Simulated Bell-like Correlations from Geometric Probability

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Abstract

Simulating Bell correlations by Monte Carlo methods can be time-consuming due to the large number of trials required to produce reliable statistics. For a noisy vector model, formulating the vector threshold crossing in terms of geometric probability can eliminate the need for trials, with inferred probabilities replacing statistical frequencies.

The noisy vector model for simulating quantum correlations

A classical model for producing Bell-like correlations, demonstrated by noisy images of polarized coins, was recently introduced by McEachern [1], who reasoned that quantum correlations arise from a process from which it is possible to obtain only one bit of information per sample. The noisy vector model [2] is a simplified variation of the coin model that satisfies McEachern's one-bit criterion.

Both models exploit detection and post-selection loopholes. The correlations can be estimated through many random trials in the manner of a Monte Carlo experiment. A small but consistent difference from the sinusoidal pattern expected from quantum mechanics becomes apparent after a large number of trials.

The noisy vector model using geometric probability, about to be described, has the advantage of high accuracy while requiring less computation than the Monte Carlo method, and rules out the possibility that the difference might be due to an artifact of pseudo-random number generation. See [3] for C code to calculate the probability functions.

Geometric Probability

One of the earliest problems in geometric probability is Buffon's needle, the aim being to calculate the probability that a needle dropped on a planked floor would cross one of the lines formed by adjacent planks. To demonstrate the concept with a simple case, assume that the length, r, of the needle is less than the width of the planks, so that a needle can only cross one line. Suppose we are given the distance, $x < r$, from the sharp point of the needle to the closest line. Imagine

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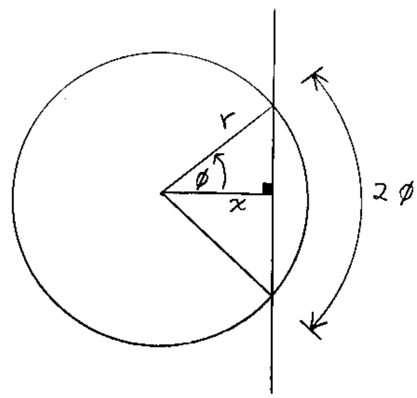
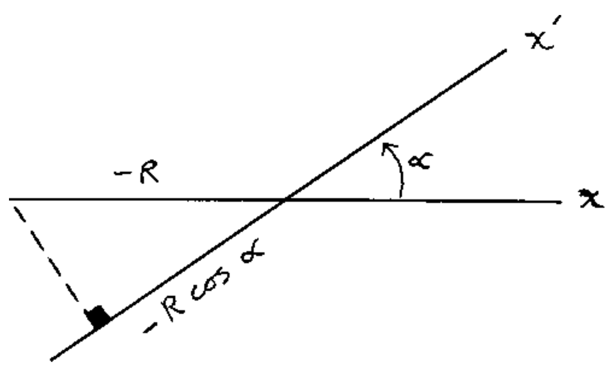
the circle formed by rotating the needle about that point. The probability that the needle will cross the line corresponds to the angular proportion of the circle determined by its intersection with the line, as in the illustration below,

$$P_{\text{cross}} = \frac{2\phi}{2\pi} = \frac{1}{\pi}\arccos\left(\frac{x}{r}\right) \tag{1}$$

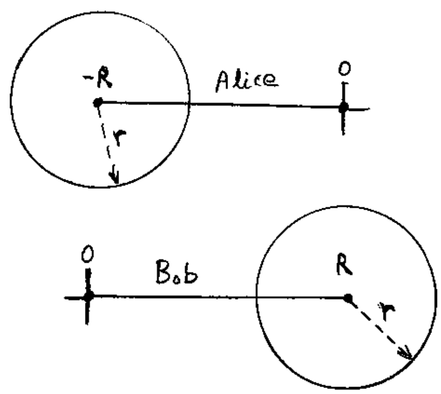


Noisy Vector Model and Geometric Probability

The noisy vector model consists of two parts: a constant signal vector which is always aligned with the x-axis, added to a randomly oriented noise vector. The coordinates of the signal vector sent to Alice are $(-R, 0)$, while the signal sent to Bob is $(R, 0)$. The signal magnitude is taken to be $R = 1$. The noise consists of a randomly oriented vector of constant magnitude, $r = 1/3$, which corresponds to model 3 in [2]. The threshold, $\epsilon = 1/4$, is the minimum value required for a projection measurement to be detected as positive, with a symmetrical condition for negative polarity detection. The projection of Alice's vector onto Alice's instrument rotated by an angle, α, is shown below for the noiseless case.

