Improved First Estimates to the Solution of Kepler's Equation

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The manuscripts provides a novel starting guess for the solution of Kepler's equation for unknown eccentric anomaly E given the eccentricity e and the mean anomaly M of an elliptical orbit.

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KEPLER'S EQUATION

Mean and Eccentric Anomaly

The track of the orbit for a 2-body potential proportional to the inverse distance of the two bodies leads to solutions which may be ellipses with eccentricity $0 \le e \le 1$. The time dependence is described by the parameter M of the mean anomaly, which is an angle measured from the center of the ellipse, and which is a product of a parameter n called the mean motion (essentially the square root of the coupling parameter in the numerator of the 2-body potential divided by the cube of the major semi-axis) and a time elapsed since some reference epoch [1]:

$$M = n(t - t_0). \tag{1}$$

For the manuscript at hand, M and e are considered fixed parameters. To compute the circular coordinates of distance and true anomaly of the body at that time in the reference frame centered at the ellipse, one encounters Kepler's equation

$$E = M + e \sin E \tag{2}$$
$$E - M - e \sin E = 0. \tag{3}$$

E and M are angles measured in radian in the range $-\pi \le E, M \le \pi$. To simplify the notation, we discuss only the cases where $M \ge 0$, because the parity

$$E(-M) = -E(M), \tag{4}$$

—equivalent to flipping the entire orbit along the major axis of the ellipse—allows to recover solutions for negative M as well.

The Inverse Problem

The numerical problem considered here is to find the root of the function

$$f(E) = E - M - e \sin E \tag{5}$$

in an efficient and numerically stable fashion.

NEWTON METHODS

The simplest technique of solving (2) is a fixed point iteration [2]

$$E^{(i+1)} = M + e \sin E^{(i)}; \quad E^{(0)} = M. \tag{6}$$

For faster convergence this is commonly replaced by a first-order Newton iteration

$$E^{(i+1)} = E^{(i)} - \frac{f}{f'} \tag{7}$$

or a second-order Newton iteration [3–6]

$$E^{(i+1)} = E^{(i)} - \frac{f}{f'(1 - \frac{ff''}{2f'^2})} = E^{(i)} - \frac{2ff'}{2f'^2 - ff''}, \tag{8}$$

where the function and its derivatives with respect to the unknown E are

$$f \equiv E - e \sin E - M; \quad f' \equiv 1 - e \cos E; \quad f'' \equiv e \sin E. \tag{9}$$

Note that, since the evaluation of the trigonometric functions is expensive compared to the fundamental operations [7, 8], the second-order iteration is preferred since the $\sin E$ in f'' is already calculated in conjunction with f.

INITIAL VALUE PROBLEM

Standard Initial Guesses

If the initial guess is the second step of (6),

$$E^{(0)} = M + e \sin M, \tag{10}$$

and the iteration (7) is used with $e > 0.99$, a known problem is that the iterations may converge to secondary roots of the equation with the wrong sign [9]. This is basically triggered by starting with an underestimate of E as illustrated in Figure 1.

A well-known remedy is to start with the initial guess

$$E^{(0)} = \pi \tag{11}$$

which is known to converge [10, 11]. The speed of convergence with the two basic Newton methods is illustrated in Figures 2 and 2.

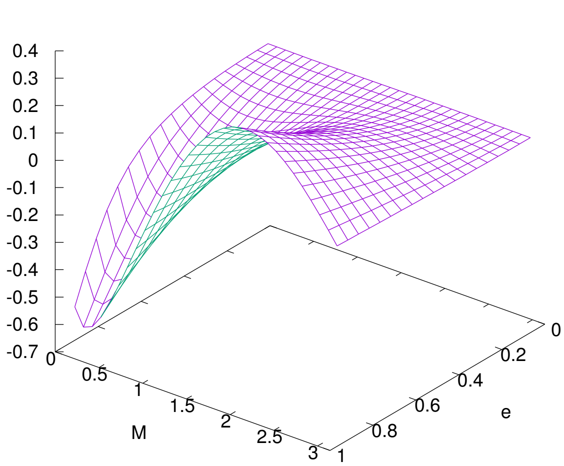

FIG. 1. Mismatch $E^{(0)} - E$ of the initial estimate (10).

