_2}, \cdots) = 0$, set $f(x_1, \cdots x_n)$ an arbitrary known function, $f \in C^m(D)$, then substitute from $u = f(x_1, \cdots x_n)$ and its partial derivatives into $F = 0$*
1. In case of getting $0 = 0$, then $u = f(x_1, \cdots x_n)$ is the solution of $F = 0$,

2. In case of getting $k = 0$, but in fact $k \neq 0$, then $u = f(x_1, \cdots x_n)$ is not the solution of $F = 0$,

3. In case of getting $h(x_1, \cdots x_l) = 0, (l \leq n)$, then $u = f(x_1, \cdots x_n)$ is the solution of $F = 0$ under the condition of $h(x_1, \cdots x_l) = 0$.

Among the three conclusions of Axiom 1, the first two are obvious and the relevant cases could be illustrated easily, we mainly analyse the third conclusion, such as

Example 5.1.
$$z_x^3 + z_y^3 + z^3 = x^3 y^3. \tag{316}$$
We may choose an arbitrary binary first differentiable function, for instance substitute $z = xy$ into Eq. (316)
$$z_x^3 + z_y^3 + z^3 = y^3 + x^3 + x^3 y^3 = x^3 y^3 \Longrightarrow y = -x.$$
So $z = xy$ is a solution of Eq. (316) under the condition of $y = -x$. Note $z = -x^2$ is not a particular solution of Eq. (316), in 3-dimensional space, $z = -x^2$ is a three-dimensional surface perpendicular to the y-axis, $z = xy$ under the condition of $y = -x$ is a three-dimensional curve:

$$\begin{cases} x = x \\ y = -x \\ z = xy = -x^2 \end{cases}$$

or set $x = t$,then:
$$\begin{cases} x = t \\ y = -t \\ z = -t^2 \end{cases}$$
So the geometric meaning and the mathematical meaning of these two cases are completely different.

Substituting from any function of n variables into an arbitrary PDE of n variables may obtain other special circumstances, such as denominator equal zero and so on, the probabilities of these circumstances are very low like geting $0 = 0$ or $k = 0$, so Axiom 1 reveals that almost any mth-differentiable function of n variables is a conditional solution of an arbitrary mth-order PDE of n variables.

Axiom 1 may be extended to ordinary differential equations (ODEs), so we present Axiom 2.

Axiom 2. *In the domain $D, (D \subset \mathbb{R}^1)$, any established mth-order ordinary differential e-quation (ODE) $F(x, y, y^{(1)}, y^{(2)} \cdots y^{(m)}) = 0$, set $f(x)$ known and $f \in C^m(D)$, then substitute*

from $y = f(x)$ and its derivatives into $F = 0$
1. In case of getting $0 = 0$, then $y = f(x)$ is the solution of $F = 0$,

2. In case of getting $k = 0$, but in fact $k \neq 0$, then $y = f(x)$ is not the solution of $F = 0$,

3. In case of getting $g(x) = 0$, then $y = f(x)$ is the solution of $F = 0$ under the condition of $g(x) = 0$,

4. In case of getting $x = k_1, k_2 \cdots k_l (l \geq 1)$, then $y = f(x)$ is the discrete solution of $F = 0$ under the condition of $x = k_1, k_2 \cdots k_l$.

The probability of the above first and second results is very little, so Axiom 2 reveals that almost every unary mth-differentiable function is a conditional solution of an arbitrary mth-order ODE. Such as Abel equation.

Example 5.2.

$$y' + y^3 + a(x) = 0, \tag{317}$$

where $a(x)$ is an arbitrary unary functions, discretionarily set $y = cx$, then

$$y' + y^3 + a(x) = c + c^3x^3 + a(x) = 0 \Longrightarrow a(x) = -c - c^3x^3.$$

Therefore, under the condition of $a(x) = -c - c^3x^3$, the particular solution of the Eq. (317) is $y = cx$.

Example 5.3.

$$y' + y^3 + x^3 = 0, \tag{318}$$

set $y = cx$, then

$$y' + y^3 + x^3 = c + \left(c^3 + 1\right)x^3 = 0 \Longrightarrow x = \left(\frac{-c}{c^3 + 1}\right)^{\frac{1}{3}}.$$

Therefore, on the point $x = \left(\frac{-c}{c^3+1}\right)^{\frac{1}{3}}$. the particular solution of the Eq. (318) is $y = cx$. In some specific case, the general solution of the Abel equation may be referred to [101, 102].

