An efficient computational method for handling singular second-order, three points Volterra integrodifferential equations

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Abstract. In this paper, a powerful computational algorithm is developed for the solution of classes of singular second-order, three-point Volterra integrodifferential equations in favorable reproducing kernel Hilbert spaces. The solutions is represented in the form of series in the Hilbert space $W_2^3[0,1]$ with easily computable components. In finding the computational solutions, we use generating the orthogonal basis from the obtained kernel functions such that the orthonormal basis is constructing in order to formulate and utilize the solutions. Numerical experiments are carried where two smooth reproducing kernel functions are used throughout the evolution of the algorithm to obtain the required nodal values of the unknown variables. Error estimates are proven that it converge to zero in the sense of the space norm. Several computational simulation experiments are given to show the good performance of the proposed procedure. Finally, the utilized results show that the present algorithm and simulated annealing provide a good scheduling methodology to multipoint singular boundary value problems restricted by Volterra operator.

Keywords: Reproducing kernel method; Multipoint boundary conditions; Singular boundary value problem; Integrodifferential equation of Volterra type

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1 Introduction

The multipoint singular boundary value problems (BVPs) arise in a variety of differential applied mathematics and physics such as gas dynamics, nuclear physics, chemical reaction, studies of atomic structures, and atomic calculations. For instance, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multipoint singular BVP as in [1]. Many problems in the theory of elastic stability can be handled using multipoint singular BVPs as in [2]. In optimal bridge design, large size bridges are sometimes contrived with multipoint supports, which corresponds to a multipoint singular BVP as in [3]. Therefore, it appears to be very important to develop numerical or analytical methods for solving such problems.

Most scientific problems and phenomenons in different fields of sciences and engineering occur nonlinearly with a set of finite singularity. To set the scene, we know that except a limited number of these problems and phenomenons, most of them do not have analytical solutions. So these nonlinear singular equations should be solved using numerical methods or other analytical methods. Anyhow, when applied to the multipoint singular BVPs, classical numerical methods designed for regular BVPs suffer from a loss of accuracy or may even fail to converge [4–6], because of the singularity; whilst analytical methods commonly used to solve nonlinear singular differential equations are very restricted and numerical techniques involving discretization of the variables on the other hand gives rise to rounding off errors. As a result, there are some restrictions to solve these multipoint singular

BVPs, firstly, we encountered with the nonlinearity of equations, secondly, these equations are singular BVPs with multipoint boundary conditions.

The aim of this work is to give an effective algorithm for solving second-order singular ordinary differential equations subject to three-point boundary conditions based on the reproducing kernel theory. More specifically, we provide the analytical-numerical solutions for the following singular differential-operator equation:

$$\begin{aligned} & u''(x)+\frac{a(x)}{p(x)}u'(x)+\frac{b(x)}{q(x)}u(x)=[Tu](x)+f(x,u(x)), \\ & [Tu](x)=\int_0^x k(x,t)\,G(u(t))\,dt, \end{aligned} \tag{1}$$

subject to the three-point boundary conditions

$$\begin{aligned} & u(0)=0, \\ & u(1)-\alpha u(\eta)=0,\ 0<\eta<1,\ \alpha>0,\ \alpha\eta<1, \end{aligned} \tag{2}$$

where $0<t<x<1$, $k(x,t)$ is a continuous arbitrary kernel function over the square $0<t<x<1$, $u\in W_2^3[0,1]$ is an unknown function to be determined, $f(x,w_1)$, and $G(w_2)$ are continuous terms in $W_2^1[0,1]$ as $w_i=w_i(x)\in W_2^3[0,1]$, $0\le x\le 1$, $-\infty<w_i<\infty$, $i=1,2$, and $W_2^1[0,1], W_2^3[0,1]$ are reproducing kernel spaces. Here, the two real-valued functions $p(x), q(x)$ are continuous and may be equal to zero at $\{x_i\}_{i=1}^m\in[0,1]$ which make Eq. (1) to be singular at $x=x_i$, while on the other hand, $a(x), b(x)$ are continuous real-valued functions on $[0,1]$.

Investigation about second-order, three-point singular BVPs restricted by Volterra operator numerically is scarce and missing, but on the other side, the solvability analysis of second-order, three-point singular BVPs have been studied by many authors. The reader is asked to refer to [7–12] in order to know more details about these analyzes, including their modifications and conditions for use, their scientific applications, their importance and characteristics, their relationship including the differences, and others.

Reproducing kernel theory has important application in numerical analysis, computational mathematics, image processing, machine learning, finance, and probability and statistics [13–16]. Recently, a lot of research work has been devoted to the applications of the RKHS algorithm for wide classes of stochastic and deterministic problems involving operator equations, differential equations, integral equations, and integro-differential equations. The RKHS algorithm was successfully used by many authors to investigate several scientific applications side by side with their theories. The reader is kindly requested to go through [17–37] in order to know more details about the RKHS algorithm, including its modification and scientific applications, its characteristics and symmetric kernel functions, and others.

The plan of the paper is the following: in Section 2, an algorithm for solving Eqs. (1) and (2) in the space $W_2^3[0,1]$ is introduced. In Section 3, two reproducing kernel functions are building in appropriate inner product spaces in order to apply our algorithm. In Section 4, the Gram-Schmidt orthogonalization process is introduced in order to provide the theoretic basis of the algorithm. In Section 5, the n-truncation numerical solution is proved to converge to the analytical solution uniformly, whilst an error bound is provided for the present RKHS algorithm. Numerical results are presented in Section 6. Finally, in Section 7 some concluding remarks are presented.

2 Solution representation with problem formulation

Problem formulation is normally the most important part of the process. It is the determination of the discretized representation of Eq. (1), the construction of the appropriate reproducing kernel spaces, the selection of the smooth kernel functions, and the separation of multipoint boundary conditions of Eq. (2).

Anyhow, in order to apply our RKHS algorithm, we multiplying both sides of Eq. (1) by $p(x)q(x)$ to obtain the following realistic equivalent form:

$$P(x)u''(x)+Q(x)u'(x)+R(x)u(x)=F(x,u(x),[Tu](x)), \tag{3}$$

such that $F(x,u(x),[Tu](x))=p(x)q(x)([Tu](x)+f(x,u(x)))$, $P(x)=p(x)q(x)$, $Q(x)=a(x)q(x)$, and $R(x)=b(x)p(x)$. Obviously, the form of Eqs. (1) and (3) are equivalent; therefore, it suffices for us to solve Eq. (3) subject to multipoint boundary conditions of Eq. (2).

Actually, it is very difficult to expand the application of RKHS algorithm to three-point boundary conditions directly, since it is very difficult to obtain a reproducing kernel function that satisfying these points. Next, we shall introduce an effective iterative algorithm for solving second-order, three-point singular BVPs restricted by Volterra operator.