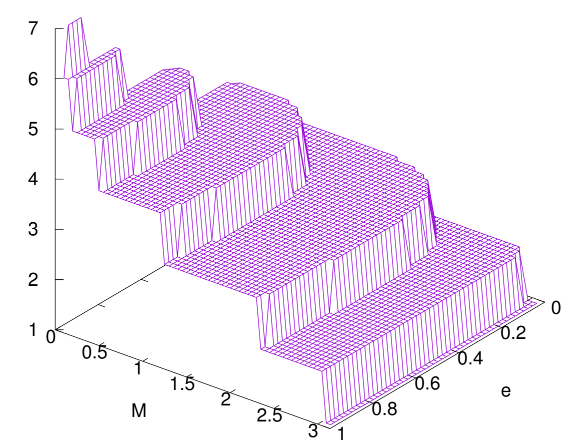

FIG. 2. The number of iterations needed for a relative accuracy of 10^{-12} in E starting from (11) iterating with (7).

Improved Initial Value, Version 1

A starting value of E is obtained by inserting the approximation

$$\sin E \approx 1 - \frac{4}{\pi^2}(E - \pi/2)^2 \tag{12}$$

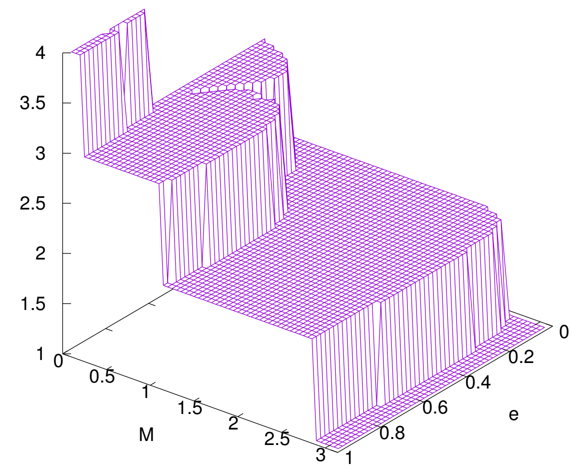

FIG. 3. The number of iterations needed for a relative accuracy of 10^{-12} in E starting from (11) iterating with (8).

into the equation. [Similar approximations could be obtained by truncating the Chebyshev series approximation of the $\sin E$ after the T_2 term [12, 13].]

This converts (2) to a quadratic equation for E

$$E^{(0)} = M + e\left[1 - \frac{4}{\pi^2}(E^{(0)} - \pi/2)^2\right], \tag{13}$$

which is solved by

$$\bar{e} \equiv \frac{\pi}{4e} - 1; \tag{14}$$

$$E^{(0)} = \frac{\pi}{2}\bar{e}\left[\operatorname{sgn}(\bar{e})\sqrt{1 + \frac{M}{e\bar{e}^2}} - 1\right]. \tag{15}$$

The error of this estimate relative to the accurate solution is shown in Figure 4. It increases where $M/e \to 0$ and $\bar{e} \to 0$. The figure shows that the (15) has the same benefit as (11) of approximating the solution from above, therefore converging [10], but being more accurate. In consequence, the convergence is faster, as demonstrated in Figure 5 in comparison with Figure 3.

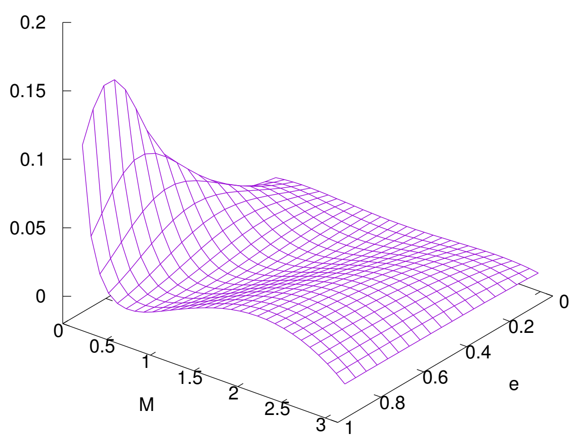

FIG. 4. Mismatch $E^{(0)} - E$ of the initial estimate (15).