We know that a univariate function satisfying certain conditions can be expanded into a Taylor series or a Fourier series. When we consider the Cauchy problem of Eq. (38), we assume that the conditions are (41) and (42). Now we propose a **conjecture** about the n-ary function:

Conjecture 5. *A n-ary function satisfying certain conditions can be expanded into a series:*

$$f(x_1, x_2, \ldots x_n) = \sum_{i=1}^{s} \varphi_i \left(k_{i_1}x_1 + k_{i_2}x_2 + \ldots + k_{i_n}x_n + k_{i_{n+1}}\right), (1 \leq s \leq \infty) \tag{319}$$

Where φ_i are arbitrarily determined unary functions and k_{i_j} are arbitrarily determined parameters.

For the unary function, Conjecture 5 is obviously correct, because the Taylor series and Fourier series are all special cases of $\sum_{i=1}^{s} \varphi_i (k_{i_1}x + k_{i_2})$ for the n-ary function ($n \geq 2$), how to strictly prove Conjecture 5 is a new mathematical problem. Even if some $f(x_1, x_2, \cdots x_n)$ cannot be strictly expanded to (318), since φ_i, k_{i_j} can be arbitrarily chosen, it can be further envisioned that $f(x_1, x_2, \cdots x_n)$ should be approximatively replaced to $\sum\limits_{i=1}^{s} \varphi_i\left(k_{i_1}x_1 + k_{i_2}x_2 + \ldots + k_{i_n}x_n + k_{i_{n+1}}\right)$ in a restricted domain.

5.2. General Equations and Restricted Equations

In this book, we have solved general solutions of many important PDEs, such as the concise general solution of acoustic equation in Cartesian coordinate system is

$$\begin{aligned} p =& f_1\left(k_1x + k_2y + k_3z + c_0t\sqrt{k_1^2 + k_2^2 + k_3^2} + k_4\right) \\ &+ f_2\left(k_5x + k_6y + k_7z - c_0t\sqrt{k_5^2 + k_6^2 + k_7^2} + k_8\right) + k_9x + k_{10}y + k_{11}z + k_{12}t + k_{13}. \end{aligned} \tag{63}$$

If we set $k_3 = k_7 = k_{11} = 0$, then

$$\begin{aligned} p =& f_1\left(k_1x + k_2y + c_0t\sqrt{k_1^2 + k_2^2} + k_4\right) + f_2\left(k_5x + k_6y - c_0t\sqrt{k_5^2 + k_6^2} + k_8\right) \\ &+ k_9x + k_{10}y + k_{12}t + k_{13} \end{aligned} \tag{320}$$

(320) is essentially the general solution (58) of 2D wave equation

$$\begin{aligned} u =& f_1\left(k_1x + k_2y + at\sqrt{k_1^2 + k_2^2} + k_3\right) + f_2\left(k_4x + k_5y - at\sqrt{k_4^2 + k_5^2} + k_6\right) \\ &+ k_7x + k_8y + k_9t + k_{10}. \end{aligned} \tag{58}$$

If we set $k_2 = k_3 = k_6 = k_7 = k_{11} = 0$, then

$$p = f_1\left(k_1x + c_0k_1t + k_4\right) + f_2\left(k_5x - c_0k_5t + k_8\right) + k_9x + k_{12}t + k_{13} \tag{321}$$

(321) is essentially the general solution (48) of the 1D wave equation

$$u = f_1\left(k_1x + k_1at + k_2\right) + f_2\left(k_3x - k_3at + k_4\right) + k_5x + k_6t + k_7, \tag{48}$$

Such as the concise general solution of 3D heat equation in Cartesian coordinate system is

$$\begin{aligned} u =& e^{\frac{k_0(k_0t + k_1x + k_2y + k_3z)}{\left(k_1^2 + k_2^2 + k_3^2\right)a^2}} h_1\left(\frac{2k_0\left(\sqrt{-l_2^2 - l_3^2}k_1 + l_2k_2 + l_3k_3\right)}{k_1^2 + k_2^2 + k_3^2}t + \sqrt{-l_2^2 - l_3^2}x + l_2y + l_3z + l_4\right) \\ &+ e^{\frac{k_{10}(k_{10}t + k_{11}x + k_{12}y + k_{13}z)}{\left(k_{11}^2 + k_{12}^2 + k_{13}^2\right)a^2}} \\ &h_2\left(\frac{2k_{10}\left(-\sqrt{-l_{12}^2 - l_{13}^2}k_{11} + l_{12}k_{12} + l_{13}k_{13}\right)}{k_{11}^2 + k_{12}^2 + k_{13}^2}t - \sqrt{-l_{12}^2 - l_{13}^2}x + l_{12}y + l_{13}z + l_{14}\right) \end{aligned} \tag{123}$$

If we set $k_0 = k_{10} = 0$, then

$$u = h_1\left(\sqrt{-l_2^2 - l_3^2}x + l_2y + l_3z + l_4\right) + h_2\left(-\sqrt{-l_{12}^2 - l_{13}^2}x + l_{12}y + l_{13}z + l_{14}\right), \tag{322}$$

Contrast (322) with (16)

$$\begin{aligned} u =& f_1(x\sqrt{-k_2^2 - k_3^2} + k_1y + k_2z + k_3) \\ &+ f_2(-x\sqrt{-k_4^2 - k_5^2} + k_4y + k_5z + k_6) + k_7x + k_8y + k_9z + k_{10}, \end{aligned} \tag{16}$$

(322) is a part of the general solution of Laplace equation.

