An extension to the theory of trigonometric functions as exact periodic solutions to quadratic Liénard type equations

D. K. K. Adjaï [a], L. H. Koudahoun [a], J. Akande [a], Y. J. F. Kpomahou [b] and M. D. Monsia [a1]

[a] *Department of Physics, University of Abomey-Calavi, Abomey-Calavi, 01.B.P. 526, Cotonou, BENIN*
E-mail: monsiadelphin@yahoo.fr

[b] *Department of Industrial and Technical Sciences, ENSET-Lokossa, University of Abomey, Abomey, BENIN*
E-mail : kpomaf@yahoo.fr

This paper slightly extends the theory of exact trigonometric periodic solutions to quadratic Liénard type equations introduced earlier by the authors of the present contribution. The extended theory is used to determine the general periodic solutions to the Duffing equation and to some Painlevé-Gambier equations as illustrative examples. Finally the mathematical equivalence between the Duffing equation and the Painlevé-Gambier XIX equation has been highlighted by means of the proposed extended theory.

Keywords: Duffing equation, Painlevé-Gambier equations, Liénard equations, periodic solution, Jacobian elliptic functions.

1 Introduction

The fundamental problem of finding general periodic solutions to nonlinear differential equations constitutes yet an active research field of mathematics since such solutions are only possible for a few number of nonlinear differential equations. As many practical problems are modeled in terms of nonlinear equations, there appears then logic to be interested to mathematical theories which may provide with a relative simplicity general periodic solutions to these equations. In the matter the theory of exact trigonometric periodic solutions to quadratic Liénard type differential equations introduced recently by the authors of this paper seems to belong to this class of mathematical theories [1]. Indeed, this theory has the ability to linearize some classes of Liénard type nonlinear differential equations in order to establish exact analytical solutions by means of the generalized Sundman transformation, and conversely to highlight some families of Liénard type equations from linear differential equations. So an interesting finding of this theory was the detection of a general class of quadratic Liénard type equations whose solutions are trigonometric periodic functions. In this regard it is appropriate to state the fundamental question: Is it possible to extend this theory by using a forcing function? The present work assumes this possibility. This fact may give the advantage to detect large classes of Liénard type equations for which exact analytical solutions may be calculated. To do so, a brief review of the theory introduced by Monsia and his coworkers is given (section 2) and secondly, the extended theory under question is carried out (section 3). Finally the general periodic solutions to Duffing equation and some Painlevé-Gambier equations are determined (section 4) and a conclusion for the research contribution is formulated.

[1]Corresponding author.E-mail: monsiadelphin@yahoo.fr

2 Theory by Akande et al. [1]

Let us consider the general second order linear differential equation as

$$y''(\tau) + by'(\tau) + a^2 y(\tau) = 0 \tag{2.1}$$

where prime denotes differentiation with respect to τ, a and b are arbitrary parameters. By application of the generalized Sundman transformation

$$y(\tau) = F(t,x), \qquad d\tau = G(t,x)dt, \qquad G(t,x)\frac{\partial F(t,x)}{\partial x} \neq 0 \tag{2.2}$$

with

$$F(t,x) = \int g(x)^l dx, \qquad G(t,x) = \exp(\gamma\varphi(x))$$

where l and γ are arbitrary parameters, and $g(x) \neq 0$ and $\varphi(x)$ are arbitrary functions of x , to (2.1), one may obtain

$$\ddot{x} + (l\frac{g'(x)}{g(x)} - \gamma\varphi'(x))\dot{x}^2 + b\dot{x}\exp(\gamma\varphi(x)) + \frac{a^2\exp(2\gamma\varphi(x))\int g(x)^l dx}{g(x)^l} = 0 \tag{2.3}$$

as general class of mixed Liénard type differential equations.

