SEPARABLE QUANTUM STATES ARE EASIER TO SYNTHESIZE

Dhananjay P. Mehendale*

Department of Physics, Savitribai Phule University of Pune, Pune, India-411007.

(Dated: April 28, 2017)

Abstract

An important application of Grover's search algorithm [2] in the domain of experimental physics is its use in the synthesis of any selected superposition state [3]. This paper is about showing the utility of factorization using [1] of the quantum state to be synthesized. We first factorize the given quantum state to be synthesized when it is factorable. We then make use of these factors and construct the corresponding operators useful for synthesis of those factors. We then build the operator called *synthesizer* by taking tensor product of these operators constructed using factors and useful for synthesis of those factors. We then apply the synthesizer made up of the tensor product of the operators that we built using the corresponding factors on the suitable register whose all the qubits have been initialized to $|0\rangle$. Further, this register is also made up of tensor product of registers of suitable lengths and the first qubit of all these registers is ancilla qubit initialized to $|0\rangle$. We show that we can achieve the speeding up of the process of synthesizing the desired quantum state with our *modified algorithm* when the state is factorable and has at least two factors. It is shown here that the greater the number of factors of the quantum state, the easier it is to synthesize. We will see that in fact the task of synthesizing an n-qubit quantum state which is completely factorable into n single qubit factors is exponentially easier than the task of synthesizing an n-qubit completely entangled quantum state having no factors.

* dhananjay.p.mehendale@gmail.com

A well known application of Grover's search algorithm [2] in the domain of experimental physics is to prepare any desired n-qubit pure quantum state [3] which can be any arbitrary superposition of computational basis states of length n, i.e. containing n qubits.

We are going to use the following simple definition about the action of tensor operators in this paper, so, we state it explicitely.

Let A and B be linear transformations or *operators* from vector spaces V and W respectively. Then the action of operator $A \otimes B$ is defined by

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$$

where $|v\rangle \epsilon V$ and $|w\rangle \epsilon W$.

To generalize, let $A_i, i = 1, 2, \cdots, n$ be linear transformations or *operators* from vector spaces $V_i, i = 1, 2, \cdots, n$ respectively. Then the action of operator $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ is defined by

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_n)(|v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle) = A_1|v_1\rangle \otimes A_2|v_2\rangle \otimes \cdots \otimes A_n|v_n\rangle$$

where $|v_i\rangle \epsilon V_i$ for all $i = 1, 2, \cdots, n$.

This definition appears very natural and this observation is at the heart of our *modified algorithm* for synthesizing the desired quantum state. It allows us to decompose an operation on an entire quantum state into operations on individual factors of that quantum state, i.e. on the individual components, which not only makes the construction of our quantum algorithm much simpler but also can cause the exponential rise in its speed! We show that the speed of synthesizing the desired n-qubit quantum state can be increased exponentially by using our modified quantum algorithm and we can attain the task of synthesizing the desired quantum state almost instantaneously when the desired n-qubit state is completely factorable into n single qubit factors. Thus, this modified quantum algorithm that we propose here will very much simplify the task of synthesis for completely separable states but remains as it was for those quantum states which do not factor (since they do not have any factors) using [1]. Throughout the paper we will adapt standard Dirac notation used in standard quantum physics. We will denote a vector v in the vector space V by $|v\rangle$. We interpret a linear operator O simply acting on a vector v, as $O|v\rangle$.

The well known application of Grover's search algorithm [2] in the domain of experimental physics is its use in the synthesis of any desired superposition [3]. A systematic discussion of the algorithm for this important application in the area of experimental quantum physics can be found in [4]. It is important to note that this algorithm [3] for systematically manufacturing the desired quantum state does not take into consideration the nature of the desired quantum state, i.e. it does not take into account whether the desired n-qubit quantum state is factorable and has any factors or it is not at all factorable, etc. In this paper we begin by applying the algorithm developed in [1] to the desired quantum state to be synthesized and carry out its full factorization and see the advantage of this factorization when more than one factors exist.