Now let us ask a question: Why there is such a relation in the general solutions of these different equations?

In order to solve this problem, we first propose three new defines: **general equations, restricted equations and homologous restricted equations.**

If the equation $F_i = 0$ *is a special case of the equation* $F = 0, (i = 1, 2, \cdots)$*, then* $F_i = 0$ *is the restricted equation of* $F = 0$*,* $F = 0$ *is the general equation of* $F_i = 0$*,* $F_i = 0$ *between each other known as the homologous restricted equations.*

In theory, any equation can have infinite general equations; if an equation contains arbitrary known functions or parameters, the equation can have infinite restricted equations.

Using the above defines, we propose **Axiom 3:**

Axiom 3: *If there is no meaningless case, the solution of a general equation is known, then the solutions of all its restricted equations are known; if the solution of a restricted equation is unknown, then the solutions of all its general equations are unknown.*

Axiom 3 is not hard to be understood. Since restricted equations are special cases of their general equations, the solutions of restricted equations are also special cases of the solutions of their general equations. Unless nonsensical cases occur, the solutions of all the restricted equations can be directly obtained by the known solutions of their general equations. On the other hand, if the solution of a restricted equation is not solved, it is impossible to solve the solutions of its general equations which are more complex.

According to the above defines and laws, we can explain why the solutions of some PDEs are similar, or even there are some definite relationships within them. Because they are homologous restricted equations, such as one-dimensional, two-dimensional and three-dimensional wave equation are all the restricted equations of Eq. (38); the heat equation and the Laplace equation are both the restricted equations of Eq. (111).

Since there is $k_7x + k_8y + k_9z + k_{10}$ in the general solution of the Laplace equation, we can make a preliminary judgment that these terms may be absent in the general solutions of (111) and (122).

5.3. Conclusion

In this paper, we have proposed ten new concepts, three new axioms and eight new theorems, they are:

concise general solution (CGS); series general solution (SGS); independent variable transformational equations (IVTEs); dependent variable transformational equations (DVTEs); symmetric vector partial differential equations (SVPDEs); corresponding scalar equation (CSE); independent variable transformation vector equation (IVTVE); general equations; restricted equations; homologous restricted equations.

Axiom 1. *In the domain* $D, (D \subset \mathbb{R}^n)$*, any established mth-order PDE with n variables* $F(x_1, \cdots x_n, u,$
$u_{x_1}, \cdots u_{x_n}, u_{x_1x_2}, \cdots) = 0$*, set* $f(x_1, \cdots x_n)$ *an arbitrary known function,* $f \in C^m(D)$*, then substitute from* $u = f(x_1, \cdots x_n)$ *and its partial derivatives into* $F = 0$
1. In case of getting $0 = 0$*, then* $u = f(x_1, \cdots x_n)$ *is the solution of* $F = 0$*,*

2. In case of getting $k = 0$, but in fact $k \neq 0$, then $u = f(x_1, \cdots x_n)$ is not the solution of $F = 0$,

3. In case of getting $h(x_1, \cdots x_l) = 0, (l \leq n)$, then $u = f(x_1, \cdots x_n)$ is the solution of $F = 0$ under the condition of $h(x_1, \cdots x_l) = 0$.

Axiom 2. *In the domain $D, (D \subset \mathbb{R}^1)$, any established mth-order ordinary differential e-quation (ODE) $F(x, y, y^{(1)}, y^{(2)} \cdots y^{(m)}) = 0$, set $f(x)$ known and $f \in C^m(D)$, then substitute from $y = f(x)$ and its derivatives into $F = 0$*
1. In case of getting $0 = 0$, then $y = f(x)$ is the solution of $F = 0$,

2. In case of getting $k = 0$, but in fact $k \neq 0$, then $y = f(x)$ is not the solution of $F = 0$,

3. In case of getting $g(x) = 0$, then $y = f(x)$ is the solution of $F = 0$ under the condition of $g(x) = 0$,

4. In case of getting $x = k_1, k_2 \cdots k_l (l \geq 1)$, then $y = f(x)$ is the discrete solution of $F = 0$ under the condition of $x = k_1, k_2 \cdots k_l$.

Above two new axioms reveal that almost any mth-differentiable function with n variables is a conditional solution of an arbitrary mth-order PDE with n variables and almost any unary mth-differentiable function is a conditional solution of an arbitrary mth-order ODE.

