Cosmology on the Back of an Envelope

John R. Berryhill

Abstract

In the twentieth century, mathematical models of the cosmos proliferated unconstrained by sufficient evidence to choose among them. But a number of recent large observational collaborations have redefined the parameters of cosmology: In the twenty-first century, spatial curvature is out, and the cosmological constant is in. A straightforward model incorporating these essential points has been available for decades and is updated here. An explicit, closed-form, solution is presented, together with useful formulas and graphs. The presentation is accessible with first-year physics and calculus.

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If the large-scale motion in the universe is isotropic and homogeneous, then there also is a universal time *t,* which is the same for all observers who see the universe as isotropic and homogeneous (our co-moving partners.) Moreover, the only pattern of motion that will perpetuate an isotropic homogeneous universe is simply a magnification factor *S* synchronized to that universal time.

We begin with a God's-eye view (or 3-D map) of the universe at one instant of *t*. The position of a co-moving object within that map at that time is $\boldsymbol{x}$. Thereafter, its position $\boldsymbol{r}$ is[1]

$$\boldsymbol{r}=S(t)\boldsymbol{x}.$$

It is moving with velocity

$$\boldsymbol{v}=\dot{\boldsymbol{r}}=\dot{S}\,\boldsymbol{x}=(\frac{\dot{S}}{S})\boldsymbol{r}.$$

It is experiencing acceleration

$$\boldsymbol{a}=\dot{\boldsymbol{v}}=\ddot{\boldsymbol{r}}=(\frac{\ddot{S}}{S})\boldsymbol{r}.$$

In other words, its motion is determined entirely by the scale factor *S*. Velocity $\boldsymbol{v}$ is strictly proportional to distance $\boldsymbol{r}$, and the factor $(\dot{S}/S)$ is known as Hubble's constant *H*. Its present-day empirical value H_0 is around 21 km/s/Mly. Its inverse $1/H_0$ has the dimensions of time and is often cited as "the age of the universe," about 14 billion years. (But we'll see that we can do better.)

If *H* truly were constant, we would have

$$S(t)=S_0\,\mathrm{e}^{Ht}$$

and

$$\ddot{S}=H^2 S.$$

Defining $\lambda\equiv H^2$, we would expect an acceleration

$$\boldsymbol{a}=(\frac{\ddot{S}}{S})\boldsymbol{r}=\lambda\boldsymbol{r}$$

proportional to distance. Using today's value $H_0^2\approx 4.8\times10^{-36}s^{-2}$, a co-moving object one million light-years away would experience an acceleration of $4.6\times10^{-15}g's$, *away* from us. On the scale of the cosmos, one encounters unimaginably vast distances and imperceptibly small accelerations.[2]

The effect of gravity also is proportional to distance, but in the opposite direction. A co-moving mass *m* at distance $\boldsymbol{r}$ sits on the surface of an imaginary sphere of volume $V=\frac{4}{3}\pi r^3$, $r = |\boldsymbol{r}|$. The mass contained in the sphere is $M=\rho V$, where ρ is the average density of the universe (today a few hydrogen atoms per cubic meter.) The gravitational force felt by *m* is

$$m\boldsymbol{a}=-mGM\,\boldsymbol{r}/r^3=-m\frac{4}{3}\pi\rho G\boldsymbol{r}.$$

Dividing through by *m*, we find an acceleration proportional to $\boldsymbol{r}$ and to density.

Density $\rho(t)$ varies as $\rho_0/S^3(t)$, since if matter is conserved, $\rho|\boldsymbol{r}|^3=\rho_0|\boldsymbol{x}|^3=\rho_0|\boldsymbol{r}|^3/S^3$. If *S* increases without limit, eventually gravity loses out to the expansive force λ. Until then, we must write[3]

$$\frac{\ddot{S}}{S} = \lambda - \frac{4}{3}\pi G \rho_0 / S^3.$$

For a co-moving object of mass *m*, we calculate its kinetic energy

$$E_K = \frac{1}{2} m |\boldsymbol{v}|^2 = \frac{1}{2} m \left(\frac{\dot{S}}{S}\right)^2 r^2,$$