The parametric choice $b = 0$ leads to

$$\ddot{x} + (l\frac{g'(x)}{g(x)} - \gamma\varphi'(x))\dot{x}^2 + \frac{a^2\exp(2\gamma\varphi(x))\int g(x)^l dx}{g(x)^l} = 0 \tag{2.4}$$

as general class of quadratic Liénard type nonlinear equations. An interesting case of (2.4) is obtained by considering $l = 0$, viz

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + a^2 x\exp(2\gamma\varphi(x)) = 0 \tag{2.5}$$

The importance of (2.5) is that it exhibits trigonometric functions as exact periodic solutions but with amplitude-dependent frequency. In [1] it is shown for the first time that some existing Liénard type equations may exhibit exact trigonometric periodic solutions. One may also find that the quadratic Liénard type equation

$$\ddot{x} + \frac{1}{2}\frac{\dot{x}^2}{1+x} + \frac{x}{1+x} = 0 \tag{2.6}$$

used to model oscillation of a liquid column in a U-tube [2] belongs to the class of equations defined by (2.5) by putting $\varphi(x) = \frac{1}{2}\ln(1+x)$, and $\gamma = -1$ [3], so that this equation may exhibit trigonometric functions as exact periodic solution [3]. That being so the extended theory may be formulated.

3 Extended theory

The objective of this section is to extend the preceding theory to look for large classes of linearizable Liénard type equations in order to ensure exact solutions which may be expressed explicitly or by quadratures. In the previous theory the general second order linear equation is considered in the form of homogeneous equation that is with no forcing function. Instead in this section, let us consider the general second order linear equation (2.1) with a constant forcing term, that is

$$y''(\tau) + by'(\tau) + a^2 y(\tau) = c \tag{3.1}$$

where c is an arbitrary parameter. The application of the generalized Sundman transformation (2.2) to (3.1) yields

$$\ddot{x} + (l\frac{g'(x)}{g(x)} - \gamma\varphi'(x))\dot{x}^2 + b\dot{x}\exp(\gamma\varphi(x)) + \frac{a^2\exp(2\gamma\varphi(x))\int g(x)^l dx}{g(x)^l} - \frac{c\exp(2\gamma\varphi(x))}{g(x)^l} = 0 \tag{3.2}$$

The comparison of (2.3) with (3.2) shows that the constant forcing function c contributes to the general class of mixed Liénard type equation (2.3) by an additional term $-\frac{c\exp(2\gamma\varphi(x))}{g(x)^l}$. This establishes the extension of the theory by Akande et al. [1] to a wider class of Liénard type nonlinear differential equations. The question now is to know: What happens to equations (2.4) and (2.5) with respect to this additional term? Let us consider then $b = 0$ and $l = 0$. The parametric choice $b = 0$ leads to

$$\ddot{x} + (l\frac{g'(x)}{g(x)} - \gamma\varphi'(x))\dot{x}^2 + \frac{a^2\exp(2\gamma\varphi(x))\int g(x)^l dx}{g(x)^l} - \frac{c\exp(2\gamma\varphi(x))}{g(x)^l} = 0 \tag{3.3}$$

The comparison of (2.4) with (3.3) shows also the persistence of the preceding additional term in the general class of quadratic Liénard type equations represented by (3.3). The application of $l = 0$ to (3.3) gives the class of quadratic Liénard type equations

$$\ddot{x} - \gamma\varphi'(x)\dot{x}^2 + a^2 x\exp(2\gamma\varphi(x)) - c\exp(2\gamma\varphi(x)) = 0 \tag{3.4}$$

which differs from the class of quadratic Liénard type equations (2.5) by the additional term $-c\exp(2\gamma\varphi(x))$. By noting

$$y(\tau) = A_0 sin(a\tau + \alpha) + \frac{c}{a^2} \tag{3.5}$$

where $a \neq 0$, the general solution to (3.1) with $b = 0$, the general solution to (3.4) becomes

$$x(t) = A_0 sin(a\phi(t) + \alpha) + \frac{c}{a^2} \tag{3.6}$$

where A_0 and α are arbitrary parameters, and $\tau = \phi(t)$ satisfies

$$dt = \exp(-\gamma\varphi(x))d\phi(t) \tag{3.7}$$

The solution (3.6) remains of periodic form. For a convenient choice of $\varphi(x)$ and γ the solution (3.6) may exhibit harmonic periodic behavior but with a slift factor $\frac{c}{a^2}$. This extends therefore the theory by Akande et al. [1]. It is then convenient to show the ability of the current theory to provide general periodic solutions to some existing Liénard equations.

4 Applications of theory

This section is devoted to show the usefulness of the present theory by considering some well known nonlinear equations.

