The real root of the equation: $x^5+x^4+x-1=0$

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Abstract

This note presents some representations for the real root of the equation:

$$x^5+x^4+x-1=0$$

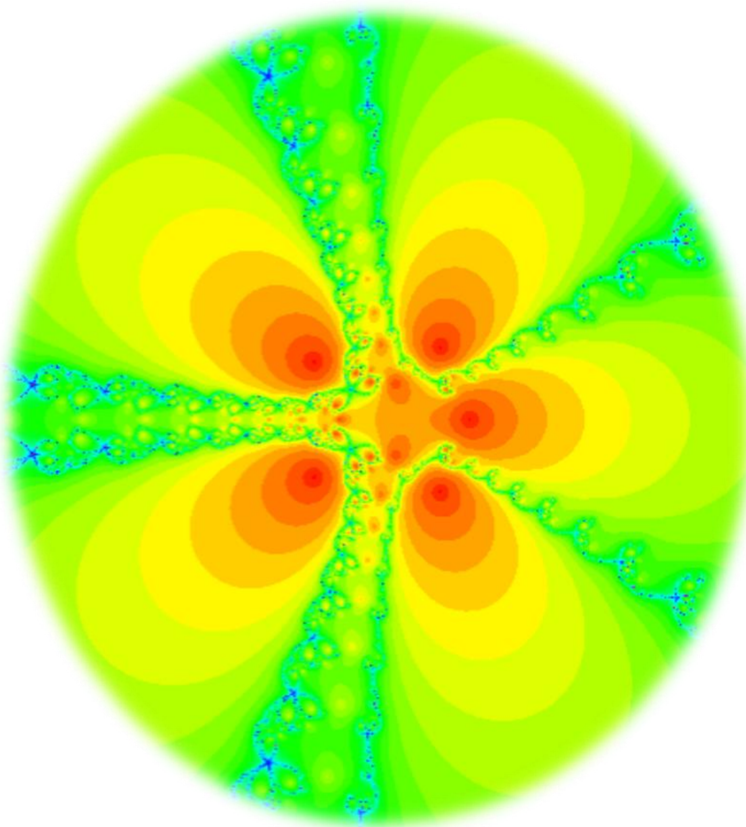

Figure 1. Newton-Julia set for: $f(z)=z^5+z^4+z-1$

1. Introduction

Let $f\left(x\right)=x^5+x^4+x-1$, $\exists! \, r\in\mathbb{R} \,/\, f\left(r\right)=0$:

$$r=0.66796070749623460575... \tag{1}$$

Complex roots:

$$x\in\mathbb{C} \,/\, f\left(x\right)=0 \Rightarrow \begin{cases} z=a+bi=0.3273...+i\times 0.8497... \\ \bar{z}=a-bi=0.3273...-i\times 0.8497... \\ w=c+di=-1.1612...+i\times 0.6758... \\ \bar{w}=c-di=-1.1612...-i\times 0.6758... \end{cases} \tag{2}$$

$$f\left(x\right)=x^5+x^4+x-1=\left(x-r\right)\left(x-z\right)\left(x-\bar{z}\right)\left(x-w\right)\left(x-\bar{w}\right) \tag{3}$$

2. Sequences for r :

$$x_{n+1}=\frac{1+x_n}{2+x_n^3+x_n^4} \, , x_1=1/2 \Rightarrow \lim_{n\to\infty} x_n=r \tag{4}$$

$$x_{n+1}=\frac{1+x_n^2+x_n^3}{1+x_n+x_n^2+x_n^3+x_n^4} \, , x_1=1 \Rightarrow \lim_{n\to\infty} x_n=r \tag{5}$$

$$x_{n+1}=\frac{1+x_n^2+x_n^3-x_n^5}{1+x_n+x_n^2+x_n^3} \, , x_1=1 \Rightarrow \lim_{n\to\infty} x_n=r \tag{6}$$

$$x_{n+1}=\frac{1+x_n^2}{1+x_n+x_n^3+x_n^4} \, , x_1=1/2 \Rightarrow \lim_{n\to\infty} x_n=r \tag{7}$$