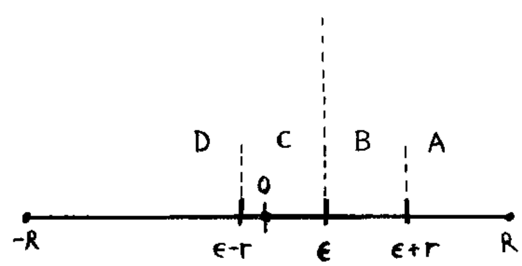


The signal with noise is a vector terminating anywhere on the circular locus of Alice's or Bob's noise vector shown below. The likelihood that the projection onto the instrument axis will cross the threshold amounts to the problem in geometric probability discussed above.



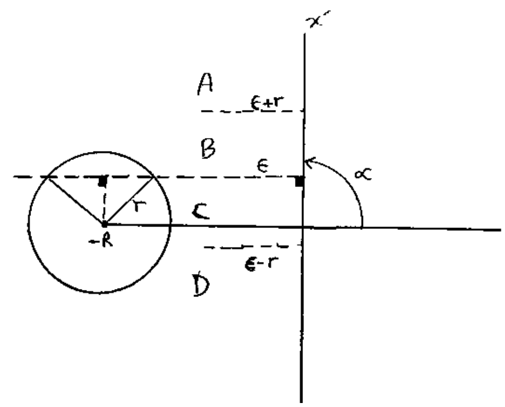
The instrument for detecting positive polarity is shown below. It can be rotated about the origin, and a detection is reported if the projection of the noisy vector meets or exceeds the threshold, ϵ. The markings on the instrument indicate the threshold, and divide the range of the projection into four regions, lettered A-D.

If the projection, p, of the noiseless signal lies in region A ($p \geq \epsilon + r$), the polarity of the noisy vector must be positive. Similarly, projections lying in region D ($p \leq \epsilon - r$) cannot be declared positive. Regions B and C are close enough to the threshold (within r of ϵ) to require more detailed treatment using geometric probability.



Below, Alice has rotated her instrument by $\alpha = 90°$, so that the projection, $p = -R\cos\alpha$, of her noiseless signal is at the origin, lying in region C ($\epsilon - r < p < \epsilon$) of the instrument. In region C, the probability of Alice indicating a positive detection is given by the geometric probability,

$$P(\text{Alice}, \alpha, +) = \frac{1}{\pi} \arccos\Big(\frac{\epsilon - p}{r}\Big) \tag{2}$$



If the projection of the noiseless signal had landed in region B ($\epsilon \le p < \epsilon + r$) the probability of Alice indicating a positive detection would be given by

$$P(\text{Alice}, \alpha, +) = 1 - \frac{1}{\pi} \arccos\Big(\frac{p - \epsilon}{r}\Big) \tag{3}$$

Calculation of Correlation

For the Monte Carlo method, observations were reduced to four totals (N_{++}, N_{+-}, N_{-+}, N_{--}) of detected polarity pairs reported by Alice and Bob for each instance of angular difference, $\theta = \alpha - \beta$. Using probabilities instead, note that Bob's angle $\beta = \alpha - \theta$, and define mean joint probabilities by averaging over all angles, α, in 1° increments as in prior examples,

$$\begin{aligned}
P(\theta, +, +) &= \Sigma_\alpha\, P(\text{Alice}, \alpha, +)\, P(\text{Bob}, \alpha - \theta, +) \,/\, 360 && (4)\\
P(\theta, +, -) &= \Sigma_\alpha\, P(\text{Alice}, \alpha, +)\, P(\text{Bob}, \alpha - \theta, -) \,/\, 360 && (5)\\
P(\theta, -, +) &= \Sigma_\alpha\, P(\text{Alice}, \alpha, -)\, P(\text{Bob}, \alpha - \theta, +) \,/\, 360 && (6)\\
P(\theta, -, -) &= \Sigma_\alpha\, P(\text{Alice}, \alpha, -)\, P(\text{Bob}, \alpha - \theta, -) \,/\, 360 && (7)
\end{aligned}$$

