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GRA for Multi Attribute Decision Making in Neutrosophic Cubic Set Environment

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Abstract. In this paper, multi attribute decision making problem based on grey relational analysis in neutrosophic cubic set environment is investigated. In the decision making situation, the attribute weights are considered as single valued neutrosophic sets. The neutrosophic weights are converted into crisp weights. Both positve and negative GRA coefficients, and weighted GRA coefficients are determined. Hamming distances for weighted GRA coefficients and standard (ideal) GRA coefficients are determined. The relative closeness coefficients are derived in order to rank the alternatives. The relative closeness coefficients are designed in ascending order. Finally, a numerical example is solved to demonstrate the applicability of the proposed approach.

Keywords: Grey relational coefficient, interval valued neutrosophic set, multi attribute decision making, neutrosophic set, neutrosophic cubic set, relative closeness coefficient

1 Introduction

In management section, banking sector, factory, plant multi attribute decision making (MADM) problems are to be extensively encountered. In a MADM situation, the most appropriate alternative is selecting from the set of alternatives based on highest degree of acceptance. In a decision making situation, decision maker (DM) considers the efficiency of each alternative with respect to each attribute. In crisp MADM, there are several approaches [1, 2, 3, 4, 5] in the literature. The weight of each attribute and the elements of decision matrix are presented by crisp numbers. But in real situation, DMs may prefer to use linguistic variables like 'good', 'bad', 'hot', 'cold', 'tall', etc. So, there is an uncertainty in decision making situation which can be mathematically explained by fuzzy set [6]. Zadeh [6] explained uncertainty mathematically by defining fuzzy set (FS). Bellman and Zadeh [7] studied decision making in fuzzy environment. Atanassov [8, 9] narrated uncertainty by introducing non-membership as independent component and defined intuitionistic fuzzy set (IFS). Degree of indeterminacy (hesitency) is not independent .

Later on DMs have recognized that indeterminacy plays an important role in decision making. Smarandache [10] incorporated indeterminacy as independent component and developed neutrosophic set (NS) and together with Wang et a. [11] defined single valued neutrosophic set (SVNS) which is an instance of neutrosophic set. Ye [12] proposed a weighted correlation coefficients for ranking the alternatives for multicriteria decision making (MCDM). Ye [13] established single valued neutrosophic cross entropy for MCDM problem. Sodenkamp [14] studied multiple-criteria decision analysis in neutrosophic environment. Mondal and Pramanik [15] defined neutrosophic tangent similarity measure and presented its application to MADM. Biswas et al. [16] studied cosine similarity measure based MADM with trapezoidal fuzzy neutrosophic numbers. Mondal and Pramanik [17] presented multi-criteria group decision making (MCGDM) approach for teacher recruitment in higher education. Mondal and Pramanik [18] studied neutrosophic decision making model of school choice. Liu and Wang [19] presented MADM method based on single-valued neutrosophic normalized weighted Bonferroni mean. Biswas et al. [20] presented TOPSIS method for MADM under single-valued neutrosophic environment. Chi and Liu [21] presented extended TOPSIS method for MADM on interval neutrosophic set. Broumi et al. [22] presented extended TOPSIS method for MADM based on interval neutrosophic uncertain linguistic variables. Nabdaban and Dzitac [23] presented a very short review of TOPSIS in neutrosophic environment. Pramanik et al. [24] studied hybrid vector similarity measures and their applications to MADM under neutrosophic environment. Biswas et al. [25] presented triangular fuzzy neutrosophic set information and its application to MADM. Sahin and Liu [26] studied maximizing deviation method for neutrosophic MADM with incomplete weight information. Ye [27] studied bidirectional projection method for MADM with neutrosophic numbers of the form *a* + *bI*, where *I* is characterized by indeterminacy. Biswas et al. [28] presented value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to MADM. Dey et al. [29] studied extended projection-based models for solving MADM problems with interval-valued neutrosophic information.

Deng [30, 31] studied grey relational analysis (GRA). Pramanik and Mukhopadhyaya [32] developed GRA based intuitionistic fuzzy multi criteria decision making (MCDM) approach for teacher selection in higher education. Dey et al. [33] established MCDM in intuitionistic fuzzy environment based on GRA for weaver selection in Khadi institution. Rao, and Singh [34] established modified GRA method for decision making in manufacturing situation. Wei [35] presented GRA method for intuitionistic fuzzy MCDM. Biswas et al. [36] studied GRA method for MADM under single valued neutrosophic assessment based on entropy. Dey et al. [37] presented extended GRA based neutrosophic MADM in interval uncertain linguistic setting. Pramanik and K. Mondal [38] employed GRA for interval neutrosophic MADM and presented numerical examples.

Several neutrosophic hybrid sets have been recently proposed in the literature, such as neutrosophic soft set proposed by Maji [39], single valued soft expert set proposed by Broumi and Smarandache [40], rough neutrosophic set proposed by Broumi, et al. [41], neutrosophic bipolar set proposed by Deli et al. [42], rough bipolar neutrosophic set proposed by Pramanik and Mondal [43], neutrosophic cubic set proposed by Jun et al. [44] and Ali et al. [45]. Jun et al. [44] presented the concept of neut-rosophic cubic set by extending the concept of cubic set proposed by Jun et al. [46] and introduced the notions of truth-internal (indeterminacy-internal, falsity-internal) neut-rosophic cubic sets and truth-external (indeterminacy-external, falsity-external) and investigated related properties. Ali et al. [45] presented concept of neutrosophic cubic set by extending the concept of cubic set [46] and defined internal neutrosophic cubic set (INCS) and external neutrosophic cubic set (ENCS). In their study, Ali et al.[45] also introduced an adjustable approach to neutrosophic cubic set based decision making.

GRA based MADM/ MCDM problems have been proposed for various neutrosophic hybrid environments [47, 48, 49, 50]. MADM with neutrosophic cubic set is yet to appear in the literature. It is an open area of research in neutrosophic cubic set environment.