$$x_{n+1}=\frac{1+x_n^2-x_n^4+x_n^6}{2} \, , x_1=1/2 \Rightarrow \lim_{n\to\infty} x_n=r \tag{8}$$

$$x_{n+1}=\frac{1+x_n-x_n^4-x_n^5}{2} \, , x_1=1/2 \Rightarrow \lim_{n\to\infty} x_n=r \tag{9}$$

$$x_{n+1}=\frac{1+2x_n-x_n^4-x_n^5}{3} \, , x_1=0 \Rightarrow \lim_{n\to\infty} x_n=r \tag{10}$$

$$x_{n+1}=\sqrt[4]{\frac{1-x_n}{1+x_n}} \, , x_1=1/2 \Rightarrow \lim_{n\to\infty} x_n=r \tag{11}$$

$$x_{n+1}=\frac{4x_n^5+3x_n^4+1}{5x_n^4+4x_n^3+1} \, , x_1=0 \Rightarrow \lim_{n\to\infty} x_n=r \tag{12}$$

3. Linear sequence for r :

$$u_{n+5}=u_{n+4}+u_{n+1}+u_n \, , n\in\mathbb{N}\cup\left\{0\right\} \tag{13}$$

$$u_0=1, u_1=2, u_2=4, u_3=6, u_4=8 \tag{14}$$

$$u_n = \{1, 2, 4, 6, 8, 11, 17, 27, \ldots\} \tag{15}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = r \tag{16}$$

4. Integral for r :

If $h(z) = \dfrac{5-4z-z^4}{z^5+z^4+z-1}$, then:

$$r = \frac{3}{8\pi}\int_0^{2\pi} h\left(\frac{3}{4}e^{ix}\right)e^{ix}dx \tag{17}$$

5. Continued radicals for r :

If $h(z) = \dfrac{1}{z^5+z^4+z-1}$, and $A = -1 + \dfrac{8\pi}{3}\left(\displaystyle\int_0^{2\pi} h\left(\frac{3}{4}e^{ix}\right)e^{ix}dx\right)^{-1}$, then

$$r = \sqrt[3]{\frac{A}{4+5\sqrt[3]{\dfrac{A}{4+5\sqrt[3]{\dfrac{A}{4+\ldots}}}}}} \tag{18}$$

$$\frac{1}{r} = \sqrt[4]{\frac{5}{A}+\frac{4}{A}\sqrt[4]{\frac{5}{A}+\frac{4}{A}\sqrt[4]{\frac{5}{A}+\ldots}}} \tag{19}$$

6. Radical for r :

If $h(z) = \dfrac{1}{z^5+z^4+z-1}$, $A = -1 + \dfrac{8\pi}{3}\left(\displaystyle\int_0^{2\pi} h\left(\frac{3}{4}e^{ix}\right)e^{ix}dx\right)^{-1}$, and

$$Y = \sqrt[3]{-A + A\sqrt{1+\frac{125A}{27}}} - \sqrt[3]{A + A\sqrt{1+\frac{125A}{27}}}$$

then

$$r = -\frac{1}{5} - \frac{1}{10}\sqrt{4+10Y} + \frac{1}{10}\sqrt{8-10Y+4\sqrt{4+10Y}+20\sqrt{5A+Y^2}} \tag{20}$$

7. Radical for r :

If $R(x,y) = -\frac{35x^3}{27} + \frac{5xy^2}{3} - \frac{7x^2}{9} + \frac{y^2}{3} + \frac{x}{9} - \frac{1}{27}$, and

$$D(x,y) = \frac{50x^6}{27} + \frac{20x^5}{9} + \frac{x^4\left(8-125y^2\right)}{27} - \frac{x^3\left(4+100y^2\right)}{27} + \frac{x^2\left(2-2y^2+80y^4\right)}{27} + \frac{32xy^4}{27} - \frac{y^6}{27} + \frac{2y^4}{27} - \frac{y^2}{27}$$

then

$$r = -\frac{1+2a}{3} + \sqrt[3]{R(a,b)+\sqrt{D(a,b)}} + \sqrt[3]{R(a,b)-\sqrt{D(a,b)}} \tag{21}$$

$$r = -\frac{1+2c}{3} + \sqrt[3]{R(c,d)+\sqrt{D(c,d)}} + \sqrt[3]{R(c,d)-\sqrt{D(c,d)}} \tag{22}$$