With these preliminaries we now proceed with our *modified algorithm* and show how the existence of factors speed up the synthesis of the desired quantum state expressed in terms of the superposition of computational basis states. Note that when the desired superposition does not have any factors then our modified algorithm reduces to (or remains identical with) the existing algorithm given in [3] but when the desired superposition to be synthesized is factorable and if it so happens that by applying procedure of full factorization in [1] it factors into say, p factors then the modified algorithm runs independently, parallely, and simultaneously, through p number of processes of separately synthesizing these p factors and the detailed discussion below will make it clear why and how much time saving will be achieved in completing the task of synthesizing the desired superposition.

Algorithm

The desired quantum state, $|\psi\rangle$, that one wishes to synthesize can be any arbitrary superposition and let it be

$$|\psi\rangle = \sum_{i_1,i_2,\ldots i_n} a_{i_1 i_2 \ldots i_n} |i_1 i_2 \ldots i_n\rangle$$

where all $a_{i_1 i_2 \ldots i_n}$ belongs to C, the field of complex numbers, and where each of $i_1, i_2 \ldots, i_n$ takes values in $\{0, 1\}$. We further assume without any loss of generality (since we can always normalize a state if and when required) that this n-qubit pure quantum state is normalized, i.e. $\sum_{i_1,i_2,\ldots i_n} |a_{i_1 i_2 \ldots i_n}|^2 = 1$. This expression for $|\psi\rangle$ as sum over computational basis states can contain in all $N = 2^n$ computational basis states, each of length n, namely, $|00\ldots0\rangle$, $|00\ldots1\rangle$, $\ldots$, $|i_1 i_2 \ldots i_n\rangle$, $\ldots$, $|11\ldots1\rangle$.

Some useful definitions are in order:

Definition 1: The factorization of any superposition state, $|\psi\rangle$, obtained by applying factorization algorithm given in [1] is called *full factorization* of that state.

Definition 2: A quantum state, $|\psi\rangle$, is called *completely entangled* if it does not factorizes into two or more factors under its full factorization.

Definition 3: A quantum state, $|\psi\rangle$, is called *completely separable* if it factorizes into tensor product of n single qubit factors under its full factorization.

Definition 4: The operator which is either a single operator when the quantum state to be synthesized has no factors, or made up of the tensor product of some p operators when the quantum state to be synthesized has some p factors, is called *Synthesizer* if when this operator is operated on the register of suitable length whose all qubits are initialized to zero leads to desired quantum state to be synthesized when all the ancellae qubits are found in state $|0\rangle$ when measured.

As stated in the abstract our goal is to show the substantial advantage of full factorization using [1] in speeding up the process of synthesizing the desired quantum state. The idea behind achieving this enormous advantage is in making use of quantum parallelism by running p processes of synthesizing the p factors independently and simultaneously when the desired quantum state has p factors!

We now proceed with the steps of our *modified algorithm*:

(i) Using [1] we carry out full factorization of the quantum state, $|\psi\rangle$, desired to be synthesized. Suppose this full factorization leads us to

$$|\psi\rangle = |\phi_{k_1}\rangle \otimes |\phi_{k_2}\rangle \otimes \ldots \otimes |\phi_{k_i}\rangle \otimes \ldots \otimes |\phi_{k_p}\rangle$$

where

$$|\phi_{k_i}\rangle = \sum_{j_1,j_2,\ldots j_{k_i}} a^i_{j_1 j_2 \ldots j_{k_i}} |j_1 j_2 \ldots j_{k_i}\rangle$$

Case (i) $p = 1$.

In this case $|\psi\rangle$ is a completely entangled state has no factors (or in other words, it has only one factor, namely, the quantum state $|\psi\rangle$ itself) and for this case the modified algorithm remains identical with the existing algorithm in [3].

Case (ii) $p > 1$.

In this case $|\psi\rangle$ is not completely entangled state and we see that we arrive at its two or more than two factors using [1].

Thus, the given quantum state factors into tensor product of p factors, namely, $|\phi_{k_1}\rangle, |\phi_{k_2}\rangle, \ldots, |\phi_{k_i}\rangle, \ldots, |\phi_{k_p}\rangle$ such that each factor in itself is a normalized quantum state.

