The asymptotic behavior of defocusing nonlinear Schrödinger equations

En-Lin Liu

October 10, 2016

Abstract

This article is concerned with the scattering problem for the defocusing nonlinear Schrödinger equations (NLS) with a power nonlinear \(|u|^p u\) where \(2/n < p < 4/n\). We show that for any initial data in \(H^{0,1}_x\), the solution will eventually scatter, i.e. \(U(-t)u(t)\) tends to some function \(u_+\) as \(t\) tends to infinity.

We consider the defocusing nonlinear Schrödinger equations (NLS)

\[
iu_t + \frac{1}{2} \Delta u = |u|^p u, \quad u(0) = u_0,
\]

(1)

where \(u\) is a complex value function \(u : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}\), \(u_0 \in H^{0,1}_x\), and \(\frac{2}{n} < p < \frac{4}{n}\).

There are many papers on the scattering theory for the NLS. For both focusing or defocusing problems, it is well known that for \(p \leq \frac{2}{n}\) there will be no scattering[1]. For \(p > \frac{2}{n}\), it is known that \(U(-t)u(t)\) converges weakly in \(H^1_x\) for any finite energy solution of NLS[7], if we assume additionally that \(u_0 \in H^{1,1}_x\), then it is know that \(U(-t)u(t)\) converges strongly in \(L^2_x[11]\). For the asymptotic completeness problem, when \(n \geq 3\), for any free solution in \(L^2_x\) or \(H^1_x\) there exists a solution of NLS which approaches the free solution in the same space as \(t\) tends to infinity[6]. In the defocusing case, if \(p > 8/ \left(\sqrt{(n+2)^2 + 8n + n - 2}\right)\), then we have the asymptotic completeness in \(H^{1,1}[4, 10, 8]\). In present paper we combine methods used in [11, 5], which gives similar result for a wider class of solutions. When \(u_0 \in H^{0,1}_x\), we have \(U(-t)u(t)\) converges strongly in \(L^2_x\) and converging rate \(t^{3/2 - np}\) which was implicitly indicate in [11]. Our main result follows:

Theorem 1 : Consider the equation (1) with \(u_0 \in H^{0,1}_x\), then there exists a unique global solution \(u\) with regularity \(U(-t)u(t) \in C(\mathbb{R}; H^{0,1}_x)\), and a function \(u_+ \in L^2_x(\mathbb{R}^n)\) satisfying

\[
\lim_{t \rightarrow \infty} \|U(-t)u(t) - u_+\|_{L^2_x} \lesssim \lim_{t \rightarrow \infty} t^{\frac{3}{2} - \frac{np}{4}} = 0.
\]

(2)

Notation:

Let \(\mathcal{F}\varphi\) and \(\hat{\varphi}\) be the Fourier transform of \(\varphi\) defined by

\[
\mathcal{F}\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x)dx.
\]
Let \( U(t) \) be the free Schrödinger group defined by
\[
U(t) \varphi = (2\pi i t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i|x-y|^2/2t} \varphi(y) dy.
\]

Note that \( U(t) = M(t)D(t)F M(t) \), where \( D(t) \) is the dilation operator \( D(t)f(x) = i^{-\frac{n}{2}} t^{-\frac{n}{2}} f \left( \frac{x}{t} \right) \), and \( M(t) = e^{\frac{i|t|^2}{2}} \). Hence \( U(-t) = M(-t)F^{-1}D^{-1}(t)M(-t) \).

Let \( P_{\leq N} \varphi, P_{\geq N} \varphi \) be the Littlewood-Paley projections:
\[
P_{\leq N} \varphi = F^{-1} \mathcal{X} \left( \frac{\xi}{N} \right) \hat{\varphi}(\xi), \quad P_{\geq N} = \varphi - P_{\leq N} \varphi
\]
where \( \mathcal{X} \) is a Schwartz radial symmetry bump function.