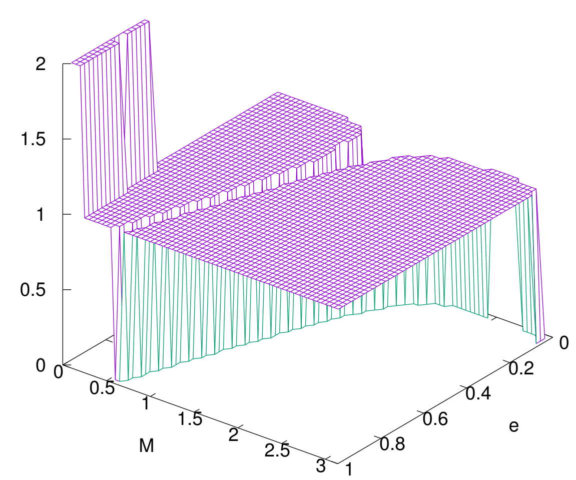

FIG. 5. The number of iterations needed for a relative accuracy of 10^{-12} in E starting from (15) iterating with (8).

Improved Inital Value, Version 2

If (3) is rewritten as

$$E-M=e\sin(M+E-M), \tag{16}$$

both sides may be expanded in a Taylor series of $E-M$,

$$E-M\approx e\sin(M)+(E-M)e\cos M-\frac{(E-M)^2}{2}e\sin M+\ldots \tag{17}$$

Keeping this series up to $O(E-M)$ yields the estimate

$$E^{(0)}=M+\frac{e\sin(M)}{1-e\cos M}. \tag{18}$$

This is basically the estimate of the second step of the fixed point iteration (6) with an enhancement factor of the second term if e or $\cos M$ are large. As pointed out earlier [3], this is also obtained applying the Newton method to the estimator $E^{(0)}=M$.

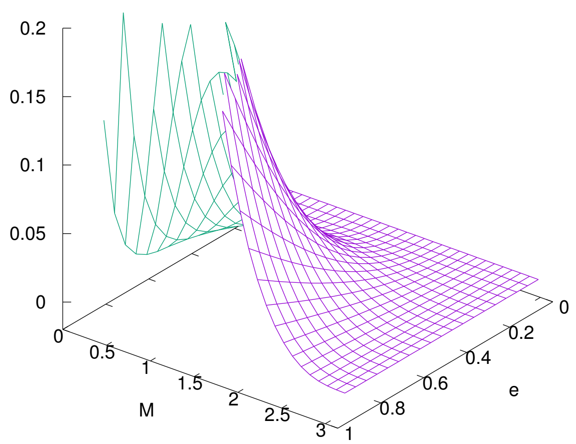

FIG. 6. Mismatch $E^{(0)}-E$ of the initial estimate (18).

If the series is kept up to $O((E-M)^2$, the associated quadratic equation leads to the estimate

$$\frac{e}{2}\sin M(E^{(0)}-M)^2+(1-e\cos M)(E^{(0)}-M)-e\sin M=0. \tag{19}$$

This quadratic equation is solved by

$$E^{(0)}-M=\frac{1-e\cos M}{e\sin M}\left[\sqrt{1+\frac{2e^2\sin^2 M}{(1-e\cos M)^2}}-1\right]. \tag{20}$$

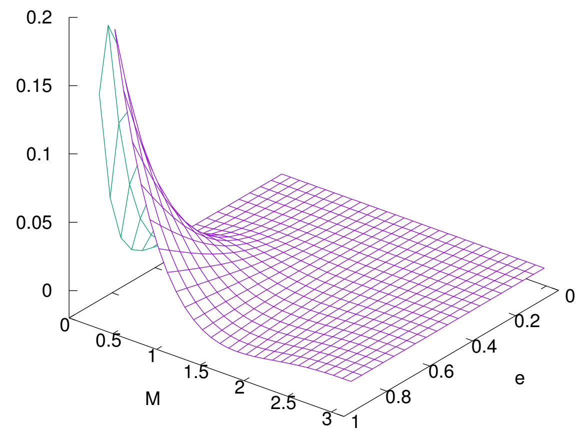

FIG. 7. Mismatch $E^{(0)}-E$ of the initial estimate (20).

Figures 6 and 7 show in comparison with Figure 4 that these approximations derived from the Taylor series of $E-M$ are not better than the one from the quadratic estimate of $\sin E$.

SUMMARY

A starting guess (15) combined with Halley's equation (8) leads to fast and stable convergence for the inverse problem of Kepler's equation for elliptic orbits.

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