Suppose state $|\phi_{k_1}\rangle$ is superposition of computational basis states of length k_1, state $|\phi_{k_2}\rangle$ is superposition of computational basis states of length k_2, $\ldots$, state $|\phi_{k_i}\rangle$ is superposition of computational basis states of length k_i, $\ldots$, state $|\phi_{k_p}\rangle$ is superposition of computational basis states of length k_p, such that $k_1 + k_2 + \ldots + k_i + \ldots + k_p = n$.

Now, our job is to make use of these p quantum states, $|\phi_{k_i}\rangle, i = 1, 2, \ldots, p$ which are factors of $|\psi\rangle$ obtained by carrying out full factorization as per [1] and to prepare p *suitable* operators, $O_{k_i}, i = 1, 2, \ldots, p$ corresponding to these p factors using [3] such that these operators, $O_{k_i}, i = 1, 2, \ldots, p$ are prepared using the the coefficients of the corresponding computational basis states that compose together these factors, $|\phi_{k_i}\rangle, i = 1, 2, \ldots, p$, of the desired quantum state to be manufactured. Note that these operators are useful in the synthesis of the factors, for example, if the operator O_{k_i} will be operated on state $|0\rangle|00\ldots0\rangle_{k_i}$ where $|0\rangle$ corresponds to ancilla qubit and $|00\ldots0\rangle_{k_i}$ corresponds to a ket vector of length k_i and further if the measurement of the first qubit (ancilla) will be carried out after operating with this operator, O_{k_i}, and if the ancilla qubit will be found to be in the state $|0\rangle$ then the remaining qubits will be in the state we wish to synthesize, i.e. $O_{k_i}|0\rangle|00\ldots0\rangle_{k_i}$ produce the factor state $|0\rangle|\phi_{k_i}\rangle$. Thus, we have prepared the operators,

$O_{k_i}, i = 1, 2, \ldots, p$, which are useful for synthesizing separately the corresponding factors, $|\phi_{k_i}\rangle, i = 1, 2, \ldots, p$.

(ii) We now build the operator, O, useful to synthesize the originally given quantum state which one desires to synthesize, namely,$|\psi\rangle$ as follows:

$$O = O_{k_1} \otimes O_{k_2} \otimes \ldots \otimes O_{k_i} \otimes \ldots \otimes O_{k_p}$$

Thus, the operator, O, is the tensor product of the operators, $O_{k_i}, i = 1, 2, \ldots, p$, which are useful for synthesizing separately the corresponding factors, $|\phi_{k_i}\rangle, i = 1, 2, \ldots, p$.

(iii) We now build a *suitable* quantum register, $|R\rangle$, as follows:

$$|R\rangle = |0\rangle|00\ldots0\rangle_{k_1} \otimes |0\rangle|00\ldots0\rangle_{k_2} \otimes \ldots \otimes |0\rangle|00\ldots0\rangle_{k_i} \otimes \ldots \otimes |0\rangle|00\ldots0\rangle_{k_p}$$

where $|0\rangle$ in each $|0\rangle|00\ldots0\rangle_{k_i}, i = 1, 2, \ldots, p$, corresponds to ancilla qubit and $|00\ldots0\rangle_{k_i}, i = 1, 2, \ldots, p$, corresponds to ket vectors of length $k_i, i = 1, 2, \ldots, p$ whose all qubits are initialized to $|0\rangle$.