Let \( H_{m,k} \) be the norm defined by
\[
\| \varphi \|^2_{H_{m,k}} = \left\| (1 - \Delta)^{m/2} \varphi \right\|^2_{L^2} + \left\| (1 + |x|^2)^{k/2} \varphi \right\|^2_{L^2}, \quad m, k \geq 0.
\]

1 Well-posedness and energy estimate.

The equation (1) is locally \( L^2_x \) well-posed with \( u_0 \in L^2_x \) by Strichartz estimate for the linear inhomogeneous problem
\[
\left( i\partial_t + \frac{1}{2}\Delta \right) u = f, \quad u(0) = u_0,
\]
which gives us
\[
\|u\|_{L^\infty_t L^2_x} + \|u\|_{L^p_t L^{2+p}_x} \lesssim \|u_0\|_{L^2_x} + \|f\|_{L^{p'}_t L^{(2+p)'}_x},
\]
where \( a = \frac{4(2+p)}{np} \) satisfying the equation \( \frac{2}{a} + \frac{n}{2+p} = \frac{n}{2} \). Applying Hölder inequality to the inhomogeneous term we obtain the unique local solution via the contraction principle in the space \( L^\infty_T (0,T;L^2_x) \cap L^p_t (0,T;L^{2+p}_x) \) provided that \( T \) is small enough. The global well-posedness of \( u \) is due to the conservation of the mass \( \|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x} \).

Denoting \( L_x u \) be the vector field \( L = x + it\nabla \), which is the conjugate of \( x \) with respect to the linear flow, \( L_x = U(t)xU(-t) \). Naturally we have
\[
\left[ i\partial_t + \frac{1}{2}\Delta, L_x \right] = 0
\]
and the equation of \( L_x u \) has the form
\[
\left( i\partial_t + \frac{1}{2}\Delta \right) L_x u = \left( 1 + \frac{p}{2} \right) |u|^p L_x u - \frac{p}{2} u^2 |u|^{p-2} \overline{L_x u},
\]
which is the linearization of (1). The well-posedness of the \( L_x u \) equation is also obtained by the same Strichartz estimate and conservation of the mass. This shows the globally well-posed for initial data in \( H^{0.1} \). See [3, 4]. Denoting
\[ w(t,v) = t^{\frac{n}{2}} e^{-it|v|^2/2} u(t,tv) , \]  
\[ (3) \]
we have \( it^{n/2} e^{-it|v|^2/2} (L_x u)(t,tv) = \partial_t w(t,v) , \) hence \( w \in C(\mathbb{R} \setminus \{0\} ; H^1_v) \) and also globally well-posed. It can also be written as \( w(t,v) = i^{-\frac{n}{2}} D^{-1}(t) M(-t) u \) and gives the differential equation
\[ iw_t + \frac{1}{2t^2} \Delta w = t^{-\frac{np}{2}} |w|^p w \]
\[ (4) \]
for \( t \in \mathbb{R} \setminus \{0\} \). Multiplying \((4)\) with \( w_t \) and takes the real part, this leads us to the following equation, the formal calculation of which can be justified by the regularizing technique of Ginibre and Velo [3]
\[ \frac{1}{4} \| L_x u(t) \|_{L^2_x}^2 + \frac{1}{2 + p} t^2 \| u(t) \|_{L^{2+p}_{t,x}}^{2+p} = \frac{4}{4+2p} \| w \|_{L^{2+p}_{t,x}}^{2+p} \]
\[ \text{(5)} \]
and use the relation \( \nabla w = -it^{n/2} L_x u(t,tv) \) to rewrite \((5)\) into the form
\[ \frac{1}{4} \| L_x u(t) \|_{L^2_x}^2 + \frac{1}{2 + p} t^2 \| u(t) \|_{L^{2+p}_{t,x}}^{2+p} = \frac{4}{4+2p} \| w \|_{L^{2+p}_{t,x}}^{2+p} \]
\[ \text{(6)} \]
Hence by Gronwall’s inequality we get the growth
\[ \| L_x u \|_{L^2_x} = \| \nabla w \|_{L^2_x} \lesssim \| xu_0 \|_{L^2_x} t^{1 - \frac{np}{4}} , \]
\[ \text{(7)} \]
and
\[ t^{\frac{np}{2}} \| u \|_{L^{2+p}_{t,x}}^{2+p} = \| w \|_{L^{2+p}_{t,x}}^{2+p} \lesssim \| xu_0 \|_{L^2_x} 1. \]
\[ \text{(8)} \]
Note that \( 0 < 1 - \frac{np}{4} < \frac{1}{2} \).

