The Simple Infinite Set

Ken Seton

Abstract

Many have suggested that the infinite set has a fundamental problem. The usual complaint rails against the actually infinite which (to critics of various finitist persuasions) unjustifiably goes beyond the finite. Here we observe the exact opposite. The problem of the infinite set defined to have an identity (content) that is specified and restricted to be forever finite.

Set theory is taken at its word. The existence of the infinite set and the representation of irrational reals as infinite sets of terms is accepted. In this context, it is shown that the standard definition of the infinite countable set is inconsistent with the existence of its own classic convergents of construction. If the set is infinite then it must be quite unlike that which set theory asserts it to be.

Set theory found itself into some trouble over a century ago trusting an unrestricted anthropic comprehension. But serious doubt is cast on the validity of infinite sets which have been defined by a comprehension which overly-restricts their content.

Keywords
Set Theory; Infinite Set Content.

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1. Set Totality Theorem

The standard definition of a countable infinite set is inconsistent with the foundation of its own infinite sequence of construction.

Definitions

[A] Let an irrational real \( \lambda_1 \) on the unit interval \([0,1]\) be specified as an infinite binary series of terms \( \lambda_{1k} \):

\[
\lambda_1 = \frac{1}{2^1}[0/1] + \frac{1}{2^2}[0/1] + \cdots + \frac{1}{2^k}[0/1] + \cdots
\]

\[
= \lambda_{11} + \lambda_{12} + \cdots + \lambda_{1k} + \cdots
\]

[B] Let the real \( \lambda_1 \) be represented by an infinite subset of the included elements of the binary base set \( \{ \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots \} \):

\[
\lambda_1 \sim \{ \lambda_{11}, \lambda_{12}, \ldots \lambda_{1k}, \ldots \} \quad (\forall k \text{ such that } \lambda_{1k} = \frac{1}{2^k} \neq 0)
\]

[C] Let the infinite sequence of rational reals:

\[1\lambda_1, 2\lambda_1, \ldots, k\lambda_1, \ldots\]

be the partial sums or convergents of construction of \( \lambda_1 \), similarly represented as (finite) subsets of the binary base set:

\[\{ \lambda_{11} \}, \{ \lambda_{11}, \lambda_{12} \}, \ldots, \{ \lambda_{11}, \lambda_{12}, \ldots, \lambda_{1k} \}, \ldots\]

[D] By standard definition and assumption, every element of the countable infinite set \( \{ \lambda_{11}, \lambda_{12}, \lambda_{13}, \ldots \} \) is indexed by a natural number:

\[
z \in \{ \lambda_{11}, \lambda_{12}, \lambda_{13}, \ldots \} \rightarrow \exists k \quad (z = \lambda_{1k} \in \{ \lambda_{11}, \lambda_{12}, \ldots \lambda_{1k} \})
\]

(1)

Theorem Proof

The real \( \lambda_1 \) is greater than each and every one of its indexed convergents:

\[
\lambda_1 > \ldots k\lambda_1 \ldots \geq 3\lambda_1 \geq 2\lambda_1 \geq 1\lambda_1
\]

\[
\rightarrow \quad \lambda_1 > (k\lambda_1 \quad \forall k)
\]

(2a)

By the representation of the reals as subsets of the binary base set:

\[
\rightarrow \quad \{ \lambda_{11}, \lambda_{12}, \lambda_{13}, \ldots \} \supset (\{ \lambda_{11}, \lambda_{12}, \ldots \lambda_{1k} \}) \forall k)
\]

(3)

By the meaning of the proper superset relation:

\[
\rightarrow \quad \exists z \in \{ \lambda_{11}, \lambda_{12}, \lambda_{13}, \ldots \} \quad (z \notin \{ \lambda_{11}, \lambda_{12}, \ldots \lambda_{1k} \} \forall k)
\]

(4)

Not every element of \( \{ \lambda_{11}, \lambda_{12}, \lambda_{13}, \ldots \} \) is indexed by a natural number.

Contradiction between (1) and (4).