(iv) We operate the operator O on the register $|R\rangle$, i.e. in other words we evaluate $O|R\rangle$. Now, using the definition about the action of the operator which is tensor product operators given at the beginning of the paper we have $O|R\rangle = (O)(|R\rangle)$ equal to

$$(O_{k_1} \otimes O_{k_2} \otimes \ldots \otimes O_{k_i} \otimes \ldots \otimes O_{k_p})(|0\rangle|00\ldots0\rangle_{k_1} \otimes |0\rangle|00\ldots0\rangle_{k_2} \otimes \ldots \otimes |0\rangle|00\ldots0\rangle_{k_i} \otimes \ldots \otimes |0\rangle|00\ldots0\rangle_{k_p})$$

equal to

$$O_{k_1}|0\rangle|00\ldots0\rangle_{k_1} \otimes O_{k_2}|0\rangle|00\ldots0\rangle_{k_2} \otimes \ldots \otimes O_{k_i}|0\rangle|00\ldots0\rangle_{k_i} \otimes \ldots \otimes O_{k_p}|0\rangle|00\ldots0\rangle_{k_p}$$

(v) We then measure together all the ancillae qubits, i.e. first qubit, $(k_1 + 1)$-th qubit, $(k_2 + 1)$-th qubit, ..., $(k_p + 1)$-th qubit in $O|R\rangle$. If we will find all these ancillae qubits in state $|0\rangle$ then the remaining qubits together will be in the state we wish to synthesize, i.e. the superposition will be projected into state

$$|\Psi\rangle = |0\rangle|\phi_{k_1}\rangle \otimes |0\rangle|\phi_{k_2}\rangle \otimes \ldots \otimes |0\rangle|\phi_{k_i}\rangle \otimes \ldots \otimes |0\rangle|\phi_{k_p}\rangle$$

where

$$|\phi_{k_i}\rangle = \sum_{j_1,j_2,\ldots j_{k_i}} a^i_{j_1 j_2 \ldots j_{k_i}} |j_1 j_2 \ldots j_{k_i}\rangle.$$

It is clear to see that the quantum state $|\Psi\rangle$ is actually the desired quantum state $|\psi\rangle$ in disguise!

Remarks

(1) *The case of completely separable quantum state:* When given n-qubit quantum state to be synthesized is completely separable into n 1-qubit factors then the synthesizer (operator) is made up of tensor product of n operators, each of these operators is made using corresponding factor among the n factors. These operators have representation in terms of 4×4 matrices and they are required to raise to power $\approx 1/2 \times \pi$, when the quantum state to be synthesized is normalized. We require to operate this operator on state $\prod^{\otimes(n)} |0\rangle|0\rangle$.

(2) *The case of completely entangled quantum state:* When given n-qubit quantum state to be synthesized is completely entangled then no simplification is possible and we need to carry out the algorithm given in [3] as it is. The synthesizer (operator) in this case is directly made using the desired quantum state and has representation in terms of $(n+1) \times (n+1)$ matrix and here the matrix is required to raise to power $\approx 1/4 \times \pi\sqrt{2^n}$, to synthesize the desired normalized quantum state to be synthesized. We require to operate in this case the operator on state $|0\rangle|00\ldots0\rangle_n$ which produces the state $|0\rangle|\psi\rangle$.

(3) *Speeding up the synthesis of desired quantum state:* It is clear to see that if desired quantum state has more number of factors then it becomes easier to synthesize it by carrying out the synthesis of its factors independently, parallely, and simultaneously with much less effort (in terms of sizes and powers of the matrices involved therein). Suppose a state to be synthesized has following factorization:

$$|\psi\rangle = |\phi_{k_1}\rangle \otimes |\phi_{k_2}\rangle \otimes \ldots \otimes |\phi_{k_i}\rangle \otimes \ldots \otimes |\phi_{k_p}\rangle$$

where $k_l = max\{k_i, i = 1, 2, \ldots, p.\}$ So, to build the operator, O_{k_l}, the size of the matrices involved and the power to which these matrices are required to raise will be largest. So, in the synthesis of $|\psi\rangle$ through parallel processing as

$$|\Psi\rangle = |0\rangle|\phi_{k_1}\rangle \otimes |0\rangle|\phi_{k_2}\rangle \otimes \ldots \otimes |0\rangle|\phi_{k_i}\rangle \otimes \ldots \otimes |0\rangle|\phi_{k_p}\rangle$$

the entire synthesis through parallel processing will complete in the time required for synthesis of $|0\rangle|\phi_{k_l}\rangle$.