Algorithm .1 To evaluate the analytical-numerical solutions of Eqs. (3) and (2):

Step A: Construct the auxiliary two-point boundary conditions as

$$u(0) = 0,\ u(1) = \gamma,$$

where γ is constant to be determined.
Step B: Solve the following equation via RKHS algorithm:

$$P(x)\,u''(x) + Q(x)\,u'(x) + R(x)\,u(x) = F(x, u(x), [Tu](x)), \tag{4}$$

subject to the two-point boundary conditions

$$u(0) = 0,\ u(1) = \gamma. \tag{5}$$

Step C: To complete the detailed process, do steps (**I, II, III, IIII**) in the following subroutine:
Step I: Define a new unknown function as

$$v(x) = u(x) - \phi(x),$$

where $\phi(x)$ satisfies $\phi(0) = 0$ and $\phi(1) = \gamma$. Hence, $\phi(x) = \gamma x$ and $v(x) = u(x) - \gamma x$.
Step II: The shape of Eq. (4) with two-point boundary conditions of Eq. (5) can be equivalently reduced to finding $v(x)$ that satisfying the equation

$$P(x)\,v''(x) + Q(x)\,v'(x) + R(x)\,v(x) = F(x, (v+\phi)(x), [Tv+\phi](x)) - \left(\phi' Q + \phi R\right)(x), \tag{6}$$

subject to the homogeneous two-point boundary conditions

$$v(0) = 0,\ v(1) = 0. \tag{7}$$

Step III: Using RKHS algorithm, the analytical-numerical solutions of Eqs. (6) and (7) can be obtained, respectively, as

$$v(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\left[F(x_k, (v+\phi)(x_k), [T(v+\phi)](x_k)) - \left(\phi' Q + \phi R\right)(x_k)\right]\bar{\psi}_i(x),$$

$$v_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\left[F(x_k, (v+\phi)(x_k), [T(v+\phi)](x_k)) - \left(\phi' Q + \phi R\right)(x_k)\right]\bar{\psi}_i(x),$$

where β_{ik} are orthogonalization coefficients and $\bar{\psi}_i(x)$ are orthonormal functions system.
Step IV: The analytical-numerical solutions of Eqs. (3) and (2) can be immediately obtained as

$$u(x) = \phi(x) + \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\left[F(x_k, u(x_k), Tu(x_k)) - \left(\phi' Q + \phi R\right)(x_k)\right]\bar{\psi}_i(x),$$

$$u_n(x) = \phi(x) + \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\left[F(x_k, u(x_k), Tu(x_k)) - \left(\phi' Q + \phi R\right)(x_k)\right]\bar{\psi}_i(x).$$

Step D: Incorporate the three-point boundary conditions of Eq. (2) into $u_n(x)$, it follows that

$$\begin{aligned} &u_n(0) = 0, \\ &u_n(1) - \alpha u_n(\eta) = 0, \end{aligned} \tag{8}$$

Step E: Solve the linear system of Eq. (8), the n-term numerical solution $u_n(x)$ for Eqs. (1) and (2) are obtained.

We mention here that, the detail mathematical descriptions of Step III and Step IV in the aforementioned subroutine of Step C can be found in detailed in Section 4.

3 The structure of appropriate inner product spaces and the main results

In this section, a method for constructing a reproducing kernel function that satisfying the two-point boundary conditions $v(0)=0$ and $v(1)=0$ is presented. By applying some good properties of the reproducing kernel space, a very simple numerical method is provided for obtaining approximation to the solution of Eqs. (3) and (2).

Prior to discussing the applicability of the RKHS algorithm on solving second-order, three-point singular BVPs restricted by Volterra operator and their associated numerical algorithm, it is necessary to present an appropriate brief introduction to preliminary topics from the reproducing kernel theory.

Definition .1 [17] Let Π be a Hilbert space of function $\theta:\Omega\to\Pi$ on a set Ω. A function $\Gamma:\Omega\times\Omega\to\mathbb{C}$ is a reproducing kernel of Π if the following conditions are satisfied. Firstly, $\Gamma(\cdot,x)\in\Pi$ for each $x\in\Omega$. Secondly, $\langle\theta(\cdot),\Gamma(\cdot,x)\rangle=\theta(x)$ for each $\theta\in\Pi$ and each $x\in\Omega$.

To solve Eqs. (6) and (7) using RKHS algorithm, we first define and construct a reproducing kernel space $W_2^3[0,1]$ in which every function satisfies the two-point boundary conditions $v(0)=0$ and $v(1)=0$. After that, we utilize a reproducing kernel space $W_2^1[0,1]$.

Definition .2 The space $W_2^3[0,1]$ is defined as $W_2^3[0,1]=\{v\mid v,v',v''$ are absolutely continuous on $[0,1]$, $v,v',v'',v'''\in L^2[0,1]$, and $v(0)=0,v(1)=0\}$. On the other hand, the inner product and the norm in $W_2^3[0,1]$ are defined, respectively, by

$$\langle v(x),w(x)\rangle_{W_2^3}=\sum_{i=0}^{2}v^{(i)}(0)\,w^{(i)}(0)+\int_0^1 v'''(x)w'''(x)dx, \tag{9}$$

and $||v||_{W_2^3}=\sqrt{\langle v(x),v(x)\rangle_{W_2^3}}$, where $v,w\in W_2^3[0,1]$.

The Hilbert space $W_2^3[0,1]$ is called a reproducing kernel if for each fixed $x\in[0,1]$, there exist $R^{\{1\}}(x,y)\in W_2^3[0,1]$ (simply $R_x^{\{1\}}(y)$) such that $\left\langle v(y),R_x^{\{1\}}(y)\right\rangle_{W_2^3}=v(x)$ for any $v(y)\in W_2^3[0,1]$ and $y\in[0,1]$.