The present paper is devoted to develop GRA method for MADM in neutrosophic cubic set environment. The attribute weights are described by single valued neutrosophic sets. Positive and negative grey relational coefficients are determined. We define ideal grey relational coefficients and relative closeness coefficients in neutrosophic cubic set environment. The ranking of alternatives is made in descending order.

The rest of the paper is designed as follows: In Section 2, some relevant definitions and properties are recalled. Section 3 presents MADM in neutrosophic cubic set environment based on GRA. In Section 4, a numerical example is solved to illustrate the proposed approach. Section 5 presents conclusions and future scope of research.

2 Preliminaries

In this section, we recall some established definitions and properties which are connected in the present article.

2.1 Definition (Fuzzy set) [6]

Let *W* be a universal set. Then a fuzzy set F over *W* can be defined by F={<w, $\mu_F(w)$ > : w ∈*W*} where $\mu_F(w): W \to [0, 1]$is called membership function of F and $\mu_F(w)$ is the degree of membership to which w ∈ F.

2.2 Definition (Interval valued fuzzy set) [52]

Let *W* be a universal set. Then, an interval valued fuzzy set F over *W* is defined by F = {[$F^-(w), F^+(w)$]/ w : w ∈ *W*}, where $F^-(w)$ and $F^+(w)$ are referred to as the lower and upper degrees of membership w ∈*W* where

$0 \le F^-(w) + F^+(w) \le 1$, respectively.

2.3 Definition (Cubic set) [46]

Let W be a non-empty set. A cubic set C in W is of the form c = { w, F(w), λ(w))/ w ∈ *W*} where F is an interval valued fuzzy set in *W* and λ is a fuzzy set in *W*.

2.4 Definition (Neutrosophic set (NS)) [10]

Let *W* be a space of points (objects) with generic element w in *W*. A neutrosophic set N in *W* is denoted by N= {< w: $T_N(w)$, $I_N(w)$, $F_N(w)$>: w∈ *W*} where T_N, I_N, F_N represent membership, indeterminacy and non-membership function respectively. T_N, I_N, F_N can be defined as follows:

$$T_N : W \to]^-0, 1^+[$$

$$I_N : W \to]^-0, 1^+[$$

$$F_A : W \to]^-0, 1^+[$$

Here, $T_N(w)$, $I_N(w)$, $F_N(w)$ are the real standard and non-standard subset of $]^-0, 1^+[$ and

$$^-0 \le T_N(w)+I_N(w)+F_N(w) \le 3^+.$$

2.5 Definition (Complement of neutrosophic set) [10]

The complement of a neutrosophic set N is denoted by N′and defined as

N′= {<w: $T_{N'}$(w), $I_{N'}$(w), $F_{N'}$(w)>, w ∈ W }

$T_{N'}(w) = \{1^+\} - T_N(w)$

$I_{N'}(w) = \{1^+\} - I_N(w)$

$F_{N'}(w) = \{1^+\} - F_N(w)$

2.6 Definition (Containment) [10, 20]

A neutrosophic set P is contained in the other neutrosophic set Q, P ⊆ Q, if and only if

$\inf(T_P) \le \inf(T_Q), \sup(T_P) \le \sup(T_Q)$,

$\inf(I_P) \ge \inf(I_Q), \sup(I_P) \ge \sup(I_Q)$.

$\inf(F_P) \ge \inf(F_Q), \sup(F_P) \ge \sup(F_Q)$.

2.7 Definition (Union) [10]

The union of two neutrosophic sets P and Q is a neutrosophic set R, written as R = P ∪ Q, whose truth-membership, indeterminacy-membership and falsity membership functions are related to those of P and Q by

$T_R(w) = T_P(w) + T_Q(w) - T_P(w) \times T_Q(w)$,

$I_R(w) = I_P(w) + I_Q(w) - I_P(w) \times I_Q(w)$,

$F_R(w) = F_P(w) + F_Q(w) - F_P(w) \times F_Q(w)$, for all $w \in W$.

2.8 Definition (Intersection) [10]

The intersection of two neutrosophic sets P and Q is a neutrosophic set C, written as R =P∪Q, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of P and Q by

$T_R(w) = T_P(w) \times T_Q(w)$,

$I_R(w) = I_P(w) \times I_Q(w)$,

$F_R(w) = F_P(w) \times F_Q(w)$, for all $w \in W$.

2.9 Definition (Hamming distance) [20, 53]

Let $P = \{\langle w_i : T_P(w_i), I_P(w_i), F_P(w_i)\rangle, i = 1, 2, \ldots, n\}$ and $Q = \{\langle w_i : T_P(w_i), I_Q(w_i), F_Q(w_i)\rangle, i = 1, 2, \ldots, n\}$ be any two neutrosophgic sets. Then the Hamming distance between P and Q can be defined as follows:

$$d(P,Q) = \sum_{i=1}^{n}\left(\left|T_P(w_i) - T_Q(w_i)\right| + \left|I_P(w_i) - I_Q(w_i)\right| + \left|F_P(w_i) - F_Q(w_i)\right|\right) \quad (1)$$

2.10 Definition (Normalized Hamming distance)

The normalized Hamming distance between two SVNSs, A and B can be defined as follows:

$${}_N d(P,Q) = \frac{1}{3n}\sum_{i=1}^{n}\left(\left|T_P(w_i) - T_Q(w_i)\right| + \left|I_P(w_i) - I_Q(w_i)\right| + \left|F_P(w_i) - F_Q(w_i)\right|\right) \quad (2)$$

2. 11 Definition (Interval neutrosophic set) [51]

Let *W* be a non-empty set. An interval neutrosophic set (INS) P in *W* is characterized by the truth-membership function P_T, the indeterminacy-membership function P_I and the falsity-membership function P_F. For each point w ∈ *W*, $P_T(w)$, $P_I(w), P_F(w)) \subseteq [0,1]$. Here P can be presented as follows:

$$P = \{< w, [P_T^L(w), P_T^U(w)], [P_I^L(w), P_I^U(w)], [P_F^L(w), P_F^U(w)] > : w \in W\}.$$

2.12 Definition (Neutrosophic cubic set) [44, 45]

Let W be a set. A neutrosophic cubic set (NCS) in *W* is a pair (P, Λ) where P = $\{\langle w, P_T(w), P_I(w), P_F(w)\rangle / w \in W\}$ is an interval neutrosophic set in *W* and $\Lambda = \{\langle w, \lambda_T(w), \lambda_I(w), \lambda_F(w)\rangle / w \in W\}$ is a neutrosophic set in W.