Example: Synthesize the following quantum state:

$$|\psi\rangle = 0.04597722|0000\rangle - 0.06896573|0001\rangle + 0.11494274i|0010\rangle - 0.16091996i|0011\rangle$$

$$-0.06896573|0100\rangle + 0.10344843|0101\rangle - 0.17241383i|0110\rangle + 0.24137956i|0111\rangle$$

$$+0.11494274i|1000\rangle - 0.17241383i|1001\rangle - 0.28735603|1010\rangle + 0.40229877|1011\rangle$$

$$-0.16091996i|1100\rangle + 0.24137956i|1101\rangle + 0.40229877|1110\rangle - 0.56321872|1111\rangle$$

Solution:

(1) Firstly we directly proceed as per [3] without seeking the factorization of $|\psi\rangle$ to build the operator O, called synthesizer:

(2) We introduce an ancilla qubit prepared in the state $|0\rangle$ and thus prepare register containing in all 5 qubits, all initialized to state $|0\rangle$, namely, $|0\rangle|0000\rangle$.

(3) We define operator

$$U_1 = I \otimes H \otimes H \otimes H \otimes H.$$

U_1 is a matrix of size 32×32.

(4) We define operator

$$U_2 : U_2|0\rangle|i_1i_2i_3i_4\rangle \rightarrow c_{i_1i_2i_3i_4}|0\rangle|i_1i_2i_3i_4\rangle + \sqrt{1-|c_{i_1i_2i_3i_4}|^2}|1\rangle|i_1i_2i_3i_4\rangle$$

plus the remaining orthonormal columns. i_j takes values in the set $\{0,1\}$ and $j \epsilon \{1,2,3,4\}$. Also, $c_{i_1i_2i_3i_4}$ are the coefficients of the respective states $|i_1i_2i_3i_4\rangle$ in $|\psi\rangle$ given above. U_2 is a matrix of size 32×32.

(5) We define operator $I_t = diag(-1,-1,\ldots-1,+1,+1,\ldots+1)$, i.e. a sequence of (-1)s 16 in number followed by a sequence of (+1)s again 16 in number along the diagonal of matrix. I_t is a matrix of size 32×32.

(6) We define operator $I_s = diag(-1, +1, +1, \ldots + 1)$, i.e. a sequence of only one (-1) followed by a sequence of (+1)s in all 31 in number along the diagonal of matrix. I_s is a matrix of size 32×32.

(7) We define operator $U = U_2.U_1$. Note that U will be a matrix of size 32×32.

(8) We define operator $Q = -(I_s.U^{-1}.I_t.U)$. Note that Q will be a matrix of size 32×32.

(9) We define $O = U.Q^m$, where $m \approx 1/4 \times \pi\sqrt{2^4}$ and compute $O|0\rangle|0000\rangle$

(10) We measure the ancilla (i.e. the first qubit). If ancilla is found in state $|0\rangle$, the remaining qubits will be in the state to be synthesized.

It is important to note that this procedure involves construction of *bigger* sized matrices (32×32 in the present example) and taking their *bigger* powers ($m \approx 1/4 \times \pi\sqrt{2^4}$ in the present example).

Now, we proceed with the factorization using [1] of given state, $|\psi\rangle$, and see that when this state factorizes then into two or more factors then how our modified algorithm helps to simplify and speedup the synthesis.

(I) We apply factorization algorithm [1] to the state $|\psi\rangle$ as given above. It can be seen that it factorizes into two identical factors, i.e. we get $|\psi\rangle = |\Theta\rangle \otimes |\Theta\rangle$, where

$$|\Theta\rangle = 0.214423|00\rangle - 0.321634|01\rangle + 0.536056i|10\rangle - 0.750479i|11\rangle.$$

(II) The full factorization of $|\psi\rangle$ produces two factors so we need to build (as per modified algorithm above) two operators using these factors and since these factors happen to be identical in the present case these operators will be identical. Thus, we get the the operator (synthesizer) to build $|\psi\rangle$ as $O = T \otimes T$ and (as per modified algorithm) we introduce two ancillae qubits both prepared in the state $|0\rangle$ and thus we prepare register containing in all 6 qubits, namely, $|R\rangle = |0\rangle|00\rangle|0\rangle|00\rangle$. We then compute

$$O|R\rangle = (O)(|R\rangle) = (T \otimes T)(|0\rangle|00\rangle \otimes |0\rangle|00\rangle) = T|0\rangle|00\rangle \otimes T|0\rangle|00\rangle$$

We now proceed to build the required operator T using the factor state $|\Theta\rangle$.