Axiom 3. *If there is no meaningless case, the solution of a general equation is known, then the solutions of all its restricted equations are known; if the solution of a restricted equation is unknown, then the solutions of all its general equations are unknown.*

Theorem 1. *In the domain $D, (D \subset \mathbb{R}^n)$, if the solution $u = f(x_1, \cdots x_n)$ of a mth-order PDE $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1 x_2}, \cdots) = 0$ is known, then the solution of its IVTE $G(y_1, \cdots y_n, u, u_{y_1}, \cdots u_{y_n}, u_{y_1 y_2}, \cdots) = 0$ is $u = f(x_1, \cdots x_n) = g(y_1, \cdots y_n)$.*

Theorem 1 reveals the law of partial differential equations solution in various orthogonal coordinate system.

Theorem 2. *In the domain $D, (D \subset \mathbb{R}^n)$, if the solution $v = f(x_1, \cdots x_n)$ of a PDE $F(x_1, \cdots x_n, v, v_{x_1}, \cdots v_{x_n}, v_{x_1 x_2}, \cdots) = 0$ is known, set $v = h(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1 x_2}, \cdots)$ then the solution of its DVTE $G(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1 x_2}, \cdots) = 0$ is the solution of $h(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1 x_2}, \cdots) = f(x_1, \cdots x_n)$.*

Theorem 3. *In the domain $D, (D \subset \mathbb{R}^1)$, if the solution $w = f(x)$ of an ODE $F(x, w, w', w'', \ldots w^{(n)}) = 0$ is known, set $w = h\left(x, y, y', y'', \ldots y^{(m)}\right)$, then the solution of its DVTE $G(x, y, y', y'', \ldots y^{(m+n)}) = 0$ is the solution of $h\left(x, y, y', y'', \ldots y^{(m)}\right) = f(x)$.*

Theorem 4. *If there are m arbitrary functions in the general solution of the CSE $F_i = 0$, then the number of arbitrary functions in the general solution of the n-dimensional SVPDE $F = 0$ is mn.*

Theorem 5. *In the domain $D, (D \subset \mathbb{R}^n)$, if the solution $u = f\left(x_1, x_2 \ldots x_n, e_{x_1}, e_{x_2}, \ldots e_{x_n}\right)$ of a mth-order vector PDE $F\left(x_1, \ldots x_n, u, \frac{\partial u}{\partial x_1}, \cdots \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \cdots\right) = 0$ is known, then the solution of* *its IVTVE* $G\left(y_1,\ldots y_n,u,\frac{\partial u}{\partial y_1},\ldots\frac{\partial u}{\partial y_n},\frac{\partial^2 u}{\partial y_1\partial y_2},\ldots\right)=0$ *is* $u=f\left(x_1,x_2\ldots x_n,e_{x_1},e_{x_2},\ldots e_{x_n}\right)=g\left(y_1,y_2,\ldots y_n,e_{y_1},e_{y_2},\ldots e_{y_n}\right)$

Theorem 6. *In the domain* $D,(D\subset\mathbb{R}^n)$*,if* $u(x_1,x_2,\cdots x_n)$ *is a first differentiable function, then*

$$\frac{\partial\int f(u)\,du}{\partial x_i}=f(u)\,u_{x_i}. \tag{281}$$

Theorem 7. *In the domain* $D,(D\subset\mathbb{R}^n)$*,any established* m*th-order PDE with* n *space variables* $F\left(x_1,\ldots x_n,u,u_{x_1},\ldots u_{x_n},u_{x_1x_2},\ldots\right)=0$*,If all the known functions satisfy* $a_i\left(x_1,\ldots x_n\right)=a_i(k_1x_1+\ldots+k_nx_n+k_{n+1})$ *where* $k_1,k_2,\cdots k_{n+1}$ *are known parameters, set* $u\left(x_1,\ldots x_n\right)=f\left(k_1x_1+\ldots+k_nx_n+k_{n+1}\right)$*, then substitute* $u=f\left(k_1x_1+\ldots+k_nx_n+k_{n+1}\right)$ *and its partial derivatives into* $F=0$

1. If $F=0$ *is a linear PDE, then it can be converted to a linear ODE,*

2. If $F=0$ *is a non-linear PDE, then it can be converted to a non-linear ODE.*

Theorem 8. *In the domain* $D,(D\subset\mathbb{R}^n)$*,any established* m*th-order PDE with* n *space variables* $F(u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)=0$*, namely in the equation there is no known function* $a_i(x_1,\cdots x_n)$ *set* $u\left(x_1,\ldots x_n\right)=f\left(k_1x_1+\ldots+k_nx_n+k_{n+1}\right)$*,where* $k_1,k_2,\cdots k_{n+1}$ *are unascertained parameters, then substitute* $u=f\left(k_1x_1+\ldots+k_nx_n+k_{n+1}\right)$ *and its partial derivatives into* $F=0$