QED.
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2 Taking Set Theory at its Word

Are we misled by the Set Totality theorem? One simple way to get closer to the argument is to consider a sequence of finite cases.

Let $\lambda_1$ be an irrational real and $1\lambda_1, 2\lambda_1, \ldots, k\lambda_1$ its first $k$ rational convergents of construction, represented by the sets of terms as already defined. We (should) have no hesitation in accepting the following implications:

$$\lambda_1 \supset 1\lambda_1 \implies \exists z \in \lambda_1 \left( z \notin 1\lambda_1 \right)$$

$$\lambda_1 \supset 2\lambda_1 \supset 1\lambda_1 \implies \exists z \in \lambda_1 \left( z \notin 2\lambda_1 \land z \notin 1\lambda_1 \right)$$

$$\vdots$$

$$\lambda_1 \supset k\lambda_1 \supset \ldots \supset 2\lambda_1 \supset 1\lambda_1 \implies \exists z \in \lambda_1 \left( z \notin k\lambda_1 \land \ldots \land z \notin 2\lambda_1 \land z \notin 1\lambda_1 \right)$$

That is, for the case of $k$ finite, no matter how large;

$$\exists z \in \lambda_1 \left( z \notin i\lambda_1 \ \forall i = 1, k \right)$$

What we understand from each small case (finite $k$) is unambiguously shown. But it is irrelevant that the case is small. There is no reason to think that the result would not apply in the infinite case. Taking set theory at its word – that the infinite collection is nothing more than a natural totality of all finitely indexed instances – why would the result be any different if we let the index domain be extended to all natural numbers?

The Set Totality theorem has been described using the case of the real quantity and the set representations of this quantity and each of its finite convergents. A familiarity with the idea of the irrational real quantity being greater than any of its own indexed rational convergents aids visualization and drives the point home.

But the theorem works for any simple infinite set $\{ 1\beta, 2\beta, \ldots \}, \{ 0, 1, 2, \ldots \}$ and even the generic $\{ s_1, s_2, s_3, \ldots \}$ for which the above logic is equally applicable.

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1 A word structure used by Prof James Franklin (UNSW) in a different context.
3 The Objection Domain

If we accept the existence of the irrational real $\lambda_1$ and its representation as an infinite series (in any base), then that real is greater than every one of its finite convergents $^k\lambda_1$. If it were otherwise, $\lambda_1$ would be rational. This establishes (2) of the theorem.

Now of course the set of non-zero terms $\{\lambda_{11}, \lambda_{12}, \lambda_{13} \ldots\}$ always contains some element in addition to the elements of any $\{\lambda_{11}, \lambda_{12}, \ldots \lambda_{1k}\}$. A proper observation is that this can be satisfied by a different indexed element for each $^k\lambda_1$ and this promotes a sceptical critique that the inference at (4) of the theorem is invalid.

The observation is correct but the objection wrong. The Set Totality theorem does not seek to find element(s) residing as-it-were closer and closer to the real $\lambda_1$, as if pursuing some prey across the plains of the potentially infinite, one-foot-after-the-other. It uses the fact that this real is a fixed quantity already established as greater than any of the indexed convergents. And one will not and cannot locate a $k$ for which this is not true. This means that it is true $\forall k$.

The meaning of the superset relation then provides the content implication for the sets of terms that correspond to the real and to its real convergents.

The objection fails because it is asserting that $\forall k$ means other than a totality of all finite index values. It says in effect that only the potential infinite is permitted. This disowns the leap that mathematical induction is able to take to the conclusion $\forall k$ and is a rejection of set theory’s leap to the actually infinite set.

On the other hand, the theorem has traction because it accepts the representation of $\lambda_1$ as an actually infinite set and the quantification $\forall$ as exhaustive, complete and decisive. It is so abundantly clear that the set representation of an irrational real $\lambda_1$ is a proper superset of all of its finitely indexed convergents ... that one is left wondering just where the objection can obtain any support at all, other than by appeal to definition and authority.