(III) We define operator

$$V_1 = I \otimes H \otimes H.$$

V_1 will be a matrix of size 8×8.

(IV) We define operator

$$V_2 : V_2|0\rangle|i_1 i_2\rangle \rightarrow d_{i_1 i_2}|0\rangle|i_1 i_2\rangle + \sqrt{1 - |d_{i_1 i_2}|^2}|1\rangle|i_1 i_2\rangle$$

plus the remaining orthonormal columns. i_j takes values in the set $\{0, 1\}$ and $j \epsilon \{1, 2\}$. Also, $d_{i_1 i_2}$ are the coefficients of the respective states $|i_1 i_2\rangle$ in $|\Theta\rangle$ given above. V_2 will be a matrix of size 8×8.

(V) We define operator $J_t = diag(-1, -1, \ldots - 1, +1, +1, \ldots + 1)$, i.e. a sequence of (-1)s, 4 in number, followed by a sequence of (+1)s again 4 in number along the diagonal of matrix. J_t will be a matrix of size 8×8.

(VI) We define operator $J_s = diag(-1, +1, +1, \ldots + 1)$, i.e. a sequence of only one (-1) followed by a sequence of (+1)s in all 7 in number along the diagonal of matrix. J_s will be a matrix of size 8×8.

(VII) We define operator $V = V_2.V_1$. Note that V will be a matrix of size 8×8.

(VIII) We define operator $S = -(J_s.V^{-1}.J_t.V)$. Note that S will be a matrix of size 8×8.

(IX) We define $T = V.S^m$, where $m \approx 1/4 \times \pi\sqrt{2^2}$ and compute $(T|0\rangle|00\rangle) \otimes (T|0\rangle|00\rangle)$

(X) We measure both the ancillae (i.e. the first qubit and fourth qubit together). If both the ancillae are found in state $|0\rangle$, the remaining qubits will be in the state to be synthesized.

For the present example the synthesis as per modified algorithm produces the state: $|\Phi\rangle = (|0\rangle|\Theta) \otimes (|0\rangle|\Theta)$, which is desired state $|\psi\rangle$ in disguise with significantly less efforts!

It is important to note that when there are factors to the state to be synthesized the process of synthesis involves construction of *smaller* sized matrices (8×8 in the present example) and taking their *smaller* powers ($m \approx 1/4 \times \pi\sqrt{2^2}$ in the present example), and the *parallel processing* (of "two" tasks in the present example) involved in the *modified algorithm* significantly speeds up of the process of synthesis.

Conclusion

We provide a *modified algorithm* for synthesizing a desired quantum state. It is shown that depending upon the nature of the quantum state to be synthesized like, whether it is completely separable, or partially factorable, or completely entangled etc. then the task of synthesizing the desired quantum state can be completed respectively exponentially faster, or faster, or with the same speed (time complexity) as that of the algorithm in [3].

Acknowledgement

I thank Dr. M. R. Modak, S. P. College, Pune-411030, India, for some useful discussions.

---

[1] D. P. Mehendale, P. S. Joag, Quant. Phys. Lett. 6, No. 1, pp 73-77, (2017).

[2] Grover L.K. Proceedings of the 28th Annual ACM Symposium on the Theory of Computing,ACM Press, New York, pp 212-219, (1996).

[3] Grover L.K. Phys. Rev. Lett. Volume 85, Issue 6, pp 1334-1337, (2000).

[4] Colin P. Williams, Explorations in Quantum Computing, second edition, Springer-Verlag London Limited, pp 256-260, (2011).