The correlation is then given by

$$C(\theta) = \frac{P(\theta,+,+) \;+\; P(\theta,-,-) \;-\; P(\theta,+,-) \;-\; P(\theta,-,+)}{P(\theta,+,+) \;+\; P(\theta,-,-) \;+\; P(\theta,+,-) \;+\; P(\theta,-,+)} \tag{8}$$

The denominator in Eq. (8) is the detection rate which, when averaged over θ, agrees with the mean detection rate reported by the Monte Carlo method.

Probabilities are shown in Fig. 1 and 2. The correlation is shown in Fig. 3

CHSH Inequality

The Clauser-Horne-Shimony-Holt inequality illustrates the procedure needed to simulate quantum measurement. Alice and Bob make two measurements each, and the inequality is based on four pairs of measurements,

$$\begin{aligned} E(\alpha,\beta) &= C(\alpha-\beta) && (9)\\ E(\alpha,\beta') &= C(\alpha-\beta') && (10)\\ E(\alpha',\beta) &= C(\alpha'-\beta) && (11)\\ E(\alpha',\beta') &= C(\alpha'-\beta') && (12) \end{aligned}$$

where the CHSH statistic

$$S = E(\alpha,\beta) - E(\alpha,\beta') + E(\alpha',\beta) + E(\alpha',\beta') \tag{13}$$

will satisfy the inequality, $|S| \leq 2$, under classical assumptions.

Equivalently, mean joint probabilities could be defined as

$$\begin{aligned} P(\alpha,\beta,+,+) &= \Sigma_\theta\, P(\mathrm{Alice},\alpha+\theta,+)\, P(\mathrm{Bob},\beta+\theta,+)\,/\,360 && (14)\\ P(\alpha,\beta,+,-) &= \Sigma_\theta\, P(\mathrm{Alice},\alpha+\theta,+)\, P(\mathrm{Bob},\beta+\theta,-)\,/\,360 && (15)\\ P(\alpha,\beta,-,+) &= \Sigma_\theta\, P(\mathrm{Alice},\alpha+\theta,-)\, P(\mathrm{Bob},\beta+\theta,+)\,/\,360 && (16)\\ P(\alpha,\beta,-,-) &= \Sigma_\theta\, P(\mathrm{Alice},\alpha+\theta,-)\, P(\mathrm{Bob},\beta+\theta,-)\,/\,360 && (17) \end{aligned}$$

with the measurement, $E(\alpha,\beta)$, calculated in the same manner as the correlation. Because of even symmetry, it is sufficient to average over 180° of phase.

For randomly selected instrument settings, $\alpha,\alpha',\beta,\beta'$, the CHSH inequality is violated in approximately 15% of trials. The CHSH statistic would never violate the inequality if the joint probabilities were not averaged over the phase, θ, in the above four equations - *i.e.,* if the classical joint probability $P(\alpha,\beta,+,+) = P(\mathrm{Alice},\alpha,+)\,P(\mathrm{Bob},\beta,+)$ was used instead of Eq. (14), with a similar simplification for the other three equations. In other words, it is phase averaging that transforms classical measurements covering all possible orientations into the quantum measurement, so that the quantum measurement depends only on the relative phase difference between Alice's and Bob's settings.

The greatest violation of the CHSH statistic for the noisy vector model is $|S| = 2.833$ at $(\alpha,\alpha',\beta,\beta') = (0°, 90°, 45°, 135°)$, which is slightly greater than $2\sqrt{2} \approx 2.828$, the maximum violation expected from quantum theory.