Theorem .1 The Hilbert space $W_2^3[0,1]$ is a complete reproducing kernel with reproducing kernel function

$$R_x^{\{1\}}(y)=\begin{cases} a_1(x)+a_2(x)y+a_3(x)y^2+a_4(x)y^3+a_5(x)y^4+a_6(x)y^5, & y\le x,\\ b_1(x)+b_2(x)y+b_3(x)y^2+b_4(x)y^3+b_5(x)y^4+b_6(x)y^5, & y>x,\end{cases} \tag{10}$$

where $a_i(x)$ and $b_i(x)$, $i=1,2,...,6$, are unknown coefficients of $R_x^{\{1\}}(y)$ and are given as

a_i's coefficients	b_i's coefficients
$a_1(x)=0,$	$b_1(x)=\frac{1}{120}x^5,$
$a_2(x)=-\frac{1}{156}x(-36+30x+10x^2-5x^3+x^4),$	$b_2(x)=-\frac{1}{312}x(-72+60x+20x^2+3x^3+2x^4),$
$a_3(x)=-\frac{1}{624}x(120-126x+10x^2-5x^3+x^4),$	$b_3(x)=-\frac{1}{624}x(120-126x-42x^2-5x^3+x^4),$
$a_4(x)=-\frac{1}{1872}x(120-126x+10x^2-5x^3+x^4),$	$b_4(x)=-\frac{1}{1872}x(120+30x+10x^2-5x^3+x^4),$
$a_5(x)=\frac{1}{3744}x(-36+30x+10x^2-5x^3+x^4),$	$b_5(x)=\frac{1}{3744}x(120+30x+10x^2-5x^3+x^4),$
$a_6(x)=\frac{1}{18720}(156-120x-30x^2-10x^3+5x^4-x^5),$	$b_6(x)=-\frac{1}{18720}x(120+30x+10x^2-5x^3+x^4).$

Proof. The proof of the completeness and reproducing property of $W_2^3[0,1]$ is similar to the proof in [18]. Now, let us find out the expression form of $R_x^{\{1\}}(y)$ in the space $W_2^3[0,1]$. It is easy to see that $\int_0^1 v'''(y)\partial_y^3 R_x^{\{1\}}(y)dy=$

$\sum\limits_{i=0}^{2}(-1)^{i}\, v^{(i)}(y)\, \partial_y^{5-i} R_x^{\{1\}}(y)\, |_{y=0}^{y=1} - \int_0^1 v(y)\, \partial_y^6 R_x^{\{1\}}(y)\, dy$. From Eq. (9), one can write

$$\begin{aligned}\left\langle v(y), R_x^{\{1\}}(y)\right\rangle_{W_2^3} &= \sum_{i=0}^{2} v^{(i)}(0)\, [\partial_y^i R_x^{\{1\}}(0) + (-1)^{i+1} \partial_y^{5-i} R_x^{\{1\}}(0)] \\ &\quad + \sum_{i=0}^{2}(-1)^i v^{(i)}(1)\, \partial_y^{5-i} R_x^{\{1\}}(1) - \int_0^1 v(y)\, \partial_y^6 R_x^{\{1\}}(y)\, dy.\end{aligned} \tag{11}$$

Since $R_x^{\{1\}}(y) \in W_2^3[0,1]$, it follows that $R_x^{\{1\}}(0) = 0$ and $R_x^{\{1\}}(1) = 0$. Again, since $v(x) \in W_2^3[0,1]$, it yield that $v(0) = 0$ and $v(1) = 0$. On the other hand, if $\partial_y^4 R_x^{\{1\}}(1) = 0$, $\partial_y^3 R_x^{\{1\}}(1) = 0$, $\partial_y^2 R_x^{\{1\}}(0) - \partial_y^3 R_x^{\{1\}}(0) = 0$, and $\partial_y^1 R_x^{\{1\}}(0) + \partial_y^4 R_x^{\{1\}}(0) = 0$, then Eq. (11) implies that $\left\langle v(y), R_x^{\{1\}}(y)\right\rangle_{W_2^3} = \int_0^1 v(y)\, (-\partial_y^6 R_x^{\{1\}}(y))dy$. Now, for any $x \in [0,1]$, if $R_x^{\{1\}}(y)$ satisfies

$$\partial_y^6 R_x^{\{1\}}(y) = -\delta(x-y)\,,\ \delta \text{ dirac-delta function}, \tag{12}$$

then $\left\langle v(y), R_x^{\{1\}}(y)\right\rangle_{W_2^3} = v(x)$. Clearly, $R_x^{\{1\}}(y)$ is the reproducing kernel function of the space $W_2^3[0,1]$. The auxiliary formula of Eq. (12) is $\lambda^6 = 0$, so, let the expression form of $R_x^{\{1\}}(y)$ be as defined in Eq. (10). For Eq. (12) let $R_x^{\{1\}}(y)$ satisfy $\partial_y^m R_x^{\{1\}}(x+0) = \partial_y^m R_x^{\{1\}}(x-0)$, $m = 0,1,2,3,4$. Integrating $\partial_y^6 R_x^{\{1\}}(y) = -\delta(x-y)$ from $x-\varepsilon$ to $x+\varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $\partial_y^5 R_x^{\{1\}}(y)$ at $y = x$ given by $\partial_y^5 R_x^{\{1\}}(x+0) - \partial_y^5 R_x^{\{1\}}(x-0) = -1$. Through the last descriptions and by using MAPLE 13 software package, the unknown coefficients $a_i(x)$ and $b_i(x)$, $i = 1,2,...,6$ of Eq. (10) can be obtained. ■

Definition .3 [17] The space $W_2^1[0,1]$ is defined as $W_2^1[0,1] = \{v \mid v$ is absolutely continuous on $[0,1]$ and $v' \in L^2[0,1]\}$. On the other hand, the inner product and the norm in $W_2^1[0,1]$ are defined, respectively, by

$$\langle v(x), w(x)\rangle_{W_2^1} = v(0)\, w(0) + \int\limits_0^1 v'(x) w'(x) dx,$$

and $||v||_{W_2^1} = \sqrt{\langle v(x), v(x)\rangle_{W_2^1}}$, where $v, w \in W_2^1[0,1]$.

Theorem .2 [17] The Hilbert space $W_2^1[0,1]$ is a complete reproducing kernel with reproducing kernel function

$$R_x^{\{2\}}(y) = \begin{cases} 1+y, & y \le x, \\ 1+x, & y > x. \end{cases}$$

4 Theoretic basis of the RKHS algorithm

Here, the formulation of a differential linear operator is presented in $W_2^3[0,1]$. After that, we use the Gram-Schmidt orthogonalization process on the orthonormal system $\left\{\overline{\psi}_i(x)\right\}_{i=1}^{\infty}$ and normalizing them on $W_2^3[0,1]$ to obtain the required orthogonalization coefficients in order to obtain the analytical-numerical solutions of Eqs. (6) and (7) using RKHS algorithm.

Let us consider the differential operator $L : W_2^3[0,1] \to W_2^1[0,1]$ such that $Lv(x) = P(x)\, v''(x) + Q(x)\, v'(x) + R(x)\, v(x)$. Thus, Eqs. (6) and (7) can be equivalently converted into the form:

$$Lv(x) = F(x, (v+\phi)(x), [T(v+\phi)](x)) - \left(\phi' Q + \phi R\right)(x)\,, \tag{13}$$

with respect to the two-point boundary conditions

$$v(0) = 0,\ v(1) = 0. \tag{14}$$

Theorem .3 The operator $L : W_2^3[0,1] \to W_2^1[0,1]$ is bounded and linear.

