Formalization of Multivariate Lagrange Interpolation

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Abstract. We generalize a simple formula for constructing the multinomial function f which interpolates a set of $(d+1)$ points in $\mathbb{R}^N$. We also provide an example of application of this method.

1 Lagrange polynomial interpolation in $\mathbb{R}^2$

Given $(d+1)$ distinct points $(x_i, y_i)_{i=0}^{d}$, where of course, $x_j \neq x_i$,

$$P(x) = \sum_{i=0}^{d} y_i \prod_{j=0, j\neq i}^{d} \frac{x - x_j}{x_i - x_j}, \tag{1}$$

is an interpolation polynomial.

Our aim is to generalize this formula to a set of vectors in $\mathbb{R}^N$, multivariate interpolations has been well documented by [1].

2 Lagrange polynomial interpolation in $\mathbb{R}^N$

Let the vector $\overrightarrow{X_k}$ belongs to the vector space $\mathbb{R}^N$ such as :

$$\overrightarrow{\mathcal{X}_k} = \begin{pmatrix} x_{k\,1} \\ ... \\ x_{k\,N} \end{pmatrix}.$$

The components of this vector are the N-components of the antecedent vector of an associated value, y_k.

We have $\mathcal{X}_d$, the set of $(d+1)$ vectors $\overrightarrow{X_k}$:

$$\mathcal{X}_d = \left\{ \overrightarrow{\mathcal{X}_0}, ..., \overrightarrow{\mathcal{X}_d} \right\}$$

and $\mathcal{Y}_d$ the set of $(d+1)$ elements, y_k :

$$\mathcal{Y}_d = \{y_0, ..., y_d\}.$$

We are trying to build a function f which to $(\overrightarrow{\mathcal{X}_k})_{k=0}^d$ associates the correspondant $(y_k)_{k=0}^d$. To do this using the Lagrange Interpolating, we can first, decompose all the $(y_k)_{k=0}^d$ in sums of $N(d+1)$ numbers, $(\lambda_{k\,j})_{k=0,n=1}^{k=d,n=N}$, hence, we decide of $(d+1)$ sets of N numbers, $(\Lambda_k)_{k=0}^d$ such as :

$$(\Lambda_k)_{k=0}^d = \left\{ \lambda_{k\,1}, ..., \lambda_{k\,N} | \sum_{i=1}^{N} \lambda_{k\,i} = y_k \right\}. \tag{2}$$

It is important to note that in some particular cases, we can't solve the system of equations to compute all the $(\Lambda_k)_{k=0}^d$ sets (see the examples). We introduce N sets, let's call them $(D_n)_{n=1}^N$, which will contain for each n, all the components $(x_{k\,n})_{k=0}^d$ present at least in double among the $(d+1)$ components at the rank j of the $(d+1)$ vectors, $(\overrightarrow{\mathcal{X}_k})_{k=0}^d$. We have the following relations :

$$(D_n)_{n=1}^N \triangleq \bigcup_{\substack{k=0 \\ \mathrm{Card}(\sqcup_{v=0}^d \{x_{v\,n}\} \setminus \{x_{k\,n}\}) \neq d}}^{d} \{x_{k\,n}\}. \tag{3}$$

We set $S_{-1\,n} \triangleq \varnothing$, by the Lagrange Interpolation, we have :

$$P_{k\,n}(x_n) \triangleq \prod_{\substack{i=0 \\ i \neq k \\ S_{i\,n} \triangleq S_{i-1\,n} \cup \{x_{i\,n}\} \\ x_{i\,n} \notin S_{i-1\,n}}}^{d} \frac{x_n - x_{i\,n}}{x_{k\,n} - x_{i\,n}}, \tag{4}$$

this polynomial expression (here, the conditions under the product are necessary, we must compute the product for the unique $(x_{i,n})_{i=0}^d$) fullfills the following necessary conditions :

$$\begin{cases} P_{k\,n}(x_n = x_{k\,n}) = 1, \\ \forall j \in [\![0; d]\!] \setminus \{k\}, P_{k\,n}(x_n = x_{j\,n}) = 0. \end{cases}$$

One may notice that the choice of the sets $(\Lambda_k)_{k=0}^d$ may lead to polynomials $P_{k\,n}$ of degree 0 for some n (the reason is our sets can fullfills the following condition : $\exists n \in [\![1; N]\!] | \mathrm{Card}(\bigcup_{k=0}^d \{\lambda_{k\,n}\}) = 1$) and therefore that f is free from x_n. We write this for practical reasons, $\mathcal{P}_{-1\,n} = \varnothing$, recall that for the $(x_{k\,n} \in D_n)_{n=1}^N$, we must sum only one time each polynomial expression representing them (otherwise, we could multiplicate each polynomial by $\frac{1}{c_n}$ where c_n would be the number of occurrences in the vectors of X_d of a same component $x_{k\,n}$, this, of course, would only works if c_n is the same for all $k \in [\![0; d]\!]$, then $c_n = d + 1 - \mathrm{Card}(\sqcup_{v=0}^d \{x_{v\,n}\} \setminus \{x_{k\,n}\})$).