References

[1] McEachern R.H., A Classical System for Producing "Quantum Correlations". (2016)
http://vixra.org/abs/1609.0129

[2] Walker C., How Well Do Classically Produced Correlations Match Quantum Theory? (2017)
http://vixra.org/abs/1701.0621

[3] Walker C., C functions for producing Bell-like correlations using geometric probability. (2017)
https://sites.google.com/site/quantcorr

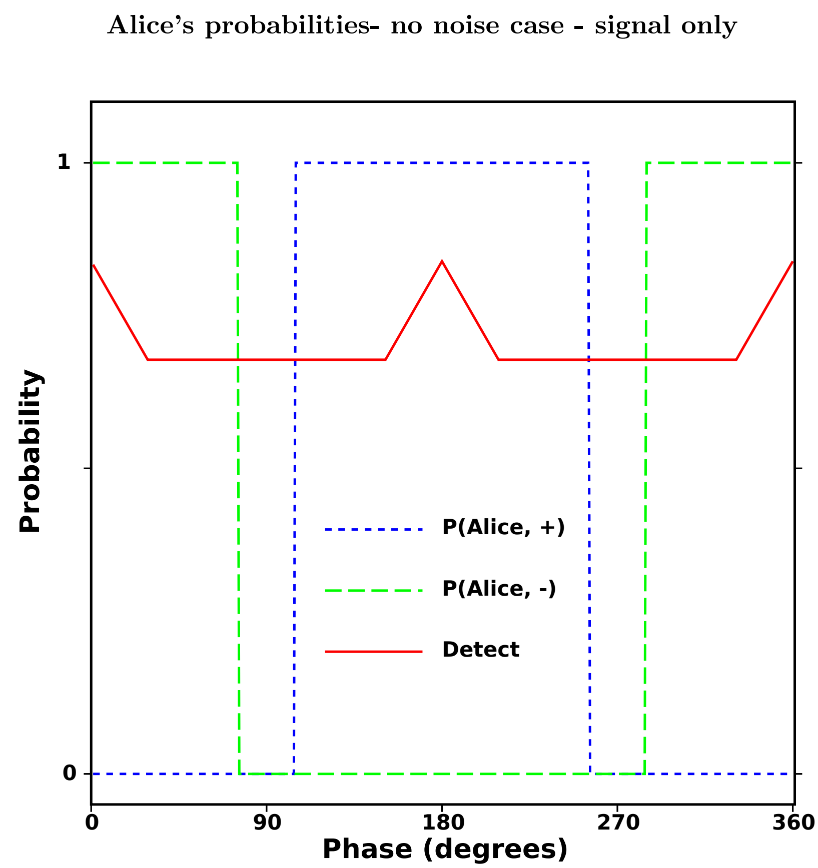


Figure 1: Noisy vector model 3 (signal only), with parameters $R = 1$, $r = 0$, $\epsilon = 1/4$. Probability that Alice will detect a positive (or negative) polarity as a function of instrument angle. No noise, but still has threshold. Solid red curve is joint probability that Alice's measurement and Bob's measurement both exceed the detection threshold.