Proof. Clearly, $\|Lv\|^2_{W_2^1} \leq M \|v\|^2_{W_2^3}$, where $M > 0$. From the definition of $W_2^1[0,1]$, we have $\|Lv\|^2_{W_2^1} = \langle Lv(x), Lv(x)\rangle_{W_2^1} = [(Lv)(0)]^2 + \int_0^1 [(Lv)'(x)]^2 dx$. By the Schwarz inequality and reproducing properties $v(x) = \left\langle v(y), R_x^{\{1\}}(y)\right\rangle_{W_2^3}$, $(Lv)(x) = \left\langle v(y), (LR_x^{\{1\}})(y)\right\rangle_{W_2^3}$, and $(Lv)'(x) = \left\langle v(y), (LR_x^{\{1\}})'(y)\right\rangle_{W_2^3}$, we get

$$|(Lv)(x)| = \left|\left\langle v(x), (LR_x^{\{1\}})(x)\right\rangle_{W_2^3}\right| \leq \left\|LR_x^{\{1\}}\right\|_{W_2^3} \|v\|_{W_2^3} = M_1 \|v\|_{W_2^3},$$

$$|(Lv)'(x)| = \left|\left\langle v(x), (LR_x^{\{1\}})'(x)\right\rangle_{W_2^3}\right| \leq \left\|(LR_x^{\{1\}})'\right\|_{W_2^3} \|v\|_{W_2^3} = M_2 \|v\|_{W_2^3},$$

where $M_i > 0$, $i = 1, 2$. Thus, $\|Lv\|^2_{W_2^1} = [(Lv)(0)]^2 + \int_0^1 [(Lv)'(x)]^2 dx \leq (M_1^2 + M_2^2)\|v\|^2_{W_2^3}$. The linearity part is obvious. ■

To construct an orthogonal function system of $W_2^3[0,1]$; put $\varphi_i(x) = R_{x_i}^{\{2\}}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and L^* is the adjoint operator of L. In other words, $\langle v(x), \psi_i(x)\rangle_{W_2^3} = \langle v(x), L^*\varphi_i(x)\rangle_{W_2^3} = \langle Lv(x), \varphi_i(x)\rangle_{W_2^1} = Lv(x_i)$, $i = 1, 2, \ldots$. The orthonormal function system $\left\{\overline{\psi}_i(x)\right\}_{i=1}^{\infty}$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \tag{15}$$

where $\beta_{ij} = \frac{1}{\|\psi_1\|_{W_2^3}}$ for $i = j = 1$, $\beta_{ij} = \frac{1}{\sqrt{\|\psi_i\|^2_{W_2^3} - \sum_{k=1}^{i-1}\left(\langle\psi_i(x), \overline{\psi}_k(x)\rangle_{W_2^3}\right)^2}}$ for $i = j \neq 1$, and $\beta_{ij} = -\frac{1}{\sqrt{\|\psi_i\|^2_{W_2^3} - \sum_{k=1}^{i-1}(c_{ik})^2}}$ $\times \sum_{k=j}^{i-1} \langle\psi_i(x), \overline{\psi}_k(x)\rangle_{W_2^3} \beta_{kj}$ for $i > j$.

Lemma .1 If $v \in W_2^3[0,1]$, then $|v(x)| \leq \frac{7}{2}\|v\|_{W_2^3}$, $|v'(x)| \leq 3\|v\|_{W_2^3}$, and $|v''(x)| \leq 2\|v\|_{W_2^3}$.

Proof. Noting that $v''(x) - v''(0) = \int_0^x v'''(p)\,dp$, where $v''(x)$ is absolute continuous on $[0,1]$. If this is integrated again from 0 to x, the result is $v'(x)$ itself as; $v'(x) - v'(0) - v''(0)x = \int_0^x \left(\int_0^z v_s'''(p)\,dp\right)dz$. Again, integrated from 0 to x, yield that $v(x) - v(0) - v'(0)x - \frac{1}{2}v''(0)x^2 = \int_0^x \left(\int_0^w \left(\int_0^z v'''(p)\,dp\right)dz\right)dw$. So, $|v(x)| \leq |v(0)| + |v'(0)||x| + \frac{1}{2}|v''(0)||x|^2 + \int_0^1 |v'''(p)|\,dp$ or $|v(x)| \leq |v(0)| + |v'(0)| + \frac{1}{2}|v''(0)| + \int_0^1 |v'''(p)|\,dp$. By using Holder's inequality and Eq. (9), we can note the following relation inequalities:

$$|v(0)| = \sqrt{v^2(0)} \leq \sqrt{\sum_{i=0}^{2}\left(v^{(i)}(0)\right)^2 + \int_0^1 (v''(x))^2\,dx} = \|v\|_{W_2^3},$$

$$|v'(0)| = \sqrt{(v'(0))^2} \leq \sqrt{\sum_{i=0}^{2}\left(v^{(i)}(0)\right)^2 + \int_a^b (v''(x))^2\,dx} = \|v\|_{W_2^3},$$

$$|v''(0)| = \sqrt{(v''(0))^2} \leq \sqrt{\sum_{i=0}^{2}\left(v^{(i)}(0)\right)^2 + \int_0^1 (v''(x))^2\,dx} = \|v\|_{W_2^3},$$

$$\int_0^1 |v'''(p)|\,dp \leq \sqrt{\int_0^1 (v'''(p))^2\,dp \int_0^1 (1)^2\,dp} \leq \sqrt{\sum_{i=0}^{2}\left(v^{(i)}(0)\right)^2 + \int_0^1 (v''(x))^2\,dx} = \|v\|_{W_2^3}.$$

Thus, $|v(x)| \leq \frac{7}{2}\|v\|_{W_2^3}$. For the second part, since $v'(x) = v'(0) + v''(0)x + \int_0^x \left(\int_0^z v'''(p)\,dp\right)dz$, this means that $|v'(x)| \leq |v'(0)| + |v''(0)| + \int_0^1 |v'''(p)|\,dp$. Thus, one can find $|v'(x)| \leq 3\|v\|_{W_2^3}$. In the third part, clearly, $v''(x) - v''(0) = \int_0^x v'''(p)\,dp$, which yield that $|v''(x)| \leq |v''(0)| + \int_0^1 |v'''(p)|\,dp$. Hence, one can write $|v''(x)| \leq 2\|v\|_{W_2^3}$. ■

Theorem .4 If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete function system of the space $W_2^3[0,1]$.