4.1 Duffing equation

The Duffing equation is one of the most investigated equations from mathematical as well as physical viewpoint. The motivation results from the fact that this equation arises in mathematical modeling of many problems of mechanics and quantum mechanics, for example. The Duffing equation is of the form [4]

$$\ddot{x} + \omega_0^2 x + \beta x^3 = 0 \tag{4.1}$$

where ω_0 and β are arbitrary parameters.

From the mechanical viewpoint, the parameter $\beta > 0$ is called hardening parameter while $\beta < 0$ is called softening parameter, and $\beta = 0$, gives the well known linear harmonic oscillator equation. The equation (4.1) may be recovered from the general class of quadratic Liénard equations (3.3). Indeed, the functional choice $\varphi(x) = \ln(f(x))$ leads to

$$\ddot{x} + (l\frac{g'(x)}{g(x)} - \gamma\frac{f'(x)}{f(x)})\dot{x}^2 + \frac{a^2 f(x)^{2\gamma}\int g(x)^l dx}{g(x)^l} - \frac{cf(x)^{2\gamma}}{g(x)^l} = 0 \tag{4.2}$$

which for $f(x) = x^2$ and $g(x) = x$, yields

$$\ddot{x} + (l - 2\gamma)\frac{\dot{x}^2}{x} + \frac{a^2}{l+1}x^{4\gamma+1} - cx^{4\gamma-l} = 0 \tag{4.3}$$

The parametric choice $l = 2\gamma = 1$, reduces (4.3) to

$$\ddot{x} - cx + \frac{a^2}{2}x^3 = 0 \tag{4.4}$$

which is the Duffing equation (4.1) by noting $c = -\omega_0^2$, and $\frac{a^2}{2} = \beta$. So the general solution to the Duffing equation (4.1) may be written according to (3.5)

$$x(t) = \varepsilon\sqrt{\frac{-\omega_0^2}{\beta} + 2A_0\sin(a\phi(t) + \alpha)} \tag{4.5}$$

where

$$\varepsilon dt = \frac{d\phi(t)}{\sqrt{\frac{-\omega_0^2}{\beta} + 2A_0\sin(a\phi(t) + \alpha)}} \tag{4.6}$$

and $\varepsilon = \pm 1$. The evaluation of the integral resulting from (4.6)

$$J = \int \frac{d\phi(t)}{\sqrt{\frac{-\omega_0^2}{\beta} + 2A_0\sin(a\phi(t) + \alpha)}} \tag{4.7}$$

depends on the value of ω_0^2 and β. Therefore three distinct cases are considered in this paper [5].

Case 1: $\beta > 0$, and $\omega_0^2 < 0$

In this case where $\beta > 0$ and $\omega_0^2 < 0$, it is moreover assumed that $-\frac{\omega_0^2}{\beta} > 2A_0 > 0$. By noting [5]

$$p = 2\sqrt{\frac{A_0}{\frac{-\omega_0^2}{\beta} + 2A_0}}$$

that is

$$p = 2\sqrt{\frac{\beta A_0}{2\beta A_0 - \omega_0^2}} \tag{4.8}$$

and

$$\delta = \arcsin\sqrt{\frac{1 - \sin(a\phi + \alpha)}{2}} \tag{4.9}$$

the integral J becomes [5]

$$J = -\frac{1}{a}\frac{p}{\sqrt{A_0}}F(\delta, p) \tag{4.10}$$

where $F(\delta, p)$ is the elliptic integral of the first kind so that using (4.6)

$$\sin\delta = sn\left(-\frac{a\varepsilon\sqrt{A_0}}{p}(t + C), p\right) \tag{4.11}$$

where $sn(z, k)$ designates a Jacobian elliptic function and C an arbitrary parameter. In this way

$$\sqrt{\frac{1 - \sin(a\phi + \alpha)}{2}} = sn\left(-\frac{a\varepsilon\sqrt{A_0}}{p}(t + C), p\right)$$

that is to say

$$\sin(a\phi + \alpha) = 1 - 2sn^2\left(-\frac{a\varepsilon\sqrt{A_0}}{p}(t + C), p\right) \tag{4.12}$$

from which the general solution to the Duffing equation (4.1) becomes

$$x(t) = \varepsilon \left[-\frac{\omega_0^2}{\beta} + 2A_0 - 4A_0 sn^2(-\frac{a\varepsilon\sqrt{A_0}}{p}(t+C), p) \right]^{\frac{1}{2}} \tag{4.13}$$

which reduces to

$$x(t) = \frac{2\varepsilon\sqrt{A_0}}{p} \left[1 - p^2 sn^2(-\frac{a\varepsilon\sqrt{A_0}}{p}(t+C), p) \right]^{\frac{1}{2}} \tag{4.14}$$

