Solving Numerically a System of Coupled Riccati ODEs for Incompressible Non-Stationary 3D Navier-Stokes Equations

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Abstract

In a recent paper, Ershkov derived a system of two coupled Riccati ODEs as solution of non-stationary incompressible 3D Navier-Stokes equations. Now in this paper, we solve these coupled Riccati ODEs using: a) Maxima and b) Mathematica 11 computer algebra packages. The result seems to deserve further investigation in particular in comparison with rigid body motion, which will be discussed elsewhere.

Introduction

The Riccati equation, named after the Italian mathematician Jacopo Francesco Riccati, is a basic first-order nonlinear ordinary differential equation (ODE) that arises in different fields of mathematics and physics.[4]

In fluid mechanics, there is an essential deficiency of the analytical solutions of Navier–Stokes equations for 3D case of non-stationary flow. The Navier-Stokes system of equations for incompressible flow of Newtonian fluids should be presented in the Cartesian coordinates as below (under the proper initial conditions):[1]

$$\nabla . \vec{u} = 0, \tag{1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla p}{\rho} + \nu \cdot \nabla^2 \vec{u} + \vec{F}, \tag{2}$$

Where u is the flow velocity, a vector field; ρ is the fluid density, p is the pressure, v is the kinematic viscosity, and F represents external force (per unit mass of volume) acting on the fluid.[1]

In ref. [1], Ershkov explores the ansatz of derivation of non-stationary solution for the Navier–Stokes equations in the case of incompressible flow, which was suggested earlier. In general case, such a solution should be obtained from the mixed system of 2 coupled Riccati ordinary differential equations (in regard to the time-parameter t). But instead of solving the problem analytically, we will try to find a numerical solution.

The coupled Riccati ODEs read as follows:[1]

$$a' = \frac{w_y}{2} \cdot a^2 - (w_x \cdot b) \cdot a - \frac{w_y}{2}(b^2 - 1) + w_z \cdot b, \tag{3}$$

$$b' = -\frac{w_x}{2} \cdot b^2 - (w_y \cdot a) \cdot b - \frac{w_x}{2}(a^2 - 1) + w_z \cdot a \tag{4}$$

We are going to rewrite the above coupled equations in Maxima language.

Computer algebra solution

The above coupled Riccati ODEs (1) and (2) can be rewritten as follows:[3]

$$a(t)' = \frac{v}{2} \cdot a(t)^2 - (u \cdot b(t)) \cdot a(t) - \frac{v}{2}(b(t)^2 - 1) + w \cdot b(t), \tag{5}$$

$$b(t)' = -\frac{u}{2} \cdot b(t)^2 - (v \cdot a(t)) \cdot b(t) - \frac{u}{2}(a(t)^2 - 1) + w \cdot a(t) \tag{6}$$

We will find out the solution of the above coupled ODE equations using two computer algebra packages: Maxima, then Mathematica.

A. Solving with Maxima

Maxima expression of coupled Riccati ODEs (5) and (6) are as follows:[3]

‘diff(a(t),t)=v/2*a(t)^2-(u*b(t))*a(t)-v/2*(b(t)^2-1)+w*b(t), (7)

‘diff(b(t),t)=-u/2*b(t)^2-(v*a(t))*b(t)-u/2*(a(t)^2-1)+w*a(t). (8)

The Maxima results are as shown below:

(%i3) 'diff(a(t),t)=v/2*a(t)^2-(u*b(t))*a(t)-v/2*(b(t)^2-1)+w*b(t);

$$(\%o3)\quad \frac{d}{d\,t}\,a(t)=b(t)\,w-\frac{(b(t)^2-1)\,v}{2}+\frac{a(t)^2\,v}{2}-a(t)\,b(t)\,u$$

(%i4) 'diff(b(t),t)=-u/2*b(t)^2-(v*a(t))*b(t)-u/2*(a(t)^2-1)+w*a(t);

$$(\%o4)\quad \frac{d}{d\,t}\,b(t)=a(t)\,w-a(t)\,b(t)\,v-\frac{b(t)^2\,u}{2}-\frac{(a(t)^2-1)\,u}{2}$$

(%i5) desolve([%o3,%o4],[a(t),b(t)]);

(%o5) [a(t)=ilt(-(

((laplace(b(t)^2,t,g34120)-laplace(a(t)^2,t,g34120))*v+2*laplace(a(t)*b(t),t,g34120)*u-2*a(0))*

g34120^2+(

(2*laplace(a(t)*b(t),t,g34120)*v+(laplace(b(t)^2,t,g34120)+laplace(a(t)^2,t,g34120))*u-2*b(0))*

w-v)*g34120-u*w)/(2*g34120^3-2*w^2*g34120),g34120,t),b(t)=ilt(-(

(2*laplace(a(t)*b(t),t,g34120)*v+(laplace(b(t)^2,t,g34120)+laplace(a(t)^2,t,g34120))*u-2*b(0))*

g34120^2+(

((laplace(b(t)^2,t,g34120)-laplace(a(t)^2,t,g34120))*v+2*laplace(a(t)*b(t),t,g34120)*u-2*a(0))*

w-u)*g34120-v*w)/(2*g34120^3-2*w^2*g34120),g34120,t)]

It is clear that the above result is undecipherable, so now we will try to compute the solution using NDSolve function in Mathematica.