Proof. Clearly, $\psi_i(x) = L^*\varphi_i(x) = \left\langle L^*\varphi_i(x), R_x^{\{1\}}(y)\right\rangle_{W_2^3} = \left\langle \varphi_i(x), L_y R_x^{\{1\}}(y)\right\rangle_{W_2^1} = L_y R_x^{\{1\}}(y)\Big|_{y=x_i} \in W_2^3[0,1]$, so, $\psi_i(x) = L_y R_x^{\{1\}}(y)\Big|_{y=x_i}$. For each fixed $v \in W_2^3[0,1]$, let $\langle v(x), \psi_i(x)\rangle_{W_2^3} = 0$, so, $\langle v(x), \psi_i(x)\rangle_{W_2^3} = \langle v(x), L^*\varphi_i(x)\rangle_{W_2^3} = \langle Lv(x), \varphi_i(x)\rangle_{W_2^1} = Lv(x_i) = 0$. But since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, therefore $Lv(x) = 0$. It follows that $v(x) = 0$ from the existence of L^{-1}. ■

5 Implementation of the iterative method

In this section, a procedure of obtaining the analytical solution of Eqs. (13) and (14) is represented in the series form in the space $W_2^3[0,1]$, whilst, the numerical solution is obtained by taking finitely many terms in this series representation form. After that, the numerical solution and its first (second) derivative are proved to converging uniformly to the analytical solution and its first (second) derivative, respectively.

Theorem .5 For each $v \in W_2^3[0,1]$, $\sum_{i=1}^{\infty}\left\langle v(x), \bar{\psi}_i(x)\right\rangle \bar{\psi}_i(x)$ is convergent in the sense of the norm of $W_2^3[0,1]$. On the other hand, if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then the analytical solution of Eqs. (13) and (14) is

$$v(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left[F(x_k, (v+\phi)(x_k), [T(v+\phi)](x_k)) - \left(\phi' Q + \phi R\right)(x_k)\right]\bar{\psi}_i(x). \tag{16}$$

Proof. Using Eq. (15), it easy to see that

$$\begin{aligned}
v(x) &= L^{-1}F(x, (v+\phi)(x), [T(v+\phi)](x)) - \left(\phi' Q + \phi R\right)(x) \\
&= \sum_{i=1}^{\infty}\left\langle v(x), \bar{\psi}_i(x)\right\rangle_{W_2^3}\bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle v(x), \psi_k(x)\right\rangle_{W_2^3}\bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle v(x), L^*\varphi_k(x)\right\rangle_{W_2^3}\bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle Lv(x), \varphi_k(x)\right\rangle_{W_2^1}\bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle F(x, (v+\phi)(x), [T(v+\phi)](x)) - \left(\phi' Q + \phi R\right)(x), \varphi_k(x)\right\rangle_{W_2^1}\bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left[F(x_k, (v+\phi)(x_k), [T(v+\phi)](x_k)) - \left(\phi' Q + \phi R\right)(x_k)\right]\bar{\psi}_i(x).
\end{aligned}$$

Hence, Eq. (16) is the analytical solution of Eqs. (13) and (14). ■

Let $\left\{\bar{\psi}_i(x)\right\}_{i=1}^{\infty}$ be the normal orthogonal system derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$, then according to Eq. (16), the analytical solution of Eqs. (13) and (14) can be denoted by

$$v(x) = \sum_{i=1}^{\infty} B_i\bar{\psi}_i(x), \tag{17}$$

where $B_i = \sum_{k=1}^{i}\beta_{ik}\left[F(x_k, (v_{k-1}+\phi)(x_k), [T(v_{k-1}+\phi)](x_k)) - \left(\phi' Q + \phi R\right)(x_k)\right]$. In fact, B_i in Eq. (17) are unknown, we will approximate B_i using known A_i. For a numerical computations, we define the initial function $v_0(x_1) = 0$, put $v_0(x_1) = v(x_1)$, and define the n-term approximation $v_n(x)$ to $v(x)$ as

$$v_n(x) = \sum_{i=1}^{n} A_i\bar{\psi}_i(x), \tag{18}$$

$$A_i = \sum_{k=1}^{i}\beta_{ik}\left[F(x_k, (v_{k-1}+\phi)(x_k), [T(v_{k-1}+\phi)](x_k)) - \left(\phi' Q + \phi R\right)(x_k)\right]. \tag{19}$$

The convergence of the analytical-numerical solutions will discuss next. For any $x \in [0,1]$ and according to Lemma .1, we get

$$\left|v_n^{(i)}(x) - v^{(i)}(x)\right| \le M_i \|v_n - v\|_{W_2^3}, \ i = 0,1,2, \tag{20}$$

where $M_0 = \frac{7}{3}$, $M_1 = 3$, $M_2 = 2$. Hence, if $\|v_n - v\|_{W_2^3} \to 0$ as $n \to \infty$, the numerical solution $v_n(x)$ and $v_n^{(i)}(x)$, $i = 0,1,2$ are converge uniformly to the analytical solution $v(x)$ and $v^{(i)}(x)$, $i = 0,1,2$, respectively.

Theorem .6 v_n satisfies $Lv_n(x_j) = F(x_j, (v_{j-1}+\phi)(x_j), [T(v_{j-1}+\phi)](x_j)) - \left(\phi' Q + \phi R\right)(x_j)$, $j = 1,2,3,....$

Proof. If $A_i = \sum_{k=1}^{i} \beta_{ik} \left[F(x_k, (v_{k-1}+\phi)(x_k), [T(v_{k-1}+\phi)](x_k)) - \left(\phi' Q + \phi R\right)(x_k)\right]$, then $v_n(x) = \sum_{i=1}^{n} A_i \bar{\psi}_i(x)$. Combining the reproducing and symmetric properties of $R_x^{\{1\}}(y)$, it follows that $Lv_n(x_j) = \sum_{i=1}^{n} A_i L\bar{\psi}_i(x_j) = \sum_{i=1}^{n} A_i \left\langle L\bar{\psi}_i(x), \varphi_j(x)\right\rangle_{W_2^1} = \sum_{i=1}^{n} A_i \left\langle \bar{\psi}_i(x), L_j^* \varphi(x)\right\rangle_{W_2^3} = \sum_{i=1}^{n} A_i \left\langle \bar{\psi}_i(x), \psi_j(x)\right\rangle_{W_2^3}$. Using the orthogonality of $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$, yields that $\sum_{l=1}^{j} \beta_{jl} Lv_n(x_l) = \sum_{i=1}^{n} A_i \left\langle \bar{\psi}_i(x), \sum_{l=1}^{j} \beta_{jl} \psi_l(x)\right\rangle_{W_2^3} = \sum_{i=1}^{n} A_i \left\langle \bar{\psi}_i(x), \bar{\psi}_j(x)\right\rangle_{W_2^3} = \sum_{l=1}^{j} \beta_{jl} \times \left[F(x_l, (v_{l-1}+\phi)(x_l), [T(v_{l-1}+\phi)](x_l)) - \left(\phi' Q + \phi R\right)(x_l)\right]$. Take $j = 1$, one gets $Lv_n(x_1) = F(x_1, (v_0+\phi)(x_1), [T(v_0+\phi)](x_1)) - \left(\phi' Q + \phi R\right)(x_1)$. Take $j = 2$, one gets $Lv_n(x_2) = F(x_2, (v_1+\phi)(x_2), [T(v_1+\phi)](x_2)) - \left(\phi' Q + \phi R\right)(x_2)$. Using mathematical induction, one get $Lv_n(x_j) = F(x_j, (v_{j-1}+\phi)(x_j), [T(v_{j-1}+\phi)](x_j)) - \left(\phi' Q + \phi R\right)(x_j)$. ■