Using the identity $k^2 sn^2(z,k) + dn^2(z,k) = 1$, the relation (4.14) becomes immediately

$$x(t) = \frac{2\varepsilon\sqrt{A_0}}{p} dn \left(\frac{a\varepsilon\sqrt{A_0}}{p}(t+C), p \right) \tag{4.15}$$

Knowing that $a = \varepsilon\sqrt{2\beta}$, (4.15) may be written in the form

$$x(t) = \frac{2\varepsilon\sqrt{A_0}}{p} dn \left(\frac{\sqrt{2\beta A_0}}{p}(t+C), p \right) \tag{4.16}$$

By making $A = \frac{2\sqrt{A_0}}{p}$, and $\Omega = \frac{\sqrt{2\beta A_0}}{p}$, that is $\Omega = \frac{A}{2}\sqrt{2\beta}$, the general solution to (4.1) definitively reads

$$x(t) = \varepsilon A dn\left(\Omega(t+C), p\right) \tag{4.17}$$

where $dn(z,k)$ is a Jacobian elliptic function, and $p^2 = \frac{2(\beta A^2 + \omega_0^2)}{\beta A^2}$.

Case 2: $\beta < 0$, and $\omega_0^2 > 0$

For $\beta < 0$ and $\omega_0^2 > 0$, it is also assumed that $-\frac{\omega_0^2}{\beta} > 2A_0 > 0$. So, the general solution to the Duffing equation (4.1) takes, using (4.17), the form

$$x(t) = \varepsilon A dc\left(\Omega(t+C), \sqrt{1-p^2}\right) \tag{4.18}$$

where $A = \frac{2\sqrt{A_0}}{p}$, and $\Omega = \frac{A}{2}\sqrt{2|\beta|}$. The function dc is a Jacobian elliptic function.

Case 3: $\beta > 0$, and $\omega_0^2 > 0$

In this case, it is assumed that $0 < |-\frac{\omega_0^2}{\beta}| < 2A_0$, that is $0 < \frac{\omega_0^2}{\beta} < 2A_0$. So, the integral J may be written as [5]

$$J = -\frac{\sqrt{A_0}}{aA_0} F(\delta, \frac{1}{p}) \tag{4.19}$$

where

$$\delta = \arcsin\left[\sqrt{\frac{2\beta A_0(1-\sin(a\phi+\alpha))}{2\beta A_0 - \omega_0^2}} \right] \tag{4.20}$$

p has the preceding value, and $F(\delta, \frac{1}{p})$ denotes the elliptic integral of the first kind. In this regard the following relationship may be written, taking into consideration (4.6)

$$\sqrt{\frac{2\beta A_0(1-\sin(a\phi+\alpha))}{2\beta A_0 - \omega_0^2}} = sn\left[-a\varepsilon\sqrt{A_0}(t+C), \frac{1}{p} \right] \tag{4.21}$$

that is

$$\sin(a\phi+\alpha) = 1 - \frac{2\beta A_0 - \omega_0^2}{2\beta A_0} sn^2\left[-a\varepsilon\sqrt{A_0}(t+C), \frac{1}{p} \right] \tag{4.22}$$

which may be rewritten

$$\sin(a\phi + \alpha) = 1 - \frac{2}{p^2} sn^2 \left(-a\varepsilon\sqrt{A_0}(t + C), \frac{1}{p} \right) \tag{4.23}$$

In this perspective the general solution to (4.1) reads

$$x(t) = \varepsilon\sqrt{-\frac{\omega_0^2}{\beta} + 2A_0 - \frac{4A_0}{p^2} sn^2 \left(-a\varepsilon\sqrt{A_0}(t + C), \frac{1}{p} \right)}$$

which may take the expression

$$x(t) = \frac{2\varepsilon\sqrt{A_0}}{p} cn \left(a\sqrt{A_0}(t + C), \frac{1}{p} \right) \tag{4.24}$$