Remark:

$$x_{n+1} = \frac{4x_n^5+3x_n^4+1}{5x_n^4+4x_n^3+1}, x_1 = \frac{1}{2}+i \Rightarrow \lim_{n\to\infty} x_n = a+bi \tag{23}$$

$$x_{n+1} = \frac{4x_n^5+3x_n^4+1}{5x_n^4+4x_n^3+1}, x_1 = -1+\frac{i}{2} \Rightarrow \lim_{n\to\infty} x_n = c+di \tag{24}$$

8. The number pi :

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(r^{2n+1}+r^{8n+4}\right) \tag{25}$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(a_n+b_nr+c_nr^2+d_nr^3+e_nr^4\right) \tag{26}$$

$$\begin{pmatrix} a_{n+2} \\ b_{n+2} \\ c_{n+2} \\ d_{n+2} \\ e_{n+2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 & -3 & 5 \\ 2 & -1 & -2 & 5 & -8 \\ -1 & 2 & -1 & -2 & 5 \\ 2 & -1 & 2 & -1 & -2 \\ 0 & 2 & -3 & 5 & -6 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \\ e_{n+1} \end{pmatrix} + \begin{pmatrix} -2 & 4 & -6 & 7 & -5 \\ 2 & -6 & 10 & -13 & 12 \\ 1 & 2 & -6 & 10 & -13 \\ -2 & 1 & 2 & -6 & 10 \\ 4 & -6 & 7 & -5 & -1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \end{pmatrix} \tag{27}$$

$$a_0 = 0, b_0 = 1, c_0 = 0, d_0 = 0, e_0 = 1 \tag{28}$$

$$a_1 = 6, b_1 = -10, c_1 = 6, d_1 = -1, e_1 = -7 \tag{29}$$

$$\pi = 4\sum_{n=0}^{\infty} r^{8n}\left(A(n)+B(n)u(n-1)+C(n)u(n-2) + D(n)u(n-3)+E(n)u(n-4)\right) \tag{30}$$

$$u(n)=\begin{cases}1 & ,n\geq 0\\ 0 & ,n<0\end{cases} \tag{31}$$

$$A(n)=\frac{19}{64(8n+1)}-\frac{9}{32(8n+3)}+\frac{7(-1)^n}{16(2n+1)}+\frac{17}{64(8n+5)}+\frac{5}{32(8n+7)} \tag{32}$$

$$B(n)=\frac{137}{16(8n-7)}-\frac{9}{16(8n-5)}+\frac{87(-1)^n}{16(2n-1)}-\frac{25}{8(8n-3)}-\frac{5}{8n-1} \tag{33}$$

$$C(n)=\frac{645}{32(8n-15)}+\frac{7}{2(8n-13)}-\frac{233(-1)^n}{16(2n-3)}-\frac{179}{32(8n-11)}-\frac{177}{16(8n-9)} \tag{34}$$

$$D(n)=\frac{29}{16(8n-23)}+\frac{5}{16(8n-21)}+\frac{21(-1)^n}{2n-5}-\frac{1}{2(8n-19)}-\frac{1}{8n-17} \tag{35}$$

$$E(n)=\frac{11}{64(8n-31)}+\frac{1}{32(8n-29)}-\frac{(-1)^n}{8(2n-7)}-\frac{3}{64(8n-27)}-\frac{3}{32(8n-25)} \tag{36}$$

References

1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions, Nueva York: Dover , 1965.
2. Erdélyi, A., W. Magnus, F. Oberhettinger y F.G. Tricomi, Higher Transcendeltal Functions, 3 vols. Nueva York: McGraw-Hill, 1953, 1955.