From Eq. (20), since $v_n(x)$ converge uniformly to $v(x)$, $v(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i(x)$. Therefore, $v_n(x) = P_n v(x)$, where P_n is an orthogonal projector from $W_2^3[0,1]$ to $\mathrm{Span}\{\psi_1, \psi_2, ..., \psi_n\}$. Thus one can write $Lv_n(x_j) = \left\langle Lv_n(x), \varphi_j(x)\right\rangle_{W_2^1} = \left\langle v_n(x), L_j^* \varphi(x)\right\rangle_{W_2^3} = \left\langle P_n v(x), \psi_j(x)\right\rangle_{W_2^3} = \left\langle v(x), P_n \psi_j(x)\right\rangle_{W_2^3} = \left\langle v(x), \psi_j(x)\right\rangle_{W_2^3} = \left\langle Lv(x), \varphi_j(x)\right\rangle_{W_2^1} = Lv(x_j)$.

Theorem .7 If $\|v_{n-1} - v\|_{W_2^3} \to 0$, $x_n \to y$ as $n \to \infty$, and $F(x, w_1, w_2)$ is continuous in $[0,1]$ with respect to x, w_i, $i = 1,2$, then $F(x_n, (v_{n-1}+\phi)(x_n), [T(v_{n-1}+\phi)](x_n)) - \left(\phi' Q + \phi R\right)(x_n) \to F(y, (v+\phi)(y), [T(v+\phi)](y)) - \left(\phi' Q + \phi R\right)(y)$ as $n \to \infty$.

Proof. We want to show that $v_{n-1}(x_n) \to v(y)$. Since,

$$\begin{aligned} |v_{n-1}(x_n) - v(y)| &= |v_{n-1}(x_n) - v_{n-1}(y) + v_{n-1}(y) - v(y)| \\ &\le |v_{n-1}(x_n) - v_{n-1}(y)| + |v_{n-1}(y) - v(y)| \\ &\le \left|(v_{n-1})'(\xi)\right| |x_n - y| + |v_{n-1}(y) - v(y)|, \ \xi \text{ lies between } x_n \text{ and } y. \end{aligned}$$

From Lemma .1, $|v_{n-1}(y) - v(y)| \le \frac{7}{2} \|v_{n-1} - v\|_{W_2^3}$ which gives $|v_{n-1}(y) - v(y)| \to 0$ as $n \to \infty$, whilst, $\left|(v_{n-1})'(\xi)\right| \le 3 \|v_{n-1}\|_{W_2^3}$. In terms of the boundedness of $\|v_{n-1}\|_{W_2^3}$, we get $|v_{n-1}(x_n) - v(y)| \to 0$ as $n \to \infty$. As a result, by means of the continuation of F the result is obtained directly. ■

Theorem .8 If $||v_n||_{W_2^3}$ is bounded and $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then the n-term numerical solution $v_n(x)$ in the iterative formula of Eq. (18) converges to the analytical solution $v(x)$ of Eqs. (13) and (14) in the space $W_2^3[0,1]$ and $v(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i(x)$, where A_i is given by Eq. (19).

Proof. The proof is straightforward. ■

If $\delta_n = ||v - v_n||_{W_2^3}$, where $v(x)$ and $v_n(x)$ are given by Eqs. (17) and (18), respectively, then $\delta_n^2 = \left\|\sum_{i=n+1}^{\infty} A_i \bar{\psi}_i\right\|_{W_2^3}^2 = \sum_{i=n+1}^{\infty} (A_i)^2$ and $\delta_{n-1}^2 = \left\|\sum_{i=n}^{\infty} A_i \bar{\psi}_i\right\|_{W_2^3}^2 = \sum_{i=n}^{\infty} (A_i)^2$. Thus, $\delta_{n-1} \ge \delta_n$, and consequently $\{\delta_n\}$ are monotone decreasing in the sense of $\|\cdot\|_{W_2^3}$. By Theorem .5, $\sum_{i=1}^{\infty} A_i \bar{\psi}_i(x)$ is convergent, so, $\delta_n^2 = \sum_{i=n+1}^{\infty} (A_i)^2 \to 0$ or $\delta_n \to 0$ as $n \to \infty$.

6 Numerical examples

In order to solve multipoint singular BVPs restricted by Volterra operator numerically and to show behavior, properties, efficiency, and applicability of the present RKHS algorithm, four multipoint singular BVPs restricted by Volterra operator will be solved numerically in this section. Here, all the symbolic and numerical computations were performed by using MAPLE 13 software package.

Using RKHS algorithm, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$, applying $R_x^{\{1\}}(y)$ and $R_x^{\{2\}}(y)$ on $[0,1]$ in which Algorithm .1 is used throughout the computations; some tabulate data is presented and discussed quantitatively at some selected grid points on $[0,1]$ to illustrate the numerical solutions for the following multipoint singular BVPs restricted by the given Volterra operator.

Example 1 Consider the singularities at two endpoint of $[0,1]$:

$$u''(x) + \frac{1}{\sin(x)} u'(x) - \frac{1}{x(x-1)} u(x) = [Tu](x) + f(x),$$

$$[Tu](x) = \int_0^x (x+1) t u(t) dt,$$

subject to the three-point boundary conditions

$$u(0) = 0,$$
$$u(1) - 4u\left(\tfrac{1}{9}\right) = 0,$$

where $0 < t < x < 1$, the analytical solution is $u(x) = x(x-1)\left(x - \frac{1}{9}\right)\cos(x)$.

Example 2 Consider the singularities at two endpoint of $[0,1]$:

$$u''(x) - \frac{1}{x^2(1-x)^2} u'(x) + \frac{1}{\sinh(x)} u(x) = u^2(x) + \sinh^{-1}(u(x)) + [Tu](x) + f(x),$$

$$[Tu](x) = \int_0^x (x-t) u^2(t)\, dt,$$

subject to the three-point boundary conditions

$$u(0) = 0,$$
$$u(1) - u\left(\tfrac{1}{2}\right) = 0,$$

where $0 < t < x < 1$, the analytical solution is $u(x) = \left(x - \frac{1}{2}\right)^2 (x-1)^2 \sinh(x)$.