B. Solving with Mathematica 11

First, equations (5) and (6) can be rewritten in the form as follows:

$$x(t)' = \frac{v}{2} \cdot x(t)^2 - (u \cdot y(t)) \cdot x(t) - \frac{v}{2}(y(t)^2 - 1) + w \cdot y(t), \qquad (9)$$

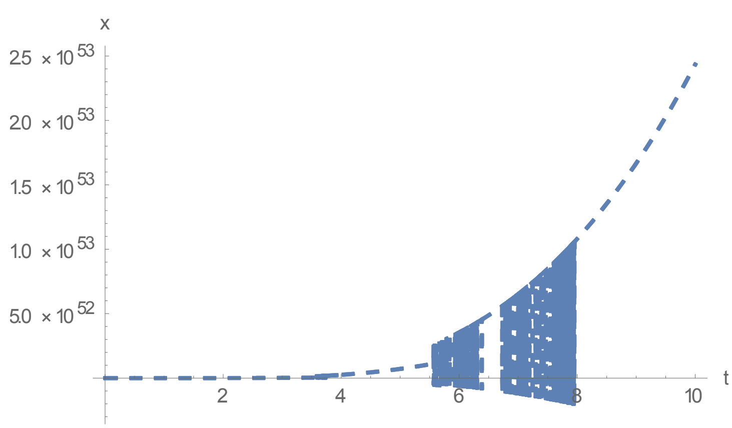
$$y(t)' = -\frac{u}{2} \cdot y(t)^2 - (v \cdot x(t)) \cdot y(t) - \frac{u}{2}(x(t)^2 - 1) + w \cdot x(t) \qquad (10)$$

Then we can put the above equations into Mathematica expression:

```
v=1;
u=1;
w=1;
{xans6[t_], vans6[t_]}=
 {x[t],y[t]}/.Flatten[NDSolve[{x'[t]==(v/2)*x[t]^2-(u*y[t])*x[t]-(v/2)*(y[t]^2-1)+w*y[t], y'[t]==-
(u/2)*y[t]^2-(v*x[t])*y[t]-(u/2)*(x[t]^2-1)+w*x[t], x[0]==1,y[0]==0}, {x[t],y[t]},{t,0,10}]]

graphx6 = Plot[xans6[t],{t,0,10}, AxesLabel->{"t","x"},PlotStyle-
>Dashing[{0.02,0.02}]];
Show[graphx6,graphx6]
```

The result is as shown below:



Concluding remarks

Using Maxima package we solve the two coupled Riccati ODEs as solution of non-stationary 3D Navier-Stokes equations.

However, we admit that the obtained computer solution is not easily plotted graphically using Maxima, therefore we decided to verify this result with other computer algebra package, i.e. Mathematica.

The result seems quite interesting to compare with solution of rigid body motion as described elsewhere by Ershkov. (to be discussed later on)

Acknowledgement

The authors (VC & SE) would like to express their gratitude to Jesus Christ and Holy Spirit who have empowered them to solve this notoriously difficult problem. *Soli Deo Gloria!*

References:

[1] Sergey V. Ershkov. Non-stationary Riccati-type flows for incompressible 3D Navier–Stokes equations. *Computers and Mathematics with Applications* 71 (2016) 1392–1404

[2] Sergey V. Ershkov. A procedure for the construction of non-stationary Riccati-type flows for incompressible 3D Navier–Stokes Equations. *Rend. Circ. Mat. Palermo* (2016) 65:73–85

[3] Victor Christianto & Florentin Smarandache. A note on computer solution of wireless Energy transmit via Magnetic Resonance. *Progress in Physics* January 2008, Vol. 1. http://www.ptep-online.com

[4] William T. Reid. *Riccati Differential Equations*. New York: Academic Press Inc., 1972

[5] L.D. Landau, E.M. Lifshitz, *Fluid mechanics, Course of Theoretical Physics 6,* 2nd revised ed., Pergamon Press, ISBN: 0-08-033932-8, 1987.

[6] I. Shingareva & Carlos Lizarraga-Celaya. *Solving Nonlinear Partial Differential Equations with Maple and Mathematica*.

[7] Patrick T. Tam. *A Physicist's guide to Mathematica.* 2nd ed. Amsterdam: Academic Press an imprint of Elsevier, 2008. 749 p.

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VC & SE

In[223]:=
```
v = 1;
u = 1;
w = 1;
{xans6[t_], vans6[t_]} =
 {x[t], y[t]} /. Flatten[
   NDSolve[{x'[t] == (v/2) * x[t]^2 - (u * y[t]) * x[t] - (v/2) * (y[t]^2 - 1) + w * y[t],
     y'[t] == - (u/2) * y[t]^2 - (v * x[t]) * y[t] - (u/2) * (x[t]^2 - 1) + w * x[t],
     x[0] == 1, y[0] == 0}, {x[t], y[t]}, {t, 0, 10}]]
```

NDSolve: At t == 1.5642693104281447`, step size is effectively zero; singularity or stiff system suspected.

Out[226]=
```
{InterpolatingFunction[ Domain: {{0., 1.56}} Output: scalar ][t],
 InterpolatingFunction[ Domain: {{0., 1.56}} Output: scalar ][t]}
```

In[229]:=
```
graphx6 =
  Plot[xans6[t], {t, 0, 10}, AxesLabel → {"t", "x"}, PlotStyle → Dashing[{0.02, 0.02}]];
Show[graphx6, graphx6]
```

Out[230]=