Knowing $a = \varepsilon\sqrt{2\beta}$, $A = \frac{2\sqrt{A_0}}{p}$, and $\Omega = a\sqrt{A_0}$, that is $\Omega = \frac{pA\sqrt{2\beta}}{2}$, or $\Omega = \sqrt{\beta A^2 + \omega_0^2}$, the general solution to Duffing equation (4.1) reduces immediately to the form

$$x(t) = \varepsilon A cn \left(\Omega(t + C), \frac{1}{p} \right) \tag{4.25}$$

where $cn(z, k)$ denotes a Jacobian elliptic function It would be now interesting to consider in the sequel of this work some Painlevé-Gambier equations.

4.2 Painlevé-Gambier XII equation

This subsection is intended to carry out the general solution to the Painlevé-Gambier XII equation [6]

$$\ddot{x} - \frac{\dot{x}^2}{x} - qx^3 - \beta x^2 - r - \frac{\delta}{x} = 0 \tag{4.26}$$

for $r = \delta = 0$, that is to say

$$\ddot{x} - \frac{\dot{x}^2}{x} - qx^3 - \beta x^2 = 0 \tag{4.27}$$

The Painlevé-Gambier equation (4.27) may be found from (3.4) by choosing $\varphi(x) = \ln x^2$, $\gamma = \frac{1}{2}$, $a^2 = -q$, and $c = \beta$. Therefore the general solution to (4.27) takes the expression

$$x(t) = A_0 \sin(\varepsilon\sqrt{-q}\phi(t) + \alpha) - \frac{\beta}{q} \tag{4.28}$$

where $\varepsilon = \pm 1$, and $dt = \frac{d\phi(t)}{x(t)}$ that is

$$dt = \frac{d\phi(t)}{A_0 \sin(\varepsilon\sqrt{-q}\phi(t) + \alpha) - \frac{\beta}{q}} \tag{4.29}$$

By integration the quantity

$$J = \int \frac{d\phi(t)}{A_0 \sin(\varepsilon\sqrt{-q}\phi(t) + \alpha) - \frac{\beta}{q}} \tag{4.30}$$

leads to consider two distinct cases following the value of $\frac{qA_0}{\beta}$.

Case 1: $\frac{q^2A_0^2}{\beta^2} < 1$

In this case the integral J becomes [5]

$$J = -\frac{2q}{\beta\varepsilon\sqrt{\frac{q^3A_0^2}{\beta^2} - q}} tg^{-1} \left[\frac{tg(\frac{(\varepsilon\sqrt{-q}\phi(t)+\alpha)}{2}) - \frac{qA_0}{\beta}}{\sqrt{1 - \frac{q^2A_0^2}{\beta^2}}} \right] \tag{4.31}$$

such that

$$\varepsilon\sqrt{-q}\phi(t)+\alpha = 2tg^{-1}\left[\sqrt{1-\frac{q^2A_0^2}{\beta^2}}tg\left(-\frac{\beta\varepsilon\sqrt{\frac{q^3A_0^2}{\beta^2}-q}(t+C)}{2q}\right)+\frac{qA_0}{\beta}\right] \tag{4.32}$$

So the general solution to (4.27) becomes

$$x(t) = A_0\sin\left[2tg^{-1}\left[\sqrt{1-\frac{q^2A_0^2}{\beta^2}}tg\left(-\frac{\beta\varepsilon\sqrt{\frac{q^3A_0^2}{\beta^2}-q}(t+C)}{2q}\right)+\frac{qA_0}{\beta}\right]\right]-\frac{\beta}{q} \tag{4.33}$$

The solution $x(t)$ is real for $q<0$. In this way, the solution (4.33) may clearly exhibit a harmonic periodic behavior with a shift factor $-\frac{\beta}{q}$.