3 GRA for MADM in neutrosophic cubic set environment

We consider a MADM problem with r alternatives {A_1, A_2, …, A_r} and s attributes {C_1, C_2, …, C_s}. Every attribute is not equally important to decision maker. Decision maker provides the neutrosophic weights for each attribute. Let $W = \{w_1, w_2, \ldots, w_s\}^T$ be the neutrosophic weights of the attributes.

Step 1 Construction of decision matrix

Step1.The decision matrix (see Table 1) is constructed as follows:

Table 1: Decision matrix

$$A = (a_{ij})_{r\times s} = \begin{pmatrix} & C_1 & C_2 & \ldots & C_s \\ A_1 & (A_{11}, \Lambda_{11}) & (A_{12}, \Lambda_{12}) & \ldots & (A_{1s}, \Lambda_{1s}) \\ A_2 & (A_{21}, \Lambda_{21}) & (A_{22}, \Lambda_{22}) & \ldots & (A_{2s}, \Lambda_{2s}) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ A_r & (A_{r1}, \Lambda_{r1}) & (A_{r2}, \Lambda_{r2}) & \ldots & (A_{rs}, \Lambda_{rs}) \end{pmatrix}_{r\times s}$$

Here $a_{ij} = (A_{ij}, \Lambda_{ij})$, $A_{ij} = \left([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U]\right)$, $\Lambda_{ij} = (T_{ij}, I_{ij}, F_{ij})$, a_{ij} means the rating of alternative A_i with respect to the attribute C_j. Each weight component w_j of attribute C_j has been taken as neutrosophic set and

$$w_j = (T_j, I_j, F_j),\ A_{ij} = \left([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U]\right)$$

are interval neutrosophic set and $\Lambda_{ij} = (T_{ij}, I_{ij}, F_{ij})$ is a neutrosophic set.

Step 2 Crispification of neutrosophic weight set

Let $w_j = (T_j, I_j, F_j)$ be the j – th neutrosophic weight for the attribute C_j. The equivalent crisp weight of C_j is defined as follows:

$$w_j^c = \frac{\sqrt{T_j^2 + I_j^2 + F_j^2}}{\sum_{j=1}^{n}\sqrt{T_j^2 + I_j^2 + F_j^2}} \text{ and } \sum_{j=1}^{s} w_j^c = 1.$$

Step 3 Conversion of interval neutrosophic set into neutrosophic set decision matrix

In the decision matrix (1), each $A_{ij} = ([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U])$ is an INS. Taking mid value of each interval the decision matrix reduces to single valued neutrosophic decision matrix (See Table 2).

Table 2: Neutrosophic decision matrix

$$M = (m_{ij})_{r\times s} = \begin{pmatrix} & C_1 & C_2 & \dots & C_s \\ A_1 & (M_{11}, \Lambda_{11}) & (M_{12}, \Lambda_{12}) & \dots & (M_{1s}, \Lambda_{1s}) \\ A_2 & (M_{21}, \Lambda_{21}) & (M_{22}, \Lambda_{22}) & \dots & (M_{2s}, \Lambda_{2s}) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ A_r & (M_{r1}, \Lambda_{r1}) & (M_{r2}, \Lambda_{r2}) & \dots & (M_{rs}, \Lambda_{rs}) \end{pmatrix}_{r\times s}$$

where each $m_{ij} = (M_{ij}, \Lambda_{ij})$ and

$$M_{ij} = \left(\frac{T_{ij}^L + T_{ij}^U}{2}, \frac{I_{ij}^L + I_{ij}^U}{2}, \frac{F_{ij}^L + F_{ij}^U}{2}\right) = (T_{ij}^m, I_{ij}^m, F_{ij}^m).$$

Step 4 Some definitions of GRA method for MADM with NCS

The GRA method for MADM with NCS can be presented in the following steps:

Step 4.1 Definition:

The ideal neutrosophic estimates reliability solution (INERS) can be denoted as

$(M^+, \Lambda^+) = [(M_1^+, \Lambda_1^+), (M_2^+, \Lambda_2^+), \dots, (M_q^+, \Lambda_q^+)]$

and defined as $M_j^+ = (T_j^+, I_j^+, F_j^+)$, where $T_j^+ = \max_i T_{ij}^m$, $I_j^{m+} = \min_i I_{ij}^m$, $F_j^{m+} = \min_i F_{ij}^m$ and $\Lambda_j^+ = (T_j^+, I_j^+, F_j^+)$ where $T_j^+ = \max_i T_{ij}, I_j^+ = \min_i I_{ij}, F_j^+ = \min_i F_{ij}$ in the neutrosophic cubic decision matrix $M = (m_{ij})_{p\times q}$, i = 1,2,...,r and j = 1, 2, ..., s.

Step 4.2 Definition:

The ideal neutrosophic estimates unreliability solution (INEURS) can be denoted as

$(M^-, \Lambda^-) = [(M_1^-, \Lambda_1^-), (M_2^-, \Lambda_2^-), \dots, (M_s^-, \Lambda_s^-)]$

and defined as $M_j^- = (T_j^{m-}, I_j^{m-}, F_j^{m-})$ where $T_j^{m-} = \min_i T_{ij}^m$, $I_j^{m-} = \max_i I_{ij}^m$, $F_j^{m-} = \max_i F_{ij}^m$ and $\Lambda_j^- = (T_j^-, I_j^-, F_j^-)$ where $T_j^- = \min_i T_{ij}, I_j^- = \max_i I_{ij}, F_j^- = \max_i F_{ij}$ in the neutrosophic cubic decision matrix $M = (m_{ij})_{r\times s}$, i = 1,2,...,r and j = 1, 2, ...,s.