1. If $F=0$ *is a linear PDE, then it can be converted to a linear ODE,*

2. If $F=0$ *is a non-linear PDE, then it can be converted to a non-linear ODE.*

In Chapter 2, we indicate that Transformational Method 1-4 are specific applications of Theorem 2. In fact, Transformational Method 1-4 are also concrete applications of Axiom 1. According to Axiom 1, the complete representation of the four transformation methods should be:

Transformational Method 1. *In the domain* $D,(D\subset\mathbb{R}^n)$*, any established* m*th-order PDE with* n *variables* $F(x_1,\cdots x_n,u,u_{x_1},\cdots u_{x_n},u_{x_1x_2},\cdots)=0$*,set* $v=v(x_1,\cdots x_n)$ *and* $u=f(v)$ *are both undetermined* m*th-differentiable functions* $(u,v\in C^m(D))$*, then substitute* $u=f(v)$ *and its partial derivatives into* $F=0$

1. In case of working out $v(x_1,\cdots x_n)$ *and* $f(v)$*, then* $u=f(v)$ *is the solution of* $F=0$*,*

2. In case of dividing out $f(v)$ *and its partial derivative, also working out* $v(x_1,\cdots x_n)$*, then* $u=f(v)$ *is the solution of* $F=0$*, and* f *is an arbitrary unary* m*th-differentiable function,*

3. In case of dividing out $f(v)$ *and its partial derivative, also getting* $k=0$*, but in fact* $k\neq 0$*, then* $u=f(v)$ *is not the solution of* $F=0$*, and* f *is an arbitrary unary* m*th-differentiable function.*

4. In case of working out $v(x_1,\cdots x_n)$ *and* $f(v)$ *under the condition of* $h(x_1,\cdots x_l)=0,(l\leq n)$*, then* $u=f(v)$ *is the solution of* $F=0$ *under the condition of* $h(x_1,\cdots x_l)=0$*,*

5. In case of dividing out $f(v)$ *and its partial derivative, also working out* $v(x_1,\cdots x_n)$ *under the condition of* $h(x_1,\cdots x_l)=0,(l\leq n)$*, then* $u=f(v)$ *is the solution of* $F=0$ *under the condition of* $h(x_1,\cdots x_l)=0$*, and* f *is an arbitrary unary* m*th-differentiable function.*

Transformational Method 2. *In the domain* $D, (D \subset \mathbb{R}^n)$*, any established mth-order PDE with n variables* $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1 x_2}, \cdots) = 0$*,set* $v = v(x_1, \cdots x_n)$ *known and* $u = f(v)$ *undetermined* $(u, v \in C^m(D))$*, then substitute* $u = f(v)$ *and its partial derivatives into* $F = 0$

1. In case of working out f*, then* $u = f(v)$ *is the solution of* $F = 0$*,*

2. In case of dividing out f *and its partial derivative, also getting* $0 = 0$*, then* $u = f(v)$ *is the solution of* $F = 0$*, and* f *is an arbitrary unary mth-differentiable function,*

3. In case of dividing out f *and its partial derivative, also getting* $k = 0$*, but in fact* $k \neq 0$*, then* $u = f(v)$ *is not the solution of* $F = 0$*, and* f *is an arbitrary unary mth-differentiable function*

4. In case of working out f *under the condition of* $h(x_1, \cdots x_l) = 0, (l \leq n)$*, then* $u = f(v)$ *is the solution of* $F = 0$ *under the condition of* $h(x_1, \cdots x_l) = 0$*,*

5. In case of dividing out f *and its partial derivative, also getting* $h(x_1, \cdots x_l) = 0, (l \leq n)$*, then* $u = f(v)$ *is the solution of* $F = 0$ *under the condition of* $h(x_1, \cdots x_l) = 0$*, and* f *is an arbitrary unary mth-differentiable function.*

Transformational Method 3. *In the domain* $D, (D \subset \mathbb{R}^n)$*, any established mth-order PDE with n variables* $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1 x_2}, \cdots) = 0$*,setting* $f(v), g(x_1, \cdots x_n)$ *and* $v(x_1, \cdots x_n)$ *are all undetermined function,* $g, v \in C^m(D)$*, then substitute* $u = gf(v)$ *and its partial derivatives into* $F = 0$

1. In case of working out f, g *and* v*, then* $u = gf(v)$ *is the solution of* $F = 0$*,*

2. In case of dividing out f *and its partial derivative, also working out* g *and* v*, then* $u = gf(v)$ *is the solution of* $F = 0$*, and* f *is an arbitrary unary mth-differentiable function,*