Case 2: $\frac{q^2A_0^2}{\beta^2} > 1$

This case corresponds to [5]

$$J = -\frac{q}{\beta\varepsilon\sqrt{q-\frac{q^3A_0^2}{\beta^2}}}\ln\left[\frac{tg(\frac{(\varepsilon\sqrt{-q}\phi(t)+\alpha)}{2})-\frac{qA_0}{\beta}-\sqrt{\frac{q^2A_0^2}{\beta^2}-1}}{tg(\frac{(\varepsilon\sqrt{-q}\phi(t)+\alpha)}{2})-\frac{qA_0}{\beta}+\sqrt{\frac{q^2A_0^2}{\beta^2}-1}}\right] \tag{4.34}$$

which gives according to (4.29)

$$\varepsilon\sqrt{-q}\phi(t)+\alpha = 2tg^{-1}\left[\frac{\sqrt{\frac{q^2A_0^2}{\beta^2}-1}\left(1+\exp(-\frac{\beta\varepsilon}{q}\sqrt{q-\frac{q^3A_0^2}{\beta^2}}(t+C))\right)}{(1-\exp(-\frac{\beta\varepsilon}{q}\sqrt{q-\frac{q^3A_0^2}{\beta^2}}(t+C)))}+\frac{qA_0}{\beta}\right] \tag{4.35}$$

Therefore the general solution $x(t)$ may be expressed in the form

$$x(t) = A_0\sin\left[2tg^{-1}\left[\frac{\sqrt{\frac{q^2A_0^2}{\beta^2}-1}\left(1+\exp\left(-\frac{\beta\varepsilon}{q}\sqrt{q-\frac{q^3A_0^2}{\beta^2}}(t+C)\right)\right)}{\left(1-\exp(-\frac{\beta\varepsilon}{q}\sqrt{q-\frac{q^3A_0^2}{\beta^2}}(t+C))\right)}+\frac{qA_0}{\beta}\right]\right]-\frac{\beta}{q} \tag{4.36}$$

For $q<0$, the solution (4.36) is real and then may exhibit also a harmonic periodic behavior with a shift factor $-\frac{\beta}{q}$.

Consider now as final illustrative example a generalized Painlevé-Gambier XIX equation.

4.3 Generalized Painlevé-Gambier XIX equation

The Painlevé-Gambier XIX equation is of the form [6]

$$\ddot{x}-\frac{1}{2}\frac{\dot{x}^2}{x}-4x^2-2x=0 \tag{4.37}$$

A general form of (4.37) may be written as

$$\ddot{x}-\frac{1}{2}\frac{\dot{x}^2}{x}+a^2x^2-cx=0 \tag{4.38}$$

so that for $a^2=-4$, and $c=2$, one may recover the Painlevé-Gambier XIX equation. The equation (4.38) may be obtained from (3.4) by setting $\gamma=\frac{1}{4}$, and $\varphi(x)=\ln x^2$. Thus the solution to (4.38) immediately takes the expression following (3.5)

$$x(t) = A_0\sin(a\phi(t)+\alpha)+\frac{c}{a^2} \tag{4.39}$$

where

$$dt = \frac{d\phi(t)}{\sqrt{A_0 \sin(a\phi(t) + \alpha) + \frac{c}{a^2}}} \tag{4.40}$$

This equation is of the same form as (4.6). So, three distinct cases may be distinguished

Case 1: $a^2 > 0$, $c > 0$, and $\frac{c}{a^2} > A_0 > 0$

In this case [5]

$$p = \sqrt{\frac{2a^2 A_0}{a^2 A_0 + c}} \tag{4.41}$$

and

$$\delta = \arcsin\left(\sqrt{\frac{1 - \sin(a\phi + \alpha)}{2}}\right) \tag{4.42}$$

so the integral

$$J = \int \frac{d\phi(t)}{\sqrt{A_0 \sin(a\phi(t) + \alpha) + \frac{c}{a^2}}} \tag{4.43}$$

becomes

$$J = -\frac{p\sqrt{2}}{a\sqrt{A_0}} F(\delta, p) \tag{4.44}$$

where $F(\delta, p)$ designates the elliptic integral of the first kind. So according to (4.40) one may write