Step 4.3 Definition:

The grey relational coefficients of each alternative from INERS can be defined as:

$$(\eta_{ij}^+, \xi_{ij}^+) = \left(\frac{\min_i \min_j \delta_{ij}^+ + \lambda \max_i \max_j \delta_{ij}^+}{\delta_{ij}^+ + \lambda \max_i \max_j \delta_{ij}^+}, \frac{\min_i \min_j \Omega_{ij}^+ + \lambda \max_i \max_j \Omega_{ij}^+}{\Omega_{ij}^+ + \lambda \max_i \max_j \Omega_{ij}^+}\right)$$

Here,

$$\delta_{ij}^+ = d(M_j^+, M_{ij}) = \sum_{i=1}^{r}\left(\left|T_j^{m+} - T_{ij}^m\right| + \left|I_j^{m+} - I_{ij}^m\right| + \left|F_j^{m+} - F_{ij}^m\right|\right)$$

and $\Omega_{ij}^+ = d(\Lambda_j^+, \Lambda_{ij}) = \sum_{i=1}^{r}\left(\left|T_j^+ - T_{ij}\right| + \left|I_j^+ - I_{ij}\right| + \left|F_j^+ - F_{ij}\right|\right)$,

i = 1, 2 ,..., r and j = 1, 2, ..., s, $\lambda \in [0,1]$.

We call $(\eta_{ij}^+, \xi_{ij}^+)$ as positive grey relational coefficient.

Step 4.4 Definition:

The grey relational coefficient of each alternative from INEURS can be defined as:

$$(\eta_{ij}^-, \xi_{ij}^-) = \left(\frac{\min_i \min_j \delta_{ij}^- + \lambda \max_i \max_j \delta_{ij}^-}{\delta_{ij}^- + \lambda \max_i \max_j \delta_{ij}^-}, \frac{\min_i \min_j \Omega_{ij}^- + \lambda \max_i \max_j \Omega_{ij}^-}{\Omega_{ij}^- + \lambda \max_i \max_j \Omega_{ij}^-}\right)$$

Here,

$$\delta_{ij}^- = d(M_j^-, M_{ij}) = \sum_{i=1}^{r}\left(\left|T_j^{m-} - T_{ij}^m\right| + \left|I_j^{m-} - I_{ij}^m\right| + \left|F_j^{m-} - F_{ij}^m\right|\right)$$

and:

$\Omega_{ij}^- = d(\Lambda_j^-, \Lambda_{ij}) = \sum_{i=1}^{r}\left(\left|T_j^- - T_{ij}\right| + \left|I_j^- - I_{ij}\right| + \left|F_j^- - F_{ij}\right|\right)$, i = 1, 2,..., r and j = 1, 2, ..., s, $\lambda \in [0,1]$.

We call $(\eta_{ij}^-, \xi_{ij}^-)$ as negative grey relational coefficient.

λ is called distinguishable coefficient or identification coefficient and it is used to reflect the range of comparison environment that controls the level of differences of the grey relational coefficient. $\lambda = 0$ indicates comparison environment disappears and $\lambda = 1$ indicates comparison environment is unaltered. Generally, $\lambda = 0.5$ is assumed for decision making.

Step 4.5 Calculation of weighted grey relational coefficients for MADM with NCS

We can construct two $r \times s$ order matrices namely $M^{+}_{GR} = \left(\eta^{+}_{ij}, \xi^{+}_{ij}\right)_{r\times s}$ and $M^{-}_{GR} = \left(\eta^{-}_{ij}, \xi^{-}_{ij}\right)_{r\times s}$. The crisp weight is to be multiplied with the corresponding elements of M^{+}_{GR} and M^{-}_{GR} to obtain weighted matrices ${}_{w}M^{+}_{GR}$ and ${}_{w}M^{-}_{GR}$ and defined as:

$${}_{w}M^{+}_{GR} = \left(w^{c}_{j}\eta^{+}_{ij}, w^{c}_{j}\xi^{+}_{ij}\right)_{r\times s} = \left(\tilde{\eta}^{+}_{ij}, \tilde{\xi}^{+}_{ij}\right)_{r\times s}$$

and $${}_{w}M^{-}_{GR} = \left(w^{c}_{j}\eta^{-}_{ij}, w^{c}_{j}\xi^{-}_{ij}\right)_{r\times s} = \left(\tilde{\eta}^{-}_{ij}, \tilde{\xi}^{-}_{ij}\right)_{r\times s}$$

Step 4.6

From the definition of grey relational coefficient, it is clear that grey relational coefficients of both types must be less than equal to one. This claim is going to be proved in the following theorems.

Theorem 1

The positive grey relational coefficient is less than unity i.e. $\eta^{+}_{ij} \le 1$, and $\xi^{+}_{ij} \le 1$.

Proof:

From the definition

$$\eta^{+}_{ij} = \frac{\min_i \min_j \delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij}}{\delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij}}$$

Now, $\min_i \min_j \delta^{+}_{ij} \le \delta^{+}_{ij}$

$$\Rightarrow \min_i \min_j \delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij} \le \delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij}$$

$$\Rightarrow \frac{\min_i \min_j \delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij}}{\delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij}} \le 1$$

$$\Rightarrow \eta^{+}_{ij} \le 1$$

Again, from the definition, we can write:

$$\xi^{+}_{ij} = \frac{\min_i \min_j \Omega^{+}_{ij} + \lambda \max_i \max_j \Omega^{+}_{ij}}{\Omega^{+}_{ij} + \lambda^{+} \max_i \max_j \Omega^{+}_{ij}}$$

Now, $\min_i \min_j \Omega^{+}_{ij} \le \Omega^{+}_{ij}$

$$\Rightarrow \min_i \min_j \Omega^{+}_{ij} + \lambda \max_i \max_j \Omega^{+}_{ij} \le \Omega^{+}_{ij} + \lambda \max_i \max_j \Omega^{+}_{ij}$$

$$\Rightarrow \xi^{+}_{ij} = \frac{\min_i \min_j \Omega^{+}_{ij} + \lambda \max_i \max_j \Omega^{+}_{ij}}{\Omega^{+}_{ij} + \lambda \max_i \max_j \Omega^{+}_{ij}}$$

$$\Rightarrow \xi^{+}_{ij} \le 1.$$

Theorem 2

The negative grey relational coefficient is less than unity i.e. $\eta^{-}_{ij} \le 1$, $\xi^{-}_{ij} \le 1$.