its gravitational potential

$$E_G = -\frac{4}{3}\pi \rho G m r^2,$$

and its λ potential

$$E_\lambda = -\frac{1}{2} m \lambda r^2.$$

We sum $E_K + E_G + E_\lambda = 0$, remove the common factors *m* and r^2, and multiply by 2 to obtain[4]

$$\left(\frac{\dot{S}}{S}\right)^2 = \frac{8\pi}{3} G \rho_0 / S^3 + \lambda,$$

or

$$\dot{S}^2 = \frac{8\pi}{3} G \rho_0 / S + \lambda S^2.$$

There is a force in the universe that opposes gravity and which drives the galaxies to fly apart from one another. Its presence is represented by λ, which is a constant, like *G*. The conflict between λ and *G* is described by the above equations for $\ddot{S}$ and $\dot{S}$. The forces would be in balance if density took the value[5] $\rho_{SS} = \lambda / (\frac{4}{3}\pi G)$. Acceleration $\ddot{S}$ would vanish, and velocity $\dot{S}$ would be constant, with $(\dot{S}/S)^2 = 3\lambda \equiv H_{SS}^2$. But to maintain a constant density would require the ongoing creation of new matter (albeit at a rate that would prove undetectable.)

Even if *G* and λ are not in balance, this is one of the few cosmological models that has an explicit solution. It is[6]

$$S^3(t) = \frac{4}{3}\pi G \frac{\rho_0}{\lambda} [\cosh(3t\sqrt{\lambda}) - 1].$$

Recalling that $\rho(t) = \rho_0 / S^3(t)$, we divide ρ_0 by this expression for S^3 to find

$$\rho(t) = \rho_{SS} / [\cosh(3t\sqrt{\lambda}) - 1].$$

The model also provides an explicit expression for *H* as a function of *t*. By differentiating

$$3S^2\dot{S} = \frac{4}{3}\pi G \frac{\rho_0}{\lambda} (3\sqrt{\lambda}) \sinh(3t\sqrt{\lambda}),$$

and then dividing by $3S^3$, we discover that

$$\frac{\dot{S}}{S} = \sqrt{\lambda} \sinh(3t\sqrt{\lambda}) / [\cosh(3t\sqrt{\lambda}) - 1],$$

and, then from the trigonometric half-angle identities,[7]

$$H(t) = \sqrt{\lambda} \coth\left(\frac{3}{2} t \sqrt{\lambda}\right).$$

In these two equations for $H(t)$ and $\rho(t)$, we find the gross dynamics of the universe determined by the ratio of two constants, λ and *G*. The formula for $\rho(t)$ depends explicitly on λ, which enters first in the definition of ρ_{SS}, then again in the argument of the *cosh* function. The most interesting value of $\rho(t)$ is, of course, $\rho(t_{now})$, and λ enters again in the determination of $t_{now} \equiv t_0$, the age of the universe. The trick is to find the value of *t* for which $H(t) = H_0$. It is

$$t_0 = \frac{2}{3} \tanh^{-1}(\sqrt{\Omega})/(H_0 \sqrt{\Omega}).$$

In this result we have defined $\Omega \equiv \lambda / H_0^2$.

In Fig. 1, the horizontal axis encompasses all values of Ω for λ between 0 and H_0^2. The left vertical axis indicates density, in units of $10^{-29}\, g/cm^3$. The right vertical axis indicates the age of the universe, relative to $1/H_0$, and goes with the red upward-swooping $t_0(\lambda)$ curve. The green diagonal line is the graph of ρ_{SS} as a function of λ: simple proportionality. The downward-sloping blue line is $\rho(t_0)$ calculated from the formulas for $\rho(t)$ and t_0. The fact that this *is* identically a straight line is accounted for in the Appendix.

Most authors in recent years accept the estimate that Ω lies in the vicinity of 0.7, which goes back to the magnificent work of Perlmutter[8] *et al*. Fig. 1 then shows that the age of the universe t_0 is around 13.8 Gyr and $\rho(t_0)$ today is near $0.26 \times 10^{-29}\, g/cm^3$. This concomitant density is much greater than can be accounted for by a careful inventory of the known universe. There is a fading hope that this problem will be resolved in due course by exotic new discoveries. But Fig.1 suggests that it is also feasible to consider lower densities and a larger cosmological constant.

