AHMED’S INTEGRAL

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ABSTRACT

This note presents some formulas related with ahmed’s integral.

INTRODUCTION

Ahmed’s integral:

$$I=\frac{5\pi^2}{96}=\int_0^1 \frac{\tan^{-1}\sqrt{2+x^2}}{\left(1+x^2\right)\sqrt{2+x^2}}\,dx \tag{1}$$

RELATED FORMULAS

$$\frac{\pi^2}{32}=\int_0^1 \frac{1}{\left(1+x^2\right)\sqrt{2+x^2}}\tan^{-1}\left(\frac{1}{\sqrt{2+x^2}}\right)dx \tag{2}$$

$$\frac{5\pi^2}{96}=\int_0^1 \frac{1}{\left(1+x^2\right)\sqrt{2+x^2}}\sin^{-1}\left(\sqrt{\frac{2+x^2}{3+x^2}}\right)dx \tag{3}$$

$$\frac{5\pi^2}{96}=\int_0^1 \frac{1}{\left(1+x^2\right)\sqrt{2+x^2}}\cos^{-1}\left(\frac{1}{\sqrt{3+x^2}}\right)dx \tag{4}$$

$$\frac{\pi^2}{96}=\int_0^1 \frac{1}{\left(1+x^2\right)\sqrt{2+x^2}}\tan^{-1}\left(\frac{\sqrt{2+x^2}-1}{\sqrt{2+x^2}+1}\right)dx \tag{5}$$

$$\int_0^1 \frac{\left(7x+3x^3\right)\tan^{-1}\sqrt{2+x^2}}{\sqrt{2+x^2}}\,dx=\left(\frac{16\sqrt{3}-9\sqrt{2}}{12}\right)\pi-\frac{1}{2}-3\sqrt{2}\tan^{-1}\left(\left(\sqrt{2}-1\right)^2\right) \tag{6}$$

$$\int_0^1 x\tan^{-1}\sqrt{2+x^2}\,dx=\frac{7\pi}{24}-\frac{\sqrt{3}-\sqrt{2}}{2}-\frac{3}{2}\tan^{-1}\left(\left(\sqrt{2}-1\right)^2\right) \tag{7}$$

$$\int_0^1 \frac{x\tan^{-1}\sqrt{2+x^2}}{\left(2+x^2\right)^{3/2}}dx = \ln 3 - \frac{3}{2}\ln 2 + \frac{1}{\sqrt{2}}\tan^{-1}\left(\left(\sqrt{2}-1\right)^2\right) - \left(\frac{8\sqrt{3}-9\sqrt{2}}{72}\right)\pi \tag{8}$$

$$\int_0^1 \frac{x\tan^{-1}\sqrt{2+x^2}}{\sqrt{2+x^2}}dx = \left(\frac{4\sqrt{3}-3\sqrt{2}}{12}\right)\pi - \ln 2 + \frac{\ln 3}{2} - \sqrt{2}\tan^{-1}\left(\left(\sqrt{2}-1\right)^2\right) \tag{9}$$

$$\frac{5\pi^2}{96} = \int_0^1\int_0^1 \frac{1}{\left(1+x^2\right)\left(1+x^2y^2+2y^2\right)}dxdy \tag{10}$$

$$\frac{5\pi^2}{96} = \frac{1}{4}\int_0^1\int_0^1 \frac{\left(xy\right)^{-1/2}}{\left(1+x\right)\left(1+xy+2y\right)}dxdy \tag{11}$$

$$\frac{\pi^2}{288} = \int_0^1 \frac{1}{\left(1+x^2\right)\sqrt{2+x^2}}\tan^{-1}\left(\frac{\sqrt{3}-\sqrt{2+x^2}}{1+\sqrt{6+3x^2}}\right)dx \tag{12}$$

$$\frac{7\pi^2}{288} = \int_0^1 \frac{1}{\left(1+x^2\right)\sqrt{2+x^2}}\tan^{-1}\left(\frac{\sqrt{6+3x^2}-1}{\sqrt{3}+\sqrt{2+x^2}}\right)dx \tag{13}$$

$$\frac{5\pi^2}{96} = \frac{\pi}{6}\tan^{-1}\frac{3}{2} - \int_0^1 \frac{1}{\left(1+x^2\right)\sqrt{2+x^2}}\tan^{-1}\left(\frac{3-2\sqrt{2+x^2}}{2+3\sqrt{2+x^2}}\right)dx \tag{14}$$

$$\frac{5\pi^2}{96} = \frac{\pi}{6}\tan^{-1}\frac{220}{147} - \int_0^1 \frac{1}{\left(1+x^2\right)\sqrt{2+x^2}}\tan^{-1}\left(\frac{220-147\sqrt{2+x^2}}{147+220\sqrt{2+x^2}}\right)dx \tag{15}$$

$$\frac{5\pi^2}{96} = \frac{\pi}{6}\tan^{-1}\frac{1323}{884} - \int_0^1 \frac{1}{\left(1+x^2\right)\sqrt{2+x^2}}\tan^{-1}\left(\frac{1323-884\sqrt{2+x^2}}{884+1323\sqrt{2+x^2}}\right)dx \tag{16}$$

$$\int_0^1 \frac{1}{\left(1+x^2\right)\sqrt{2+x^2}}\tan^{-1}\left(\frac{\sqrt{2+x^2}-t}{1+t\sqrt{2+x^2}}\right)dx = 0 \tag{17}$$

$$t = \left(2-\sqrt{2}\right)\sqrt{2+\sqrt{2}} + \sqrt{2} - 1 = \tan\left(\frac{5\pi}{16}\right) \tag{18}$$

$$I = \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n 3^{-n}}{\left(2n+1\right)^2}F\left(\{1,1\},\left\{n+\frac{3}{2}\right\},\frac{1}{2}\right)F\left(\{1,n+1\},\left\{n+\frac{3}{2}\right\},\frac{2}{3}\right) \tag{19}$$

$$I=\frac{1}{6}\sum_{n=0}^{\infty}\frac{2^n}{2n+1}\binom{2n}{n}^{-1}\sum_{k=0}^{n}\frac{(-6)^{-k}}{2k+1}\binom{2k}{k}\binom{n}{k}^{-1}F\left(\{1,k+1\},\left\{k+\frac{3}{2}\right\},\frac{2}{3}\right) \tag{20}$$

$$I=\frac{2}{9}\sum_{n=0}^{\infty}\left(\frac{7}{9}\right)^n\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{r=0}^{m}\sum_{s=0}^{r}\binom{n}{k}\binom{k}{m}\binom{m}{r}\binom{r}{s}\frac{(-1)^k\,2^{m-r+k}3^{r-s}7^{-k}}{(2k-2m+2r+2s+1)(2m+1)} \tag{21}$$

$$I=\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-4/3)^k}{2k+1}\sum_{m=0}^{k}\binom{k}{m}\frac{2^{-m}F\left(\{1,1\},\{m+\frac{3}{2}\},\frac{1}{2}\right)}{2m+1} \tag{22}$$

Remark: $F\left(\{a,b\},\{c\},x\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n!}x^n \quad ,|x|<1$, hypergeometric function.

References

1. Ahmed, Z.: Ahmed’s Integral: the maiden solution. arXiv:1411.5169v2 [math.HO] 1 Dec 2014.