Proof:

From the definition, we can write

$$\eta^{-}_{ij} = \frac{\min_i \min_j \delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij}}{\delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij}}$$

Now, $\min_i \min_j \delta^{-}_{ij} \le \delta^{-}_{ij}$

$$\Rightarrow \min_i \min_j \delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij} \le \delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij}$$

$$\eta^{-}_{ij} = \frac{\min_i \min_j \delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij}}{\delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij}}$$

$$\Rightarrow \eta^{-}_{ij} \le 1$$

Again, from the definition

$$\xi^{-}_{ij} = \frac{\min_i \min_j \Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij}}{\Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij}}$$

Now, $\min_i \min_j \Omega^{-}_{ij} \le \Omega^{-}_{ij}$

$$\Rightarrow \min_i \min_j \Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij} \le \Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij}$$

$$\Rightarrow \xi^{-}_{ij} = \frac{\min_i \min_j \Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij}}{\Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij}}$$

$$\Rightarrow \xi^{-}_{ij} \le 1.$$

Note 1:

i. Since $\eta^{+}_{ij} \le 1$, $w^{c}_{j} \le 1$ then $\eta^{+}_{ij} w^{c}_{j} \le 1 \Rightarrow \tilde{\eta}^{+}_{ij} \le 1$

ii. Since $\eta^{-}_{ij} \le 1$, $w^{c}_{j} \le 1$ then $\eta^{-}_{ij} w^{c}_{j} \le 1 \Rightarrow \tilde{\eta}^{-}_{ij} \le 1$

iii. Since $\xi^{+}_{ij} \le 1$, $w^{c}_{j} \le 1$ then $\xi^{+}_{ij} w^{c}_{j} \le 1 \Rightarrow \tilde{\xi}^{+}_{ij} \le 1$

iv. Since $\xi^{-}_{ij} \le 1$, $w^{c}_{j} \le 1$ then $\xi^{-}_{ij} w^{c}_{j} \le 1 \Rightarrow \tilde{\xi}^{-}_{ij} \le 1$

Step 4.7

We define the ideal or standard grey relational coefficient as (1, 1). Then we construct ideal grey relational coefficient matrix of order $r\times s$ (see Table 3).

Table 3: Ideal grey relational coefficient matrix of order $r\times s$

$$I = \begin{pmatrix} (1,1)\,(1,1)\ldots(1,1) \\ (1,1)\,(1,1)\ldots(1,1) \\ \ldots\ldots\ldots\ldots\ldots \\ (1,1)\,(1,1)\ldots(1,1) \end{pmatrix}_{r\times s}$$

Step 5 Determination of Hamming distances

We find the distance d^{+}_{i} between the corresponding elements of i-th row of I and ${}_{w}M^{+}_{GR}$ by employing Hamming distance. Similarly, d_i^- can be determined between I and ${}_wM_{GR}^-$ by employing Hamming distance as follows:

$$d_i^+ = \frac{1}{2s}[\sum_{j=1}^{s}\{|1-\tilde{\eta}_{ij}^+| + |1-\tilde{\xi}_{ij}^+|\}], \text{ i = 1, 2, …, r.}$$

$$d_i^- = \frac{1}{2s}[\sum_{j=1}^{s}\{|1-\tilde{\eta}_{ij}^-| + |1-\tilde{\xi}_{ij}^-|\}], \text{ i = 1, 2, …, r.}$$

Step 6 Determination of relative closeness coefficient

The relative closeness coefficient can be calculated as:

$$\Delta_i = \frac{d_i^+}{d_i^+ + d_i^-} \quad \text{i = 1, 2, ..., r.}$$

Step 7 Ranking the alternatives

According to the relative closeness coefficient, the ranking order of all alternatives is determined. The ranking order is made according to descending order of relative closeness coefficients.

4 Numerical example

Consider a hypothetical MADM problem. The problem consists of single decision maker, three alternatives with three alternatives {A_1, A_2, A_3} and four attributes {C_1, C_2, C_3, C_4}. The solution of the problem is presented using the following steps:

Step 1. Construction of neutrosophic cubic decision matrix

The decison maker forms the decision matrix which is displayed in the Table 4, at the end of article.

Step 2. Crispification of neutrosophic weight set

The neutrosophic weights of the attributes are taken as:

$$W = \{(0.5, 0.2, 0.1), (0.6, 0.1, 0.1), (0.9, 0.2, 0.1), (0.6, 0.3, 0.4)\}^T$$

The equivalent crisp weights are

$$W^c = \{(0.1907), (0.2146), (0.3228), (0.2719)\}^T$$

Step 3 Conversion of interval neutrosophic set into neutrosophic set in decision matrix

Taking the mid value of INS in the Table 4, the new decision matrix is presented in the following Table 5, at the end of article.

Step 4 Some Definitions of GRA method for MADM with NCS

The ideal neutrosophic estimates reliability solution (INERS) (M^+, Λ^+) and the ideal neutrosophic estimates unreliability solution (INEURS) (M^-, Λ^-) are presented in the Table 6, at the end of article.