## 2 Wave packets and the asymptotic equation.

To study the global decay properties of solutions we use the method of testing by wave packets developing by Ifrim and Tataru [5]. A wave packet is an approximate solution localized in both space and frequency on the scale of the uncertainty principle. We define a wave packet \( \Psi_v \) adapted to the ray \( \Gamma_v := \{ x = vt \} \) and measure \( u \) along \( \Gamma_v \) by considering
\[ \gamma(t,v) = \int u(t,x) \overline{\Psi_v(t,x)} dx. \]
The test function \( \Psi_v \) is of the form
\[ \Psi_v(t,x) = \mathcal{X} \left( \frac{x-vt}{\sqrt{t}} \right) e^{i\phi} \]
where the phase function \( \phi = \frac{|v|^2}{2t} \). Here for the computation purpose, rewrite \( \gamma \) as
\[ \gamma = P_{\leq \sqrt{t} w} , \]
which is the same definition as the original one.
A direct computation yields
\[
 i\gamma_t = F \left[ D\mathcal{X} \left( \frac{\xi}{\sqrt{t}} \right) \cdot \frac{\xi}{2t^{\frac{3}{2}}} + \frac{|\xi|^2}{2t^2} \mathcal{X} \left( \frac{\xi}{\sqrt{t}} \right) \right] \dot{w} + t^{-\frac{np}{2}} P_{\leq \sqrt{t}} |w|^p w := I_1 + I_2. \tag{9}
\]

We apply the similar argument of Tsutsumi and Yajima [11] by computing the decaying rate of \( \|\gamma(t) - \gamma(s)\|_{L^2_x}^2 \) when \( t, s \) goes to infinity to prove that \( \gamma \) converges to some function. Since

\[
 I_1 = F \left[ D\mathcal{X} \left( \frac{\xi}{\sqrt{t}} \right) \cdot \frac{\xi}{2t^{\frac{3}{2}}} + \frac{|\xi|^2}{2t^2} \mathcal{X} \left( \frac{\xi}{\sqrt{t}} \right) \right] \dot{w},
\]

and \( \mathcal{X} \) is a Schwartz function, we get
\[
 \|I_1(t)\|_{L^2_x} \lesssim t^{-\frac{3}{2}} \|\|\| \dot{w}\|_{L^2_x} = t^{-\frac{3}{2}} \|L_x u\|_{L^2_x}. \tag{10}
\]

For the nonlinear part
\[
 I_2 = t^{-\frac{np}{2}} P_{\leq \sqrt{t}} |w|^p w,
\]

by using (8) and Hölder’s inequality, we have for any \( s \geq r \geq 1 \) and any \( T \geq 1 \)
\[
 \left| \left\langle \int_s^T I_2(\sigma) d\sigma, \gamma(T) \right\rangle \right| = \left| \int_s^T \sigma^{-\frac{np}{2}} \langle P_{\leq \sqrt{\sigma}} |w|^p w(\sigma), \gamma(T) \rangle d\sigma \right| \lesssim \int_s^T \sigma^{-\frac{np}{2}} \|w(\sigma)\|_{L^{\frac{2}{1+p}}_s}^{1+p} \|\gamma(T)\|_{L^2_\sigma, p} d\sigma \\
 \lesssim \int_s^T \sigma^{-\frac{np}{2}} \|w(\sigma)\|_{L^{\frac{2}{1+p}}_s}^{1+p} \|w(T)\|_{L^2_\sigma, p} d\sigma \\
 \lesssim \|u_0\|_{L^2_x} s^{1-\frac{np}{2}} - r^{1-\frac{np}{2}}. \tag{11}
\]

By the relation \( \gamma(T) = \gamma(1) - i \int_1^T I_1(\sigma) d\sigma - i \int_1^T I_2(\sigma) d\sigma \) which directly gives
\[
 \gamma(r) - \gamma(s) = -i \int_s^r I_1(\sigma) d\sigma - i \int_s^r I_2(\sigma) d\sigma. \tag{12}
\]