The scale factor $S(t)$ that encapsulates the history and future of the universe is plotted in Fig. 2 for three choices of the Ω parameter: 0.9 (blue), 0.7 (red), and 0.5 (green). (Recall that $\lambda = \Omega H_0^2$.) $S(t)$ is normalized so that $S(t_0) = 1$ (left-hand scale.) All three curves have $\dot{S}/S = H_0$ at t_0. The leftmost points shown for each curve represent a time 28 Myr after the "big bang," and well after the time when the cosmic background radiation originated. The age of the universe corresponding to the three curves is 18.4 Gyr, 13.8 Gyr, and 12 Gyr, respectively. The effect of an Ω as high as 0.9 is to add 5 Gyr to the age of the cosmos and to enhance future acceleration.

The right-hand scale of Fig. 2 is calibrated for redshift $z = S(t_0)/S(t) - 1$. (*E.G.*, when the light was emitted that now arrives with redshift 1, the scale factor was half what it is now.) The supernova data upon which modern cosmology is based all have redshifts less than 1; most are clustered around 0.5. In Fig. 2, there is comparatively little difference between the *S* curves for 0.7 and 0.9, when $z < 1$. When the supernova data points are plotted on a chart like this, they scatter all over the area between these curves. It suggests that a larger λ is not ruled out by the data.

The limitation on observing high-redshift objects is that the enormous distances reduce their brightness by factors of thousands and more. A recent paper[9] says that improved telescopes in the future should expect to see ten times as many galaxies as we observe today.