$\delta^+ = (\delta_{ij}^+) = (d(M_j^+, M_{ij}))\forall i, j$ is presented as below:

$$\delta^+ = \begin{pmatrix} 0.85 & 0.95 & 0.05 & 0.15 \\ 0.65 & 0 & 0.7 & 0.25 \\ 0.05 & 0.15 & 0.25 & 0.45 \end{pmatrix}$$

The $\Omega^+ = (\Omega_{ij}^+) = (d(\Lambda_j^+, \Lambda_{ij}))\forall i, j$ is presented as below:

$$\Omega^+ = \begin{pmatrix} 0.45 & 1.2 & 0.4 & 0.15 \\ 0.05 & 0.5 & 0.2 & 0.2 \\ 0.25 & 0.3 & 0.2 & 0.5 \end{pmatrix}$$

$\delta^- = (\delta_{ij}^-) = (d(M_j^-, M_{ij}))\forall i, j$ is presented as below:

$$\delta^- = \begin{pmatrix} 0.25 & 0.3 & 0.7 & 0.55 \\ 0.45 & 1.2 & 0 & 0.45 \\ 1.05 & 0.65 & 0.6 & 0.25 \end{pmatrix}$$

The $\Omega^- = (\Omega_{ij}^-) = (d(\Lambda_j^-, \Lambda_{ij}))\forall i, j$ is presented as:

The positive grey relational coefficient $M_{GR}^+ = (\eta_{ij}^+, \xi_{ij}^+)_{3\times4}$ is presented in the Table 7, at the end of article.

The negative grey relational coefficient $M_{GR}^- = (\eta_{ij}^-, \xi_{ij}^-)_{3\times4}$ is presented in the Table 8, at the end of article.

Now, we multiply the crisp weight with the corresponding elements of M_{GR}^+ and M_{GR}^- to get weighted matrices ${}_wM_{GR}^+$ and ${}_wM_{GR}^-$ and which are described in the Table 9 and 10 respectively, at the end of article.

Step 5 Determination of Hamming distances

Hamming distances are calculated as follows:

$d_1^+ = 0.84496$, $d_1^- = 0.83845625$,

$d_2^+ = 0.82444375$, $d_2^- = 0.85328875$,

$d_3^+ = 0.82368675$, $d_3^- = 0.85277$.

Step 6 Determination of relative closeness coefficient

The relative closeness coefficients are calculated as:

$$\Delta_1 = \frac{d_1^+}{d_1^+ + d_1^-} = 0.501932$$

$$\Delta_2 = \frac{d_2^+}{d_2^+ + d_2^-} = 0.491403576$$

$$\Delta_3 = \frac{d_3^+}{d_3^+ + d_3^-} = 0.49132$$

Step 7 Ranking the alternatives

The ranking of alternatives is made according to descending order of relative closeness coefficients. The ranking order is shown in the Table 11 below.

Alternatives	Ranking order
A_3	1
A_2	2
A_1	3

Conclusion

This paper develops GRA based MADM in neutrosophic cubic set environment. This is the first approach of GRA in MADM in neutrosophic cubic set environment. The proposed approach can be applied to other decision making problems such as pattern recognition, personnel selection, etc.

The proposed approach can be applied for decision making problem described by internal NCSs and external NCSs. We hope that the proposed approach will open up a new avenue of research in newly developed neutrosophic cubic set environment.

References

[1] C. L. Hwang, and K. Yoon. Multiple attribute decision making: methods and applications: a state-of-the-art survey, Springer, London, (1981).

[2] J.P. Brans, P. Vinvke, and B. Mareschal. How to select and how to rank projects: The PROMETHEE method. European Journal of Operation Research, 24(1986), 228–238.

[3] S. Opricovic. Multicriteria optimization of civil engineering systems. Faculty of Civil Engineering, Belgrade (1998).

[4] S. Opricovic, and G. H. Tzeng. Compromise solution by MCDM methods: a comparative analysis of VIKOR and TOPSIS. European Journal of Operation Research, 156 (2004), 445–455.

[5] B. Roy. The outranking approach and the foundations of ELECTRE methods. Theory Decision, 31(1991), 49–73.

[6] L. A. Zadeh. Fuzzy Sets. Information and Control, 8 (3) (1965), 338-353.

[7] R. Bellman, and L. A. Zadeh. Decision making in a fuzzy environment, Management Science, 17B (4) (1970), 141-164.

[8] K. T. Atanassov. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20 (1986), 87–96.

[9] K. T. Atanassov. On Intuitionistic fuzzy set theory, studies in fuzziness and soft computing, Springer- Verlag, Berlin (2012).

[10] F. Smarandache. A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic. Rehoboth: American Research Press (1998).

[11] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman. Single valued neutrosophic sets, Multispace and Multistructure, 4(2010), 410–413.

[12] J. Ye. Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. International Journal of General Systems, 42(4) (2013), 386-394.

[13] J. Ye. Single valued neutrosophic cross entropy for multicriteria decision-making problems. Applied Mathematical Modeling, (2013), doi:10.1016/j.apm.2013.07.020.

[14] M. Sodenkamp, Models, methods and applications of group multiple-criteria decision analysis in complex and uncertain systems. Dissertation, University of Paderborn, Germany, 2013.

[15] K. Mondal, and S. Pramanik. Neutrosophic tangent similarity measure and its application to multiple attribute decision-making. Neutrosophic Sets and Systems, 9 (2015), 85-92.

[16] P. Biswas, S. Pramanik, and B. C. Giri. Cosine similarity measure based multi-attribute decision-making with trapezoidal fuzzy neutrosophic numbers. Neutrosophic Sets and Systems, 8 (2015), 47-57.

[17] K. Mondal, and S. Pramanik. Multi-criteria group decision-making approach for teacher recruitment in higher education under simplified Neutrosophic environment. Neutrosophic Sets and Systems, 6 (2014), 28-34.

[18] K. Mondal, and S. Pramanik. Neutrosophic decision-making model of school choice. Neutrosophic Sets and Systems, 7 (2015), 62-68.

[19] P. Liu, and Y. Wang. Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean. Neural Computing and Applications, 25(7) (2014), 2001–2010.

[20] P. Biswas, S. Pramanik, and B. C. Giri. TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment. Neural Computing and Applications. doi: 10.1007/s00521-015-1891-2, 2015.

[21] P. Chi, and P. Liu. An extended TOPSIS method for the multi-attribute decision making problems on interval neutrosophic set. Neutrosophic Sets and Systems, 1 (2013), 63–70.