3. In case of getting $k = 0$*, but in fact* $k \neq 0$*, then* $u = gf(v)$ *is not the solution of* $F = 0$*,*

4. In case of working out f, g *and* v *under the condition of* $h(x_1, \cdots x_l) = 0, (l \leq n)$*, then* $u = gf(v)$ *is the solution of* $F = 0$ *under the condition of* $h(x_1, \cdots x_l) = 0$*,*

5. In case of dividing out f *and its partial derivative, also working out* g *and* v *under the condition of* $h(x_1, \cdots x_l) = 0, (l \leq n)$*, then* $u = gf(v)$ *is the solution of* $F = 0$ *under the condition of* $h(x_1, \cdots x_l) = 0$*, and* f *is an arbitrary unary mth-differentiable function.*

Transformational Method 4. *In the domain* $D, (D \subset \mathbb{R}^n)$*, any established mth-order PDE with n variables* $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1 x_2}, \cdots) = 0$*,setting* $g(x_1, \cdots x_n)$ *is known and* $f(v), v(x_1, \cdots x_n)$ *are undetermined,* $g, v \in C^m(D)$*, then substitute* $u = gf(v)$ *and its partial derivatives into* $F = 0$

1. In case of working out f *and* v*, then* $u = gf(v)$ *is the solution of* $F = 0$*,*

2. In case of dividing out f *and its partial derivative, also working out* $v(x_1, \cdots x_n)$*, then* $u = gf(v)$ *is the solution of* $F = 0$*, and* f *is an arbitrary unary mth-differentiable function,*

3. In case of getting $k = 0$*, but in fact* $k \neq 0$*, then* $u = gf(v)$ *is not the solution of* $F = 0$*,*

4. In case of working out f *and* v *under the condition of* $h(x_1, \cdots x_l) = 0, (l \leq n)$*, then* $u = gf(v)$ *is the solution of* $F = 0$ *under the condition of* $h(x_1, \cdots x_l) = 0$*,* *5. In case of dividing out f and its partial derivative, also working out $v(x_1, \cdots x_n)$ under the condition of $h(x_1, \cdots x_l) = 0, (l \le n)$, then $u = gf(v)$ is the solution of $F = 0$ under the condition of $h(x_1, \cdots x_l) = 0$, and f is an arbitrary unary mth-differentiable function.*

Using above four new transformational methods, the general solutions and the exact solutions of the Cauchy problem for the Laplace equation, 2D wave equation, the acoustic wave equation, Helmholtz equation, heat equation and the diffusion equation have been solved. In some cases, the general solutions and the exact solutions of the Cauchy problem for the Poisson equation and Schrödinger equation have been solved too. We also find a singularity of general solutions of Helmholtz equation for the first time, namely the number of arbitrary functions in the general solutions is more than 2.

For research the laws of the general solution of mth-order LPDEs with n variables, we also present 11 Propositions and 4 conjectures in Chapter 3.

Appendix

Appendix A

In (37) it can be proved that if $k_1, l_1 \neq 0$ and $k_1, l_1 \nrightarrow 0$, $c_1 v$ can be described by f_1 and f_2, set

$$k_i = l_i = C_i, (i = 2, 3, \ldots n+1).$$

Then

$$f_1 = f_1 \left(kx_1 + C_2x_2 + \ldots + C_nx_n + C_{n+1}\right),$$

$$f_2 = f_2 \left(-kx_1 + C_2x_2 + \ldots + C_nx_n + C_{n+1}\right),$$

where

$$k = \left(-\frac{a_2C_2^2 + a_3C_3^2 + \ldots + a_nC_n^2 + a_{n+1}C_2C_3}{a_1}\right)^{\frac{1}{2}}.$$

Set

$$\begin{array}{l} Ac_1(kx_1 + C_2x_2 + \ldots + C_nx_n + C_{n+1}) + Bc_1(-kx_1 + C_2x_2 + \ldots + C_nx_n \\ +C_{n+1}) = (A+B)c_1(C_1x_1 + C_2x_2 + \ldots + C_nx_n + C_{n+1}) \Rightarrow C_1 = \frac{A-B}{A+B}k. \end{array}$$

If $A = B \neq 0$, then $\frac{A-B}{A+B} = 0$. If $B = -A + 1$, then

$$\lim_{A\to\infty} \frac{A-B}{A+B} = \lim_{A\to\infty} (2A-1) \to \infty,\ \lim_{A\to\infty} \frac{A-B}{A+B} = \lim_{A\to\infty} (2A-1) \to -\infty.$$

Namely $\frac{A-B}{A+B} \in (-\infty, \infty)$, if $k \neq 0$ and $k \nrightarrow 0$, selecting A, B felicitously, C_1 may equal to arbitrary real number, so $c_1 v$ can be described by f_1, f_2, and

$$\begin{aligned} c_1 v &= c_1 \left(C_1x_1 + C_2x_2 + \ldots + C_nx_n + C_{n+1}\right) \\ &= \tfrac{Ac_1}{A+B} \left(kx_1 + C_2x_2 + \ldots + C_nx_n + C_{n+1}\right) \\ &+ \tfrac{Bc_1}{A+B} \left(-kx_1 + C_2x_2 + \ldots + C_nx_n + C_{n+1}\right), \end{aligned}$$

where $C_1 = \frac{A-B}{A+B}k$.