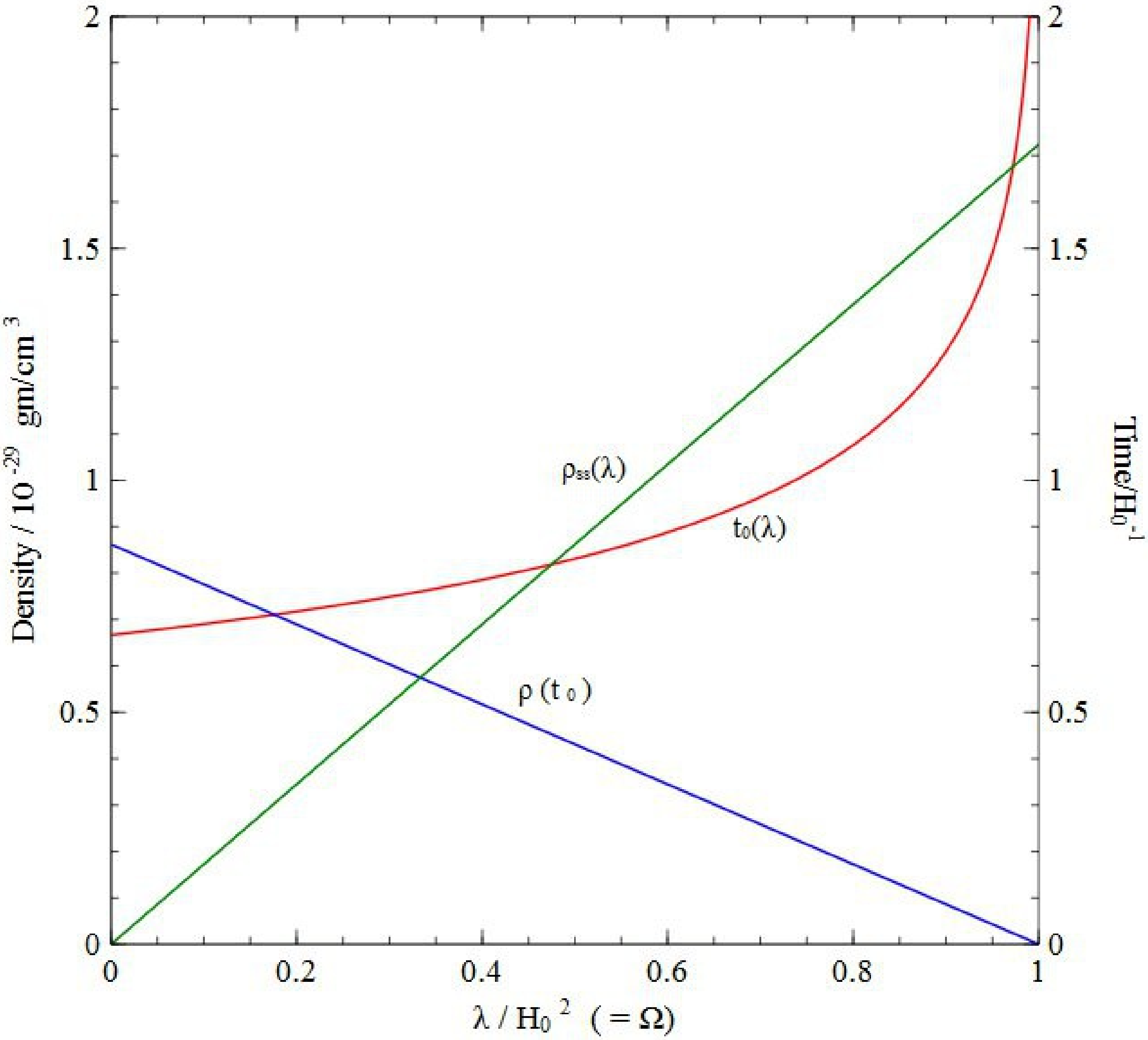

Fig. 1: Age of the universe t_0 and the corresponding density $\rho(t_0)$ versus the value of the cosmological constant λ .