[22] S. Nabdaban, and S. Dzitac. Neutrosophic TOPSIS: a general view. 6th International Conference on Computers Communications and Control, 2016, 250-253.

[23] S. Broumi, J. Ye, and F. Smnarandache. An extended TOPSIS method for multiple attribute decision making based on interval neutrosophic uncertain linguistic variables. Neutrosophic Sets and Systems, 8 (2015), 22-31.

[24] S. Pramanik, P. Biswas, and B. C. Giri. Hybrid vector similarity measures and their applications to multi-attribute decision making under neutrosophic environment. Neural Computing and Applications, (2015), doi: 10.1007/s00521-015-2125-3.

[25] P. Biswas, S. Pramanik, and B. C. Giri. Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making, NSS, 12 (2016), 20-40.

[26] R. Sahin,and P. Liu. Maximizing deviation method for neutrosophic multiple attribute decision making with incomplete weight information. Neural Computing and Applications, (2015), doi: 10.1007/s00521-015-1995-8.

[27] J. Ye. Bidirectional projection method for multiple attribute group decision making with neutrosophic numbers. Neural Computing and Applications, (2015), doi: 10.1007/s00521-015-2123-5.

[28] P. Biswas, S. Pramanik, and B. C. Giri. Value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to multi-attribute decision making. Neutrosophic Sets and Systems, 12 (2016), 127-138.

[29] P. P. Dey, S. Pramanik, and B.C. Giri. Extended projection-based models for solving multiple attribute decision making problems with interval –valued neutrosophic information. In: New Trends in Neutrosophic Theory and Applications, eds. F. Smarandache and S. Pramanik, Pons Editions, Brussels, 2016, 127- 140.
[30] J. L. Deng. Introduction to grey system theory. The Journal of Grey System, 1(1) (1989), 1–24.
[31] J. L. Deng. The primary methods of grey system theory, Huazhong University of Science and Technology Press, Wuhan, (2005).
[32] S. Pramanik, and D. Mukhopadhyaya. Grey relational analysis based intuitionistic fuzzy multi criteria group decision making approach for teacher selection in higher education. International Journal of Computer Applications, 34 (10) (2011), 21 – 29.
[33] P.P. Dey, S. Pramanik, and B.C. Giri, Multi-criteria group decision making in intuitionistic fuzzy environment based on grey relational analysis for weaver selection in Khadi institution. Journal of Applied and Quantitative Methods, 10(4) (2015), 1-14.
[34] R. V. Rao, and D. Singh. An improved grey relational analysis as a decision making method for manufacturing situations. International Journal of Decision Science, Risk and Management, 2(2010), 1–23.
[35] G. W. Wei. Grey relational analysis method for intuitionistic fuzzy multiple attribute decision making. Expert Systems with Applications, 38(2011), 11671-11677.
[36] P. Biswas, S. Pramanik, and B.C. Giri. Entropy based grey relational analysis method for multi-attribute decision – making under single valued neutrosophic assessments. Neutrosophic Sets and Systems, 2 (2014), 102 – 110.
[37] P.P. Dey, S. Pramanik, and B.C. Giri. An extended grey relational analysis based multiple attribute decision making in interval neutrosophic uncertain linguistic setting. Neutrosophic Sets and Systems, 11 (2016), 21-30.
[38] S. Pramanik, and K. Mondal. Interval neutrosophic multi-attribute decision-making based on grey relational analysis. Neutrosophic Sets and Systems, 9(2015), 13-22.
[39] P. K. Maji. Neutrosophic soft set. Annals of fuzzy Mathematics and Informatics, 5 (1) (2013), 157 – 168.
[40] S. Broumi, and F. Smarandache. Single valued neutrosophic soft expert sets and their application in decision making. Journal of New Theory, 3 (2015), 67 – 88.
[41] S. Broumi, F. Smarandache, and M. Dhar. Rough neutrosophic sets. Neutrosophic Sets and Systems, 3 (2014), 62-67.
[42] I. Deli, M. Ali, and F. Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. Proceedings of the 2015 International Conference on Advanced Mechatronic Systems, Beijing, China, August (2015), 22-24.
[43] S. Pramanik, and K. Mondal. Rough bipolar neutrosophic set. Global Journal of Engineering Science and Research Management, 3 (6) (2015), 71- 81.
[44] Y. B. Jun, F. Smarandache, and C. S. Kim. Neutrosophic cubic sets. New mathematics and natural computation, 9 (2015), 1 – 15.
[45] M. Ali, I. Deli, and F. Smarandache, The theory of neutrosophic cubic sets and their applications in pattern recognition, Journal of intelligent and fuzzy systems, 30 (2016), 1957-1963.
[46] Y. B. Jun, C. S. Kim, K. O. Yang. Cubic sets. Annals Fuzzy Mathematics Information, 4 (1) (2012), 83 – 98.
[47] S. Pramanik, and S. Dalapati. GRA based multi criteria decision making in generalized neutrosophic soft set environment. Global Journal of Engineering Science and Research Management, 3 (5) (2016), 153 - 169.
[48] P.P. Dey, S. Pramanik, and B.C. Giri. Neutrosophic soft multi attribute group decision making based on grey relational analysis method. Journal of New Results in Science, 10 (2016), 25 – 37.
[49] K. Mondal, and S. Pramanik. Rough neutrosophic multi-Attribute decision-making based on grey relational analysis. Neutrosophic Sets and Systems, 7 (2014), 8-17.
[50] S. Pramanik, and K. Mondal. Interval neutrosophic multi-Attribute decision-making based on grey relational analysis. Neutrosophic Sets and Systems, 9 (2015), 13-22.
[51] H. Wang, F. Smarandache, Y.-Q. Zhang, and R. Sunderraman, Interval Neutrosophic Sets and Logic: Theory and Applications in Computing, Hexis, Phoenix, AZ, (2005).
[52] I.B. Turksen. Interval-valued fuzzy sets based on normal forms. Fuzzy Sets and Systems, 20 (1986), 191–210.
[53] R. W. Hamming. Error detecting and error correcting codes. Bell System Technical Journal, 29 (2) (1950), 147–160.