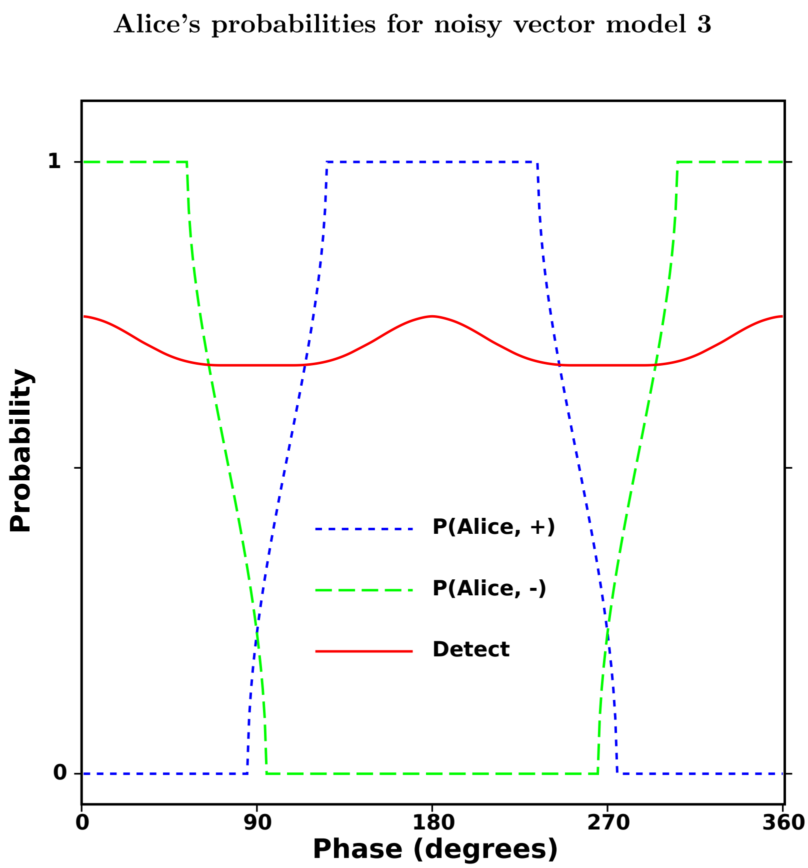


Figure 2: Noisy vector model 3 with parameters $R = 1$, $r = 1/3$, $\epsilon = 1/4$. Geometric probability that Alice will detect a positive (or negative) polarity as a function of instrument angle. Swap curves for Bob's probabilities. The expected average detection rate is 0.6952.

Correlation from geometric probability

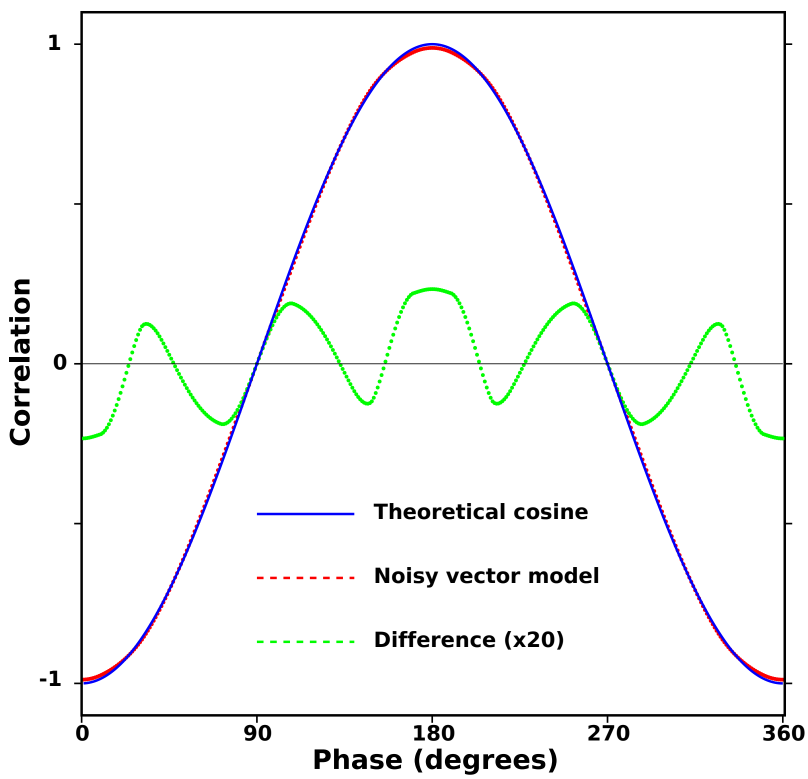


Figure 3: Noisy vector model 3 computed using geometric probability, with parameters $R = 1$, $r = 1/3$, $\epsilon = 1/4$. This shows the correlation and difference from the cosine nearly identical to the Monte Carlo simulation in [2].