Derivation of the Planck Force from Heisenberg Uncertainty Relations

In this paper I derive the expression for the Planck force from the Heisenberg uncertainty relations.

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1. Derivation of the Planck Force

Let us consider a real particle that moves along the x axis of certain reference system S under the action of a net force \( F \). Let us suppose that at certain time, \( t \), the particle has an uncertainty in momentum along the x axis of \( \Delta p_x \), an uncertainty in position of \( \Delta x \), an uncertainty in energy of \( \Delta E \) and an uncertainty in time of \( \Delta t \) (we shall use \( \Delta E \) and \( \Delta t \) later in this section). Then according to Heisenberg, the relation between \( \Delta p_x \) and \( \Delta x \) is given by

\[
\Delta p_x \Delta x \geq \frac{\hbar}{2}
\]  \hspace{1cm} (1.1)

Let's also assume that the speed of the particle is much less than the speed of light. This is

\[ v \ll c \]  \hspace{1cm} (1.2)

Because of the definition of momentum we may write
\[ m \Delta v_x \Delta x \geq \frac{\hbar}{2} \]  

(1.3)

Where \( \Delta v_x \) is the uncertainty in the velocity of the particle. Because of the uncertainty in velocity, this velocity could be anywhere, let's say, between \( v_{x1} \) and \( v_{x2} \). This means that

\[ \Delta v_x = v_{x2} - v_{x1} \]  

(1.4)

According to Heisenberg, the uncertainty, \( \Delta v_x \), in the velocity of the particle yields uncertainty, \( \Delta E \), in its energy. Let us assume that \( \Delta t \) is the uncertainty in the time for which the particle has an uncertainty in energy of \( \Delta E \). Thus, according to Heisenberg \( \Delta E \) and \( \Delta t \) are related by

**Energy-time uncertainty relation**

\[ \Delta E \Delta t \geq \frac{\hbar}{2} \]  

(1.5)

The time for which the particle has an energy equal to \( E \) (with an uncertainty in energy of \( \Delta E \)) could be anywhere between, let's say, \( t_1 \) and \( t_2 \), so that we write

\[ \Delta t = t_2 - t_1 \]  

(1.6)

Because we have both an uncertainty in the velocity of the particle and an uncertainty in the time for which this uncertainty occurs, we must have an uncertainty, \( \Delta a_x \), in the acceleration of the particle (we assume that there is a net force, \( F \), acting on the particle). Therefore, the uncertainty in acceleration is given by

\[ \Delta a_x = \frac{v_{x2} - v_{x1}}{t_2 - t_1} = \frac{\Delta v_x}{\Delta t} \]  

(1.7)

Now let us divide eq. (1.3) by \( \Delta t \). This yields

\[ m \frac{\Delta v_x}{\Delta t} \Delta x \geq \frac{\hbar}{2} \frac{1}{\Delta t} \]  

(1.8)

According to eq. (1.7) we may write

\[ m \Delta a_x \Delta x \geq \frac{\hbar}{2} \frac{1}{\Delta t} \]  

(1.9)

But, according to Newton's second law of motion, the quantity \( m \Delta a_x \) has to be the uncertainty, \( \Delta F \), in the net force, \( F \), that is exerted on the particle. This lead us to write

\[ \Delta F \Delta x \geq \frac{\hbar}{2} \frac{1}{\Delta t} \]  

(1.10)

But the first side of this inequation is the product of the uncertainty in force times the uncertainty in position: \( \Delta F \Delta x \). This must represent the uncertainty in the energy of...
the particle, \( \Delta E \), as given by relation (1.5). Now one may ask: what is the maximum uncertainty in force?. To answer this question let us write the above inequation as follows

\[
\Delta F \geq \frac{\hbar}{2 \Delta x \Delta t}
\]  

(1.11)

The uncertainty in force will be maximum when both, the uncertainty in position and the uncertainty in time are minimum. So if we denote \( \Delta x_{\min} \) the minimum uncertainty in position and \( \Delta t_{\min} \) the minimum uncertainty in time, we may write the maximum uncertainty in force, \( \Delta F_{\max} \), as follows

\[
\Delta F_{\max} \geq \frac{\hbar}{2 \Delta x_{\min} \Delta t_{\min}}
\]  

(1.12)

Now let us assume that the minimum length with physical meaning is the Planck length, \( L_p \). This length is defined as

**Planck length**

\[
L_p = \sqrt{\frac{\hbar G}{2 \pi c^3}}
\]  

(1.13)

And let us also assume that the minimum time with physical meaning is the Planck time, \( T_p \). This time is defined as

**Planck time**

\[
T_p = \sqrt{\frac{\hbar G}{2 \pi c^5}}
\]  

(1.14)

Thus, replacing the values of \( \Delta x_{\min} \) and \( \Delta t_{\min} \) by \( L_p \) and \( T_p \), respectively, we get

\[
\Delta F_{\max} \geq \frac{\hbar}{2 \sqrt{\frac{\hbar G}{2 \pi c^3}} \sqrt{\frac{\hbar G}{2 \pi c^5}}}
\]  

(1.15)

Which turns out to be

\[
\Delta F_{\max} \geq \frac{c^4}{2G}
\]  

(1.16)

Then, to make the definition of the Planck force compatible with previous definitions based on (a) Newton's second law of motion and (b) Coulomb's law, we are compelled to define the Planck force as

**Planck force**

\[
F_p = \frac{c^4}{G}
\]  

(1.17)

Thus, the maximum uncertainty in force is
\[ \Delta F_{\text{max}} \geq \frac{F_P}{2} \quad (1.18) \]

The minimum value of this uncertainty occurs for the equal sign

\[ \left( \Delta F_{\text{max}} \right)_{\text{min uncertainty}} = \frac{F_P}{2} \quad (1.19) \]

thus, the minimum uncertainty in the maximum force is equal to the Planck force divided by 2. Hence, according to Heisenberg, the Planck force is twice the minimum uncertainty for the maximum uncertainty in force.

\[ F_P = 2 \left( \Delta F_{\text{max}} \right)_{\text{minimum uncertainty}} \quad (1.20) \]

### 2. Conclusions

I have presented, in the foregoing pages, a derivation of the Planck force based on the Heisenberg uncertainty relations:

**Momentum-position uncertainty relation in the x direction**

\[ \Delta p_x \Delta x \geq \frac{\hbar}{2} \quad (2.1) = (1.1) \]

and

**Energy-time uncertainty relation**

\[ \Delta E \Delta t \geq \frac{\hbar}{2} \quad (2.2) = (1.5) \]

The conclusions that we extract from the previous mathematical approach are

(a) the definition of the Planck force:

**Planck force**

\[ F_P \equiv \frac{c^4}{G} \quad (2.3) = (1.17) \]

(b) the interpretation of the Planck force as a force that is twice the minimum uncertainty for the maximum uncertainty in force:

\[ F_P = 2 \left( \Delta F_{\text{max}} \right)_{\text{minimum uncertainty}} \quad (2.4) = (1.20) \]

The Planck force is an extremely important composite constant (or unit if you like) because allow us to derive Newton's law of universal gravitation from first principles [1] and because it has been silently incorporated by Einstein in his general relativity's field equations [1].
REFERENCES