Appendix B

The calculation of (43) as follows. In (40), set $c_1=0, k_{i_j}=l_{i_j}, (i=1,2,\cdots s, j=2,3,\cdots n+1)$

$$k_{i_1}=\left(-\left(a_2k_{i_2}^2+\ldots+a_nk_{i_n}^2+a_{n+1}k_{i_2}k_{i_3}\right)/a_1\right)^{\frac{1}{2}}. \tag{44}$$

According to (40)-(42)

$$\begin{aligned} u\left(0,x_2,\ldots x_n\right) &= \sum_{i=1}^{s}(f_{1_i}(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}})+f_{2_i}(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}})) \\ &= \sum_{i=1}^{s}\varphi_i(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}), \end{aligned}$$

$$\begin{aligned} u_{x_1}\left(0,x_2,\ldots x_n\right) &= \sum_{i=1}^{s}(k_{i_1}f_{1_i}^{'}(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}})-k_{i_1}f_{2_i}^{'}(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}})) \\ &= \sum_{i=1}^{s}(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}). \end{aligned}$$

We have

$$\begin{aligned} &f_{1_i}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)+f_{2_i}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right) \\ &=\varphi_i\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right), \end{aligned} \tag{323}$$

$$\begin{aligned} &k_{i_1}f_{1_i}^{'}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)-k_{i_1}f_{2_i}^{'}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right) \\ &=\psi_i\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right) \end{aligned} \tag{324}$$

According to (324) we get

$$\begin{aligned} &f_{1_i}(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}})-f_{2_i}(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}) \\ &=\tfrac{1}{k_{i_1}}\int_{k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}}^{k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}}\psi(\xi_i)d\xi_i+ \\ &f_{1_i}(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}})-f_{2_i}(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}) \end{aligned} \tag{325}$$

Combining (323) and (325), then

$$\begin{aligned} &f_{1_i}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right) \\ &=\tfrac{1}{2}\varphi_i\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)+\tfrac{1}{2k_{i_1}}\int_{k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}}^{k_{i_2}x_2+\ldots k_{i_n}x_n+k_{i_{n+1}}}\psi\left(\xi_i\right)d\xi_i \\ &+\tfrac{1}{2}f_{1_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right)-\tfrac{1}{2}f_{2_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right) \\ &\Rightarrow f_{1_i}\left(k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right) \\ &=\tfrac{1}{2}\varphi_i\left(k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)+\tfrac{1}{2k_{i_1}}\int_{k_{i_2}x_{2_0}+k_{i_3}x_{3_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}}^{k_{i_1}x_1+k_{i_2}x_2+\ldots k_{i_n}x_n+k_{i_{n+1}}}\psi\left(\xi_i\right)d\xi_i \\ &+\tfrac{1}{2}f_{1_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right)-\tfrac{1}{2}f_{2_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right) \end{aligned}$$

$$\begin{aligned} &f_{2_i}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right) \\ &=\tfrac{1}{2}\varphi_i\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)-\tfrac{1}{2k_{i_1}}\int_{k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}}^{k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}}\psi\left(\xi_i\right)d\xi_i- \\ &\tfrac{1}{2}f_{1_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right)+\tfrac{1}{2}f_{2_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right) \\ &\Rightarrow f_{2_i}\left(-k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right) \\ &=\tfrac{1}{2}\varphi_i\left(-k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)-\tfrac{1}{2k_{i_1}}\int_{k_{i_2}x_{2_0}+k_{i_3}x_{3_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}}^{-k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}}\psi\left(\xi_i\right)d\xi_i \\ &-\tfrac{1}{2}f_{1_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right)+\tfrac{1}{2}f_{2_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right) \end{aligned}$$

In the conditions of (41) and (42), the exact solution of Eq. (38) is

$$\begin{aligned} u =& \frac{1}{2}\sum_{i=1}^{s}(\varphi_i(k_{i_1}x_1 + k_{i_2}x_2 + \ldots + k_{i_n}x_n + k_{i_{n+1}}) \\ &+ \varphi_i(-k_{i_1}x_1 + k_{i_2}x_2 + \ldots + k_{i_n}x_n + k_{i_{n+1}}) \\ &+ \frac{1}{k_{i_1}}\int_{-k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}}^{k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}} \psi(\xi_i)d\xi_i) \end{aligned} \tag{43}$$