Example 3 Consider the singularities at multipoint of $[0,1]$:

$$u''(x) + \frac{1}{x(x-1)\left(x - \frac{1}{3}\right)} u'(x) + \frac{1}{(e^x - 1)} u(x) = \cosh(u(x)) + [Tu](x) + f(x),$$

$$[Tu](x) = \int_0^x \cosh(x)\, u^4(t)\, dt,$$

subject to the three-point boundary conditions

$$u(0) = 0,$$
$$u(1) - 2u\left(\tfrac{1}{3}\right) = 0,$$

where $0 < t < x < 1$, the analytical solution is $u(x) = x\left(x - \frac{1}{3}\right)\left(2x^2 - 3x + 1\right)$.

Example 4 Consider the singularities at multipoint of $[0,1]$:

$$u''(x)+\frac{1}{\ln(x+1)}u'(x)-\frac{1}{x\left(x-\frac{1}{4}\right)(x-1)}u(x)=u(x)e^{u(x)}+f(x),$$

$$[Tu](x)=\int_{0}^{x}e^{t+x}e^{u(t)}dt,$$

subject to the three-point boundary conditions

$$u(0)=0,$$
$$u(1)-3u\left(\tfrac{1}{4}\right)=0,$$

where $0<t<x<1$, the analytical solution is $u(x)=\ln\left(x^2(1-x)\left(x-\frac{1}{4}\right)+1\right)$.

Our next goal is to illustrate some numerical results of the RKHS solutions of the aforementioned examples in numeric values. In fact, results from numerical analysis are an approximation, in general, which can be made as accurate as desired. Because a computer has a finite word length, only a fixed number of digits are stored and used during computations. Next, the agreement between the analytical-numerical solutions is investigated for Examples 1, 2, 3, and 4 at various x in $[0,1]$ by computing the absolute errors and the relative errors of numerically approximating their analytical solutions for the corresponding equivalent equations as shown in Tables 1, 2, 3, and 4, respectively. Anyhow, it is clear from the tables that, the numerical solutions are in close agreement with the analytical solutions for all examples, while the accuracy is in advanced by using only few tens of the RKHS iterations. Indeed, we can conclude that higher accuracy can be achieved by computing further RKHS iterations.

Table 1. The analytical-numerical solutions and errors for Example 1.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.16	−0.00648674140330012	−0.00648701932919779	2.77926×10^{-7}	4.28452×10^{-5}
0.32	−0.04314675763472329	−0.04314697922423827	2.21590×10^{-7}	5.13572×10^{-6}
0.48	−0.08166976184992833	−0.08166986517758862	1.03328×10^{-7}	1.26519×10^{-6}
0.64	−0.09774018067274835	−0.09774015978092232	2.08918×10^{-8}	2.13749×10^{-7}
0.80	−0.07679256174137644	−0.07679247394675902	8.77946×10^{-8}	1.14327×10^{-6}
0.96	−0.01869522215933257	−0.01869518857715539	3.35822×10^{-8}	1.79630×10^{-6}

Table 2. The analytical-numerical solutions and errors for Example 2.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.16	0.01310653223586577	0.013106673444133368	1.41208×10^{-7}	1.07739×10^{-5}
0.32	0.00487640352847436	0.004876573498209635	1.69970×10^{-7}	3.48556×10^{-5}
0.48	0.00005393349784172	0.000054062551919420	1.29054×10^{-8}	2.39284×10^{-3}
0.64	0.00173897887325904	0.001739035091758989	5.62185×10^{-8}	3.23285×10^{-5}
0.80	0.00319718153587544	0.003197173005396070	8.53048×10^{-9}	2.66812×10^{-6}
0.96	0.00037729187128320	0.000377276827374901	1.50439×10^{-8}	3.98734×10^{-5}

Table 3. The analytical-numerical solutions and errors for Example 3.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.16	−0.01584128000000000	−0.015841501202561670	2.21203×10^{-7}	1.39637×10^{-5}
0.32	−0.00104447999999999	−0.001044838270691351	3.58271×10^{-7}	3.43013×10^{-4}
0.48	0.00146432000000000	0.001463908889641097	4.11110×10^{-7}	2.80752×10^{-4}
0.64	−0.01978367999999998	−0.019784059627530715	3.79628×10^{-7}	1.91889×10^{-5}
0.80	−0.04480000000000005	−0.044800263728179024	2.63728×10^{-7}	5.88679×10^{-6}
0.96	−0.02213887999999997	−0.022138943318272125	6.33183×10^{-8}	2.86005×10^{-6}

Table 4. The analytical-numerical solutions and errors for Example 4.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.16	−0.00193723522905100	−0.00193715122696797	8.40021×10^{-8}	4.33618×10^{-5}
0.32	0.00486239935272751	0.00486253640369627	1.37051×10^{-7}	2.81859×10^{-5}
0.48	0.02718301141037295	0.02718317043250062	1.59022×10^{-7}	5.85006×10^{-6}
0.64	0.05591504562239846	0.05591520011909257	1.54497×10^{-7}	2.76306×10^{-6}
0.80	0.06803241039182698	0.06803253422667077	1.23835×10^{-7}	1.82023×10^{-6}
0.96	0.02583677729641778	0.02583681728855300	3.99921×10^{-8}	1.54788×10^{-6}

7 Concluding remarks

In this work, we have used the reproducing kernel algorithm for solving linear and nonlinear second-order, three-point singular BVPs restricted by Volterra operator. In the meantime, we employed our algorithm and its conjugate operator to construct the complete orthonormal basis in the reproducing kernel space $W_2^3[0,1]$. By separating the multipoint boundary conditions and adding the initial and boundary conditions to the reproducing kernel space that satisfying these points, we obtain the analytical-numerical solutions of the problem. The algorithm is applied in a direct way without using linearization, perturbation, or any restrictive assumptions. It may be concluded that RKHS algorithm is very powerful and efficient in finding the analytical-numerical solutions for a wide class of multipoint singular BVPs. It is worth mentioning here that the algorithm is capable of reducing the volume of the computational work and complexity while still maintaining the high accuracy of the numerical results.

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References

[1] M. Moshinsky, Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinuas, Bol. Soc. Mat. Mexicana 7 (1950) 1-25.

[2] S. Timoshenko, Theory of Elastic Stability, McGraw–Hill, New York, 1961.

[3] Y. Lin, J. Lin, A numerical algorithm for solving a class of linear nonlocal boundary value problems, Applied Mathematics Letters 23 (2010) 997-1002.