$$-\frac{a\sqrt{A_0}}{p\sqrt{2}}(t + C) = F(\delta, p)$$

that is

$$\sin\delta = sn\left(-\frac{a\sqrt{2A_0}}{2p}(t + C), p\right) \tag{4.45}$$

In this regard, the general solution (4.39) becomes

$$x(t) = \frac{2A_0}{p^2} dn^2\left(\frac{a\sqrt{2A_0}}{2p}(t + C), p\right) \tag{4.46}$$

which may take the expression

$$x(t) = A^2 dn^2\left[\Omega(t + C), p\right] \tag{4.47}$$

where $A = \frac{\sqrt{2A_0}}{p}$, $\Omega = \frac{a\sqrt{2A_0}}{2p}$ that is $\Omega = \frac{aA}{2}$, and p may be rewritten as $p^2 = \frac{2(a^2A^2 - c)}{a^2A^2}$

Case 2: $a^2 < 0$, $c > 0$ and $0 < |\frac{c}{a^2}| < A_0$

In such a situation, the integral [5]

$$J = -\frac{1}{a}\sqrt{\frac{2}{A_0}} F(\delta, \frac{1}{p}) \tag{4.48}$$

where $\delta = \arcsin\left(\sqrt{\frac{a^2A_0(1 - \sin(a\phi + \alpha))}{a^2A_0 + c}}\right)$, and $p = \sqrt{\frac{2a^2A_0}{a^2A_0 + c}}$. So the use of (4.40) leads to

$$-a\sqrt{\frac{A_0}{2}}(t + C) = F(\delta, \frac{1}{p}) \tag{4.49}$$

which may give

$$\sqrt{\frac{a^2A_0(1-\sin(a\phi+\alpha))}{a^2A_0+c}} = sn\left(-a\sqrt{\frac{A_0}{2}}(t+C), \frac{1}{p}\right) \tag{4.50}$$

from which

$$\sin(a\phi+\alpha) = 1 - \frac{a^2A_0+c}{a^2A_0}sn^2\left(-a\sqrt{\frac{A_0}{2}}(t+C), \frac{1}{p}\right) \tag{4.51}$$

Therefore the general solution (4.39) may take the expression

$$x(t) = \frac{2A_0}{p^2}\left[1 - sn^2\left(-a\sqrt{\frac{A_0}{2}}(t+C), \frac{1}{p}\right)\right] \tag{4.52}$$

The general solution (4.52) may be also expressed as

$$x(t) = \frac{2A_0}{p^2}cn^2\left(a\sqrt{\frac{A_0}{2}}(t+C), \frac{1}{p}\right) \tag{4.53}$$

which becomes

$$x(t) = A^2cn^2\left(\frac{apA}{2}(t+C), \frac{1}{p}\right) \tag{4.54}$$

where $A^2 = \frac{2A_0}{p^2}$.

For $a^2 = i^2|a^2|$, that is $a = \pm i\sqrt{|a^2|}$, the general solution $x(t)$ may take the form

$$x(t) = A^2cn^2\left(\frac{i\sqrt{|a^2|}pA}{2}(t+C), \frac{1}{p}\right)$$

that is

$$x(t) = \frac{A^2}{cn^2\left[\frac{\sqrt{|a^2|}pA}{2}(t+C), \sqrt{1-\frac{1}{p^2}}\right]} \tag{4.55}$$

which becomes definitively

$$x(t) = \frac{A^2}{cn^2\left[\Omega(t+C), \sqrt{1-\frac{1}{p^2}}\right]} \tag{4.56}$$

where $\Omega = \frac{\sqrt{|a^2|}pA}{2}$.

By making $a^2 = -4$, and $c = 2$, the exact doubly periodic solution to Painlevé-Gambier XIX equation may be written as

$$x(t) = \frac{A^2}{cn^2\left[\sqrt{2A^2+1}(t+C), \sqrt{\frac{A^2+1}{2A^2+1}}\right]} \tag{4.57}$$

that is

$$x(t) = \frac{A^2}{cn^2\left[\Omega(t+C), \sqrt{\frac{A^2+1}{2A^2+1}}\right]} \tag{4.58}$$

where $A^2 = \frac{2A_0-1}{2}$, and $p = \frac{1}{A}\sqrt{2A^2+1}$

Case 3: $a^2 > 0$, $c < 0$ and $0 < |\frac{c}{a^2}| < A_0$

This case corresponds also to $p = \sqrt{\frac{2a^2A_0}{a^2A_0+c}}$, $\delta = \arcsin\left(\sqrt{\frac{a^2A_0(1-\sin(a\phi+\alpha))}{a^2A_0+c}}\right)$, and the integral