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Table 4: Construction of neutrosophic cubic decision matrix

$A=(a_{ij})_{3\times 4}=$

	C_1	C_2	C_3	C_4
A_1	(([0.2, 0.3],[0.3, 0.5],[0.2, 0.5]), (0.3, 0.2, 0.3))	([0.1, 0.3],[0.2, 0.4],[0.3, 0.6]), (0.2, 0.5, 0.4))	(([0.6, 0.9],[0.1, 0.2],[0, 0.2]), (0.4, 0.5, 0.1))	(([0.4, 0.7],[0.1, 0.3],[0.2, 0.3]), (0.7, 0.3, 0.2))
A_2	([0.6, 0.8],[0.4, 0.6],[0.3, 0.7]), (0.5, 0.2, 0.1))	([0.7, 0.9],[0.2, 0.3],[0.1, 0.3]), (0.7, 0.3, 0.3))	(([0.5, 0.7],[0.4, 0.6],[0.3, 0.5]), (0.4, 0.1, 0.2))	(([0.4, 0.5],[0.1, 0.3],[0.2, 0.3]), (0.6, 0.2, 0.1))
A_3	(([0.4, 0.9],[0.1, 0.4],[0, 0.2]), (0.25, 0.15, 0.1))	(([0.8, 0.9],[0.4, 0.7],[0.4, 0.6]), (0.8, 0.1, 0.2))	(([0.6, 0.9],[0.1, 0.3],[0, 0.3]), (0.5, 0.4, 0.3))	(([0.6, 0.8],[0.5, 0.7],[0.2, 0.4]), (0.5, 0.1, 0.4))

Table 5: Construction of neutrosophic decision matrix

$M=(m_{ij})_{3\times 4}=$

	C_1	C_2	C_3	C_4
A_1	((0.25, 0.4, 0.35), (0.3, 0.2, 0.3))	((0.2, 0.3, 0.45), (0.2, 0.5, 0.4))	((0.75, 0.15, 0.1), (0.4, 0.5, 0.1))	((0.55, 0.2, 0.25), (0.7, 0.3, 0.2))
A_2	((0.7,0.5, 0.5), (0.5, 0.2, 0.1))	((0.8, 0.25, 0.2), (0.7, 0.3, 0.3))	((0.6, 0.5, 0.4), (0.4, 0.1, 0.2))	((0.45, 0.2, 0.25), (0.6, 0.2, 0.1))
A_3	((0.65, 0.25, 0.1), (0.25, 0.15, 0.1))	((0.85, 0.55, 0.5), (0.8, 0.1, 0.2))	((0.75, 0.2, 0.15), (0.5, 0.4, 0.3))	((0.7, 0.6, 0.3), (0.5, 0.1, 0.4))

Table 6: The ideal neutrosophic estimates reliability solution (INERS) (M^{+}, Λ^{+}) and the ideal neutrosophic estimates unreliability solution (INEURS) (M^{-}, Λ^{-})

(M^{+}, Λ^{+})	((0.7,0.25,0.1), (0.5,0.15,0.1))	((0.85,0.25,0.2), (0.8,0.1,0.2))	((0.75,0.15,0.1), (0.5,0.1,0.1))	((0.7,0.2,0.25), (0.7,0.1,0.1))
(M^{-}, Λ^{-})	((0.25,0.5,0.5), (0.25,0.2,0.3))	((0.2,0.55,0.5), (0.2,0.5,0.4))	((0.6,0.5,0.4), (0.4,0.5,0.3))	((0.45,0.6,0.3), (0.5,0.3,0.4))

Table 7: The positive grey relational coefficient $M_{GR}^{+}=\left(\eta_{ij}^{+}, \xi_{ij}^{+}\right)_{3\times 4}$

$$M_{GR}^{+}=\begin{pmatrix} (0.3585, 0.6190) & (0.333, 0.3611) & (0.9048, 0.65) & (0.76, 0.7222) \\ (0.4222, \quad 1\ ) & (1, 0.5909) & (0.4042, 0.8125) & (0.6552, 0.8125) \\ (0.9048, 0.7647) & (0.76, 0.7222) & (0.6552, 0.8125) & (0.5135, 0.5909) \end{pmatrix}$$

Table 8: The negative grey relational coefficient $M_{GR}^{-} = \left(\eta_{ij}^{-}, \xi_{ij}^{-}\right)_{3\times4}$

$$M_{GR}^{-} = \begin{pmatrix} (0.7059, 0.5454) & (0.6667, 1) & (0.4615, 0.75) & (0.5217, 0.6) \\ (0.5714, 0.5714) & (0.3333, 0.4286) & (1, 0.5454) & (0.5714, 0.5454) \\ (0.3636, 0.7059) & (0.48, 0.3333) & (0.5, 0.75) & (0.7059, 0.75) \end{pmatrix}$$

Table 9: Weighted matrix $_{w}M_{GR}^{+}$ $_{w}M_{GR}^{+} =$

$$\begin{pmatrix} (0.06836, 0.11804) & (0.07153, 0.07749) & (0.29207, 0.20982) & (0.20664, 0.19637) \\ (0.08051, 0.1907) & (0.2146, 0.12681) & (0.13048, 0.26228) & (0.17815, 0.22092) \\ (0.17252, 0.14583) & (0.163096, 0.15498) & (0.21150, 0.26228) & (0.13962, 0.16066) \end{pmatrix}$$

Table 10: Weighted matrix $_{w}M_{GR}^{-}$

$$_{w}M_{GR}^{-} = \begin{pmatrix} (0.13461, 0.10401) & (0.14307, 0.2146) & (0.14897, 0.2421) & (0.14185, 0.16314) \\ (0.10896, 0.10896) & (0.07153, 0.08173) & (0.3228, 0.17606) & (0.15536, 0.14829) \\ (0.06934, 0.13461) & (0.10301, 0.07153) & (0.1614, 0.2421) & (0.19193, 0.20392) \end{pmatrix}$$