We set the logical proposition $\mathrm{A}_{j\,n} \triangleq (x_{j\,n} \notin D_n) \vee (P_{j\,n} \notin \mathcal{P}_{j\,n})$ where $\mathcal{P}_{j\,n}$ is defined by the following recurrence relation : $\mathcal{P}_{j\,n} \triangleq \mathcal{P}_{j-1\,n} \cup \{P_{j\,n}(x_n)\}$.

If all the conditions are met, then following (1), f is defined as the sum of the lagrange interpolation polynomials of the sets $(\Lambda_k)_{k=0}^{d}$:

$$\begin{aligned} f : \mathbb{R}^{\dim f \leq N} &\to \mathbb{R} \\ \overrightarrow{\mathcal{X}} &\mapsto \sum_{n=1}^{N} \sum_{\substack{j=0 \\ \Lambda_{j\,n}}}^{d} \lambda_{j\,n} P_{j\,n}(x_n) \ . \end{aligned} \tag{5}$$

This function satisfies the condition $\forall(\overrightarrow{\mathcal{X}} = \overrightarrow{\mathcal{X}_{k\in[\![0;d]\!]}} \in \mathcal{X}_d), f(\overrightarrow{\mathcal{X}}) = y_k \in \mathcal{Y}_d$.

More generally, if we decide of a set of $N(d+1)$ powers, let's denote it by R_d such as $R_d = \{p_{0\,1}; p_{1\,1}; ...; p_{d\,N} | (\forall i \in [\![0;d]\!], \exists (p_{i\,z} \in R_d) \neq 0)\}$ then,

$$\forall i \in [\![0;d]\!], \exists (x_z = x_{i\,z}) | \prod_{z=1}^{N} \prod_{i=0}^{d} \left(x_z - x_{i\,z} \right)^{p_{i\,z}} = 0, \tag{6}$$

which is equivalent to say that for all function ψ,

$$f(\overrightarrow{\mathcal{X}}) + \psi(\overrightarrow{\mathcal{X}}) \prod_{z=1}^{N} \prod_{i=0}^{d} \left(x_z - x_{i\,z} \right)^{p_{i\,z}}, \tag{7}$$

is also an interpolation of our set of $(d+1)$ points.

Example. Suppose we are given points $(0,0,1), (0,1,2), (1,0,2), (1,1,e)$ which lie on $(x-y)^2 + e^{xy}$. Applying (2), we must solve the following, impossible, system of equations

$$\begin{cases} x(0) + y(0) = 1 \\ x(0) + y(1) = 2 \\ x(1) + y(0) = 2 \\ x(1) + y(1) = e \end{cases} \Rightarrow \{x(0), y(0), x(1), y(1)\} \in \varnothing.$$

Hence, here, we can't apply our method ((2) is not necessarily applicable).

Now suppose that our set of points is $(0,0,1), (0,1,2), (1,2,9+e^2), (1,3,16+e^3)$, we have

$$\begin{cases} y(0) = 1 - x(0) \\ y(1) = 2 - x(0) \\ y(2) = 9 + e^2 - x(1) \\ y(3) = 16 + e^3 - x(1) \end{cases} .$$

We decide to choose $x(0) = 1$, $x(1) = 2$, we have $y(0) = 0$, $y(1) = 1$, $y(2) = 7 + e^2$, $y(3) = 14 + e^3$. Using (1) and (5), we have to compute the two interpolation polynomials. The first one interpolates the points $(0,1)$, $(1,2)$, the second must interpolate $(0,0)$, $(1,1)$, $(2, 7+e^2)$, $(3, 14+e^3)$.

$$\text{InterpolatingPolynomial}\Big[\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, x \Big] + \text{InterpolatingPolynomial}\Big[\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 7+e^2 \\ 3 & 14+e^3 \end{pmatrix} y \Big]$$

$$= x + \frac{1}{6} y(-4y^2 - 3e^2(y^2 - 4y + 3) + e^3(y^2 - 3y + 2) + 27y - 17) + 1.$$

Conclusion. We formalized a multivariate analogue of Lagrange's interpolation polynomials. We showed how to interpolate an N-variable multinomial function given a set of points. Moreover, we gave necessaries conditions for the existence of the multinomial using this method.

References

[1] T. S. Mariano Gasca. *Polynomial interpolation in several variables.* University of Zaragoza, Universität Erlangen-Nürnberg, 2001. URL: `http://pcmap.unizar.es/~gasca/investig/GSSurvey.pdf`.