$$J = -\frac{1}{a}\sqrt{\frac{2}{A_0}}F(\delta, \frac{1}{p}) \tag{4.59}$$

So the general solution (4.39) may be written as

$$x(t) = \frac{2A_0}{p^2} cn^2 \left(a\sqrt{\frac{A_0}{2}}(t + C), \frac{1}{p} \right) \tag{4.60}$$

that is

$$x(t) = A^2 cn^2 \left(\Omega(t + C), \frac{1}{p} \right) \tag{4.61}$$

where

$$A^2 = \frac{2A_0}{p^2} \tag{4.62}$$

and

$$\Omega = \frac{apA}{2} \tag{4.63}$$

The parameter p may also be expressed as $p^2 = \frac{2(a^2A^2 - c)}{a^2A^2}$.

That being so it is then possible to show the equivalence between the Duffing equation and the generalized Painlevé-Gambier XIX equation.

5 Equivalence between equations

This section is devoted to highlight the mathematical equivalence between the Duffing equation and the generalized Painlevé-Gambier XIX equation. The comparison of (4.6) with (4.40) as well as the comparison of resulting general solutions suggest this mathematical equivalence, that is to say the mapping of the Duffing equation onto the generalized Painlevé-Gambier XIX equation and vice versa, the mapping of the generalized Painlevé-Gambier XIX equation into the Duffing equation. In other words, the general solution to the Duffing equation may be obtained in terms of the solution to the generalized Painlevé-Gambier XIX equation and vice versa, the solution to the generalized Painlevé-Gambier equation may be calculated in terms of the general solution to Duffing equation. Formally consider the variable transformation

$$x^2 = 2W \tag{5.1}$$

due to the above general solutions to Duffing equation and general solutions to the generalized Painlevé-Gambier equation (4.38). The substitution of (5.1) into (4.1) yields

$$\ddot{W} - \frac{1}{2}\frac{\dot{W}^2}{W} + 4\beta W^2 + 2\omega_0^2 W = 0 \tag{5.2}$$

which is the generalized Painlevé-Gambier XIX equation (4.38) by taking $a^2 = 4\beta$, and $c = -2\omega_0^2$. So with that the equivalence between Duffing equation and the generalized Painlevé-Gambier XIX equation has been shown and a conclusion may be formulated for the present work.

Conclusion

The vital problem of finding exact periodic solutions to nonlinear differential equations is still an active research field of mathematics. A slight extension of an earlier Liénard type nonlinear differential equations theory is introduced in this paper for the determination of exact periodic solutions as well as exact trigonometric periodic solutions. By doing so the general solution to the Duffing equation as well as for some Painlevé-Gambier equations are in a clear and simple fashion determined. In this perspective as an interesting result, it has been shown that the Duffing equation is mathematically equivalent to the generalized Painlevé-Gambier XIX equation such that the solution of the last equation may be obtained in terms of the solution of the first and vice versa, the solution of the first may be determined in terms of the solution of the last equation.

References

[1] J. Akande , D.K.K. Adjaï, L.H. Koudahoun, Y.J.F. Kpomahou, M.D. Monsia, Theory of exact trigonometric periodic solutions to quadratic Liénard type equations, viXra: 1704.0199V3 (2017).

[2] W.R.Utz, periodic solutions of a nonlinear second order differential equation, SIAM J. Appl. Math, 19(1), 1970.

[3] J. Akande, D.K.K. Adjaï, L.H. Koudahoun, Biswanath Rath, Pravanjan Mallick, Rati Ranjan Sahoo, Y.J.F. Kpomahou, Marc D. Monsia, Exact quantum mechanics of quadratic Liénard type oscillator equations with bound states energy spectrum, viXra: 1702.0242V1 (2017).

[4] A.H. Nayfeh and D.T. Mook, Nonlinear Oscillations (John Wiley and Sons (New York), 1979).

[5] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Seventh edition, 2007.

[6] E.L. Ince, Ordinary differential equations, Dover, New York 1956.