[4] U.M. Ascher, R.M. Mattheij, R.D. Russell, Numerical Solution of Boundary Value Problems for Ordinary Differential Equations, Volume 13, Classics in Applied Mathematics, 1995.

[5] O. Abu Arqub, Z. Abo-Hammour, S. Momani, N. Shawagfeh, Solving singular two-point boundary value problems using continuous genetic algorithm, Abstract and Applied Analysis, Vol. 2012, Article ID 205391, 25 page, 2012. doi.10.1155/2012/205391.

[6] Z. Abo-Hammour, O. Abu Arqub, O. Alsmadi, S. Momani, A. Alsaedi, An Optimization Algorithm for Solving Systems of Singular Boundary Value Problems, Applied Mathematics and Information Sciences 8 (2014) 2809-2821.

[7] C.P. Gupta, A sharper conditions for the solvability of a three-point second order boundary value problem, Journal of Mathematical Analysis and Applications 205 (1997) 586-597.

[8] H.B. Thompson, C. Tisdell, Three-point boundary value problems for second-order, ordinary, differential equations, Mathematical and Computer Modelling 34 (2001) 311-318.

[9] P. Singh, A second-order singular three-point boundary value problem, Applied Mathematics Letters 17 (2004) 969-976.

[10] R. Ma, D. O'Regan, Solvability of singular second order m-point boundary value problems, Journal of Mathematical Analysis and Applications 301 (2005) 124-134.

[11] F. Geng, Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method, Applied Mathematics and Computation 215 (2009) 2095-2102.

[12] Y. Zhou, Y. Xu, Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations, Journal of Mathematical Analysis and Applications 320 (2006) 578-590.

[13] M. Cui, Y. Lin, Nonlinear Numerical Analysis in the Reproducing Kernel Space, Nova Science, New York, NY, USA, 2009.

[14] A. Berlinet, C.T. Agnan, Reproducing Kernel Hilbert Space in Probability and Statistics, Kluwer Academic Publishers, Boston, Mass, USA, 2004.

[15] A. Daniel, Reproducing Kernel Spaces and Applications, Springer, Basel, Switzerland, 2003.

[16] H.L. Weinert, Reproducing Kernel Hilbert Spaces: Applications in Statistical Signal Processing, Hutchinson Ross, 1982.

[17] Y. Lin, M. Cui, L. Yang, Representation of the exact solution for a kind of nonlinear partial differential equations, Applied Mathematics Letters 19 (2006) 808-813.

[18] L.H. Yang, Y. Lin, Reproducing kernel methods for solving linear initial-boundary-value problems, Electronic Journal of Differential Equations 2008 (2008) 1-11.

[19] O. Abu Arqub, M. Al-Smadi, Numerical algorithm for solving two-point, second-order periodic boundary value problems for mixed integro-differential equations, Applied Mathematics and Computation 243 (2014) 911-922.

[20] S. Momani, O. Abu Arqub, T. Hayat, H. Al-Sulami, A computational method for solving periodic boundary value problems for integro-differential equations of Fredholm-Voltera type, Applied Mathematics and Computation 240 (2014) 229-239.

[21] M. Al-Smadi, O. Abu Arqub, S. Momani, A computational method for two-point boundary value problems of fourth-order mixed integro-differential equations, Mathematical Problems in Engineering, Mathematical Problems in Engineering, vol 2013, Article ID 832074, 10 pages, 2012. doi:10.1155/2013/832074.

[22] O. Abu Arqub, M. Al-Smadi, S. Momani, Application of reproducing kernel method for solving nonlinear Fredholm-Volterra integro-differential equations, Abstract and Applied Analysis, vol. 2012, Article ID 839836, 16 pages, 2012. doi:10.1155/2012/839836.

[23] O. Abu Arqub, M. Al-Smadi, N. Shawagfeh, Solving Fredholm integro-differential equations using reproducing kernel Hilbert space method, Applied Mathematics and Computation 219 (2013) 8938-8948.

[24] N. Shawagfeh, O. Abu Arqub, S. Momani, Analytical solution of nonlinear second-order periodic boundary value problem using reproducing kernel method, Journal of Computational Analysis and Applications 16 (2014) 750-762.

[25] M. Al-Smadi, O. Abu Arqub, A. El-Ajuo, A numerical method for solving systems of first-order periodic boundary value problems, Journal of Applied Mathematics, Volume 2014, Article ID 135465, 10 pages, 2014. doi:10.1155/2014/135465.

[26] O. Abu Arqub, An iterative method for solving fourth-order boundary value problems of mixed type integro-differential equations, Journal of Computational Analysis and Applications 18 (2015) 857-874.

[27] O. Abu Arqub, M. Al-Smadi, S. Momani, T. Hayat, Numerical solutions of fuzzy differential equations using reproducing kernel Hilbert space method, Soft Computing (2015). doi:10.1007/s00500-015-1707-4.

[28] F. Geng, M. Cui, Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space, Applied Mathematics and Computation 192 (2007) 389-398.

[29] C. Li, M. Cui, The exact solution for solving a class nonlinear operator equations in the reproducing kernel space, Applied Mathematics and Computation 143 (2003) 393-399.

[30] W. Jiang, Z. Chen, Solving a system of linear Volterra integral equations using the new reproducing kernel method, Applied Mathematics and Computation 219 (2013) 10225-10230.

[31] F. Geng, M. Cui, A reproducing kernel method for solving nonlocal fractional boundary value problems, Applied Mathematics Letters 25 (2012) 818-823.

[32] F. Geng, S.P. Qian, Reproducing kernel method for singularly perturbed turning point problems having twin boundary layers, Applied Mathematics Letters 26 (2013) 998-1004.

[33] O. Abu Arqub, Reproducing kernel algorithm for the analytical-numerical solutions of nonlinear systems of singular periodic boundary value problems, Mathematical Problems in Engineering, Volume 2015 (2015), Article ID 518406, 13 pages. doi:10.1155/2015/518406.

[34] O. Abu Arqub, Adaptation of reproducing kernel algorithm for solving fuzzy Fredholm-Volterra integrodifferential equations, Neural Computing & Applications, (2015). DOI: 10.1007/s00521-015-2110-x.

[35] O. Abu Arqub, The reproducing kernel algorithm for handling differential algebraic systems of ordinary differential equations, Mathematical Methods in the Applied Sciences, In press.

[36] F. Geng, S.P. Qian, S. Li, A numerical method for singularly perturbed turning point problems with an interior layer, Journal of Computational and Applied Mathematics 255 (2014) 97-105.

[37] W. Jiang, Z. Chen, A collocation method based on reproducing kernel for a modified anomalous subdiffusion equation, Numerical Methods for Partial Differential Equations 30 (2014) 289-300.