Since \( \|\gamma(T)\|_{L^2_x} \leq \|u(T)\|_{L^2_x} = \|u_0\|_{L^2_x} \), and by (7), (10) display
\[
 \int_s^r \|I_1(\sigma)\|_{L^2_x} d\sigma \lesssim \|u_0\|_{L^2_x} \int_s^r \sigma^{-\frac{1}{2}} \|w\|_{L^{\frac{2}{1+p}}_s} d\sigma \lesssim \|u_0\|_{L^2_x} s^{1-\frac{np}{2}} - r^{1-\frac{np}{2}},
\]

and (11) which gives us
\[
 \left\langle \gamma(r) - \gamma(s), \gamma(r) - \gamma(s) \right\rangle \lesssim \|u_0\|_{L^2_x} \|\gamma(r) - \gamma(s)\|_{L^2_x} \left( s^{1-\frac{np}{2}} - r^{1-\frac{np}{2}} \right) + s^{1-\frac{np}{2}} - r^{1-\frac{np}{2}}. \tag{13}
\]

From above equations there \( \exists g \in L^p_v \) such that \( \lim_{r \to \infty} \|\gamma(t) - g\|_{L^2_v} = 0 \), moreover we have \( \|\gamma(t) - g\|_{L^2_v} \lesssim \|u_0\|_{L^2_x} t^{\frac{3}{2}-\frac{np}{4}} \|\gamma(t) - g\|_{L^2_v} + t^{1-\frac{np}{2}} \) which gives us
\[
 \lim_{t \to \infty} \|\gamma(t) - g\|_{L^2_v} \lesssim \|u_0\|_{L^2_x} \lim_{t \to \infty} t^{\frac{3}{2}-\frac{np}{4}} = 0. \tag{14}
\]
At last, if we take $u_+ = i \frac{\partial}{\partial t} F^{-1} g$, then there is the estimation
\[
\|U(-t)u(t) - u_+\|_{L^2} \leq \|i \frac{\partial}{\partial t} M(-t)F^{-1} w(t) - i \frac{\partial}{\partial t} F^{-1} g\|_{L^2} \\
\approx \|M(-t)F^{-1} (w(t) - \gamma(t))\|_{L^2} + \|M(-t)F^{-1} \gamma(t) - F^{-1} \gamma\|_{L^2} \\
+ \|F^{-1} \gamma(t) - F^{-1} g\|_{L^2} \\
:= R_1 + R_2 + R_3.
\]
It’s obvious that $R_3(t) = \|\gamma(t) - g\|_{L^2}$. For $R_1$, by direct computation which yields
\[
R_1(t) = \|w(t) - \gamma(t)\|_{L^2} = \|P_{\geq \sqrt{t}} w(t)\|_{L^2} \lesssim t^{-\frac{1}{2}} \|\nabla w\|_{L^2} \lesssim \|u_0\|_{L^2} t^{\frac{1}{2} - \frac{np}{4}}.
\]
Use the Taylor expansion of $e^{ix}$, we have that
\[
R_2(t) = \left( e^{i \frac{\|x\|^2}{2t}} - 1 \right) F^{-1} \gamma(t) \approx \left( \frac{x}{\sqrt{t}} \right) \mathcal{X} \left( \frac{x}{\sqrt{t}} \right) \hat{w}(t) \|_{L^2} \\
= t^{-\frac{1}{2}} \|\nabla w(t)\|_{L^2} \lesssim \|u_0\|_{L^2} t^{\frac{1}{2} - \frac{np}{4}}.
\]
Together by (14), (15), (16), and (17)
\[
\lim_{t \to \infty} \|U(-t)u(t) - u_+\|_{L^2} \lesssim \|u_0\|_{L^2} \lim_{t \to \infty} t^{\frac{1}{2} - \frac{np}{4}} = 0.
\]
By the time symmetry property of NLS, we have the same result when $t \to -\infty$.
From coservation of mass we have $\|u_+\|_{L^2} = \|g\|_{L^2} = \|u_0\|_{L^2}$ and (14), (18) display $\|\nabla \gamma\|_{L^2} \lesssim \|L_x u\|_{L^2} \lesssim u_0 t^{1 - \frac{np}{4}}$, $0 < \alpha \leq \frac{np}{2} - 1$
\[
\|\langle x \rangle^{\alpha} u_+\|_{L^2} = \lim_{t \to \infty} \|\langle \nabla \rangle^{\alpha} \gamma\|_{L^2} \lesssim u_0 1.
\]

References