Appendix C

Consider the following initial value problem of Eq. (150) on the condition of (169)

$$u\left(x,y,z,0\right) = e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right) + \varphi_2\left(-\sqrt{-2}x+y+z\right)\right), \tag{172}$$

$$\begin{aligned} &u_t\left(x,y,z,0\right) \\ &= e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right) + \varphi_2\left(-\sqrt{-2}x+y+z\right)\right) \\ &+ \frac{\hbar}{m}e^{x+y+z}\Big(\left(2+\sqrt{-2}\right)\varphi_1'\left(\sqrt{-2}x+y+z\right) + \left(2-\sqrt{-2}\right)\varphi_2'\left(-\sqrt{-2}x+y+z\right)\Big). \end{aligned} \tag{173}$$

Comparing (171) with (172) we have

$$k_1 = k_2 = k_3 = -\frac{1}{c}\ , l_2 = l_3 = l_{12} = l_{13} = 1\ , k_5 = l_5 = l_{15} = 0.$$

Then

$$\begin{aligned} u\left(x,y,z,t\right) =& e^{x+y+z-ck_4t}(h_1(\sqrt{-2}x+y+z+\frac{\hbar}{m}(2+\sqrt{-2})t) \\ &+ h_2(-\sqrt{-2}x+y+z+\frac{\hbar}{m}(2-\sqrt{-2})t)), \end{aligned}$$

$$\begin{aligned} (x,y,z,t) =& -ck_4e^{x+y+z-ck_4t}(h_1(\sqrt{-2}x+y+z+\frac{\hbar}{m}(2+\sqrt{-2})t) \\ &+ h_2(-\sqrt{-2}x+y+z+\frac{\hbar}{m}(2-\sqrt{-2})t)) \\ &+ e^{x+y+z-ck_4t}(\frac{\hbar}{m}(2+\sqrt{-2})h_1'(\sqrt{-2}x+y+z+\frac{\hbar}{m}(2+\sqrt{-2})t) \\ &+ \frac{\hbar}{m}(2-\sqrt{-2})h_2'(-\sqrt{-2}x+y+z+\frac{\hbar}{m}(2-\sqrt{-2})t)). \end{aligned}$$

Therefore

$$\begin{aligned} &u\left(x,y,z,0\right) = e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right) + \varphi_2\left(-\sqrt{-2}x+y+z\right)\right) \\ &= e^{x+y+z}\left(h_1\left(\sqrt{-2}x+y+z\right) + h_2\left(-\sqrt{-2}x+y+z\right)\right) \\ &\Longrightarrow h_1\left(\sqrt{-2}x+y+z\right) = \varphi_1\left(\sqrt{-2}x+y+z\right) \\ &\Longrightarrow h_1\left(\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2+\sqrt{-2}\right)t\right) = \varphi_1\left(\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2+\sqrt{-2}\right)t\right). \end{aligned}$$

Namely

$$h_1\left(\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2+\sqrt{-2}\right)t\right) = \varphi_1\left(\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2+\sqrt{-2}\right)t\right). \tag{326}$$

$$h_2\left(-\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2-\sqrt{-2}\right)t\right) = \varphi_2\left(-\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2-\sqrt{-2}\right)t\right). \tag{327}$$

Thus

$$
\begin{aligned}
&u_t\left(x,y,z,0\right) = e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right)+\varphi_2\left(-\sqrt{-2}x+y+z\right)\right)\\
&+\frac{\hbar}{m}e^{x+y+z}\left(\left(2+\sqrt{-2}\right)\varphi_1'\left(\sqrt{-2}x+y+z\right)+\left(2-\sqrt{-2}\right)\varphi_2'\left(-\sqrt{-2}x+y+z\right)\right)\\
&= -ck_4e^{x+y+z}\left(h_1\left(\sqrt{-2}x+y+z\right)+h_2\left(-\sqrt{-2}x+y+z\right)\right)\\
&+e^{x+y+z}\left(\frac{\hbar}{m}\left(2+\sqrt{-2}\right)h_1'\left(\sqrt{-2}x+y+z\right)+\frac{\hbar}{m}\left(2-\sqrt{-2}\right)h_2'\left(-\sqrt{-2}x+y+z\right)\right)\\
&\Longrightarrow k_4 = -\frac{1}{c}.
\end{aligned}
$$

So the exact solutions of the initial value problem is

$$
\begin{aligned}
u =&e^{x+y+z+t}(\varphi_1(\sqrt{-2}x+y+z+\frac{\hbar}{m}(2+\sqrt{-2})t)\\
&+\varphi_2(-\sqrt{-2}x+y+z+\frac{\hbar}{m}(2-\sqrt{-2})t)).
\end{aligned}
\tag{174}
$$

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