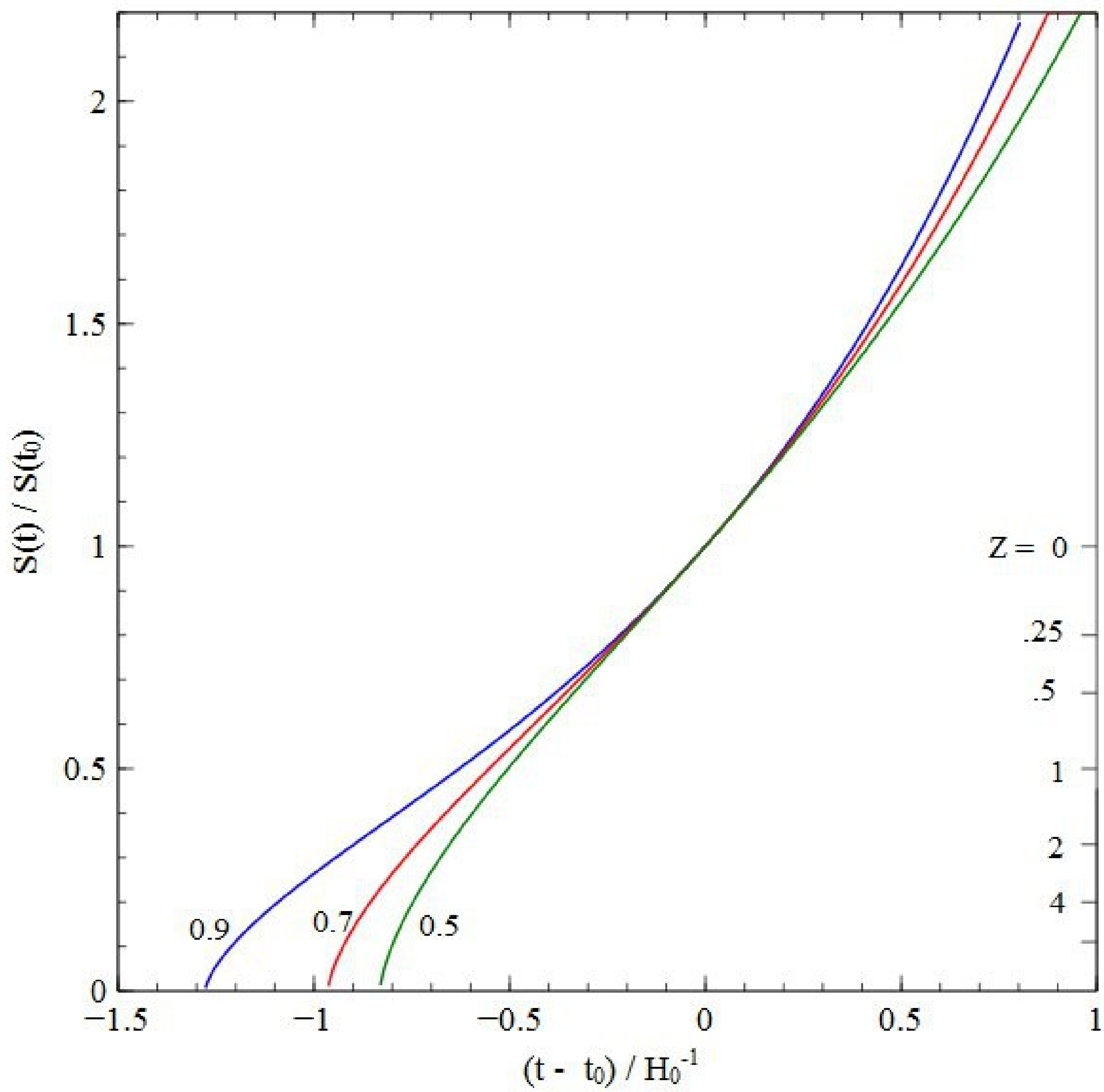

Fig. 2: Cosmic scale factor *S* versus universal time *t* for three values of Ω, and thus age t_0. Negative numbers are times before the present. The z values are the corresponding redshifts.

Appendix

In Fig.1, the downward-sloping line results from the formula

$$\rho(t_0)=\rho_{SS}/[\cosh(3\,t_0\sqrt{\lambda})-1]$$

evaluated with

$$t_0=\frac{2}{3}\tanh^{-1}(\sqrt{\Omega})/(H_0\sqrt{\Omega}).$$

To see that this makes a straight line, first we apply a trigonometric identity,

$$[\cosh(3\,t_0\sqrt{\lambda})-1]=2\sinh^2(\frac{3}{2}t_0\sqrt{\lambda}),$$

from Abramowitz and Stegun[7], (4.5.28). Since $\lambda=\Omega H_0^2,$ when we plug in t_0 we obtain

$$2\sinh^2(\frac{3}{2}[\frac{2}{3}\tanh^{-1}(\sqrt{\Omega})/(H_0\sqrt{\Omega})]H_0\sqrt{\Omega})$$
$$=\quad 2\sinh^2(\tanh^{-1}(\sqrt{\Omega})).$$

For clarity, define $z\equiv\tanh^{-1}(\sqrt{\Omega})$. Then $\tanh z=\sqrt{\Omega}$. Now

$$\tanh z=\sinh z/\cosh z \quad (4.5.3) \text{ and}$$
$$\cosh^2 z-\sinh^2 z=1 \quad (4.5.16).$$

So $\sinh^2 z=\Omega\cosh^2 z$ and $(1-\Omega)\cosh^2 z=1$. Therefore,

$$2\sinh^2(\tanh^{-1}(\sqrt{\Omega}))=2\Omega/(1-\Omega),$$

and, since $\rho_{SS}=\Omega H_0^2/(\frac{4}{3}\pi G)$, we have

$$\rho(t_0)=(1-\Omega)H_0^2/(\frac{8}{3}\pi G).$$

This describes a straight line that descends from $\rho_{ss}/2$ to 0 as Ω increases from 0 to 1. QED

Notes

1 Insiders call $\boldsymbol{x}$ a "co-moving coordinate." Nevertheless, it is a *fixed* parameter for each co-moving object.

2 $\ddot{S}$ can be determined independently of $\dot{S}$. Its value is traditionally reported in terms of a "deceleration" parameter q defined by $\ddot{S}/S=-q\,H^2$. The recently reported value is q = -.58, i.e., the increase of S is *accelerating*.

3 Insiders traditionally apply a gratuitous factor $1/3$ to λ here, for some purported convenience. Convenience is one thing, clarity is another. I prefer to recognize the actual value of λ, which is a significant universal constant.

4 If empty space had curvature, one would add or subtract something else here, depending. But we now accept that space is flat.

5 My ρ_{SS} is twice the traditionally reported ρ_C, which is a legacy from models now ruled out by observations. I prefer a notation that is self-explanatory in the present context.

6 My source for this is Sir Hermann Bondi, in *Cosmology,* second edition, reprinted 1961. See particularly p. 80-82.

7 Abramowitz and Stegun, *Handbook of Mathematical Functions*, Dover edition, 1965, §4.5 Hyperbolic Functions.

8 Saul Perlmutter in *Physics Today*, April 2003, 53-60.

9 Conselice *et al*, arXiv:1607.03909v2 [astroph.GA].