A key perspective arising from the theorem is that the projection or leap from the convergents of construction to the fixed infinite set is matched by a simultaneous projection or leap of the set’s content. The indexed sequence of set convergents is always synchronized with the content of the sequence. Given the formation of the infinite set, why would we imagine that the corresponding content of the sequence could have any imperative other than to take that same leap?

The act of making the set actually infinite has of necessity forced its identity beyond the finitely indexed realm. And because element membership is identity in set theory, a distinct set means a distinct content. The leap to the infinite cannot yield a set that is otherwise.
4. Quite Unlike that which Set Theory Asserts

There are significant consequences arising from the Set Totality theorem. On the one hand, it would be concluded by many a finitist that it is invalid nonsense to assert the existence of the single set totality of (say) all the natural numbers. Most attacks on infinite set theory come from this direction. And many paradoxes and ridiculi proffered in support arise from the inconsistency as identified by the theorem.

On the other hand, one might accept that reals such as $\frac{1}{2\pi}$ exist, with a commensurate representation as infinite sets. In this case, by the Set Totality theorem, an infinite set representing that real must contain at least one element other than those contained in any of the finitely indexed convergents of its construction. And it is clear what these [non-indexed] elements must be:

$$\{ \lambda_{11} \}, \{ \lambda_{11}, \lambda_{12} \} \ldots \sim \{ \lambda_{11}, \lambda_{12} \ldots [0] \}$$
$$\{ 1\lambda_{1} \}, \{ 1\lambda_{1}, 2\lambda_{1} \} \ldots \sim \{ 1\lambda_{1}, 2\lambda_{1} \ldots [\lambda_{1}] \}$$
$$\{ \}, \{ 0 \}, \{ 0, 1 \} \ldots \sim \{ 0, 1, 2, \ldots [\omega] \}$$

The ultimate set of the last line above, the archetype infinite set of set theory, must necessarily contain a transfinite element. It is somewhat ironic that if we embrace the actually infinite set, the formation of the set forces the presence of the transfinite in the set. The theorem shows us that the set necessarily contains a transfinite element - that is, an element not equal to any finite natural number. So let us call this element $\omega$. No longer do we have to define $\omega$ as $\{ 0, 1, 2, \ldots \}$. In this sense the Set Totality theorem provides a kind of heuristic proof that the transfinite exists, albeit by assuming that we can form the set. Firstly, the set itself can be described as transfinite, because it exists other than as one of the finitely indexed convergent sets in the sequence of its construction. But the set is (correspondingly) also transfinite because its identity (content) is not all finitely indexed per that sequence of construction.

If infinite sets are to be accepted and contradiction is to be avoided, the above sets (or any sets that contain such sequences of elements) do not exist without also containing the relevant [non-indexed] limit elements. And this has great consequence. Infinite sets are quite unlike that which set theory asserts they be. To cut to the chase:

The rationals can be listed but they cannot be formed into a single infinite set without containing all reals.

The set of all rationals is the set of all reals.
5. A Good Idea at the Time

It is easy to imagine that one defines, or granting a construction, extracts or creates the collection of finite numbers. 0, 1, 2, 3, ... Supported by familiarity and a supposed integrity of definition, one might feel that it cannot be a mistake to conceptualize the idea that there exists a totality or single set of all but only such numbers. Such a single set seems to be in complete harmony with the element contributions from the construction, so what could possibly be wrong with it? It is a set Cantor [1,P86] would have identified as

```plaintext
a collection into a whole of definite and separate objects
of our intuition or our thought.
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And the extant definition of the countable infinite set does continue to be supported essentially on the basis of our intuition or thought. It is a conceptualization that sits easily and lightly in the mind’s eye. But as with the historical troubles of set theory, it is naive to assume automatically that what we prefer, imagine or assert can just be so. It is known that an unrestricted formation of very large sets can lead to contradiction. But the formation of actually infinite sets restricted to contain only finitely indexed elements is also contradictory.

Aristotle, Gauss and many a philosopher, church father and lay thinker, known and unknown, have rejected the idea of the completed infinite sequence of natural numbers. And they are of course correct, in that each is understanding the natural numbers as an infinite list. And as a list 0, 1, 2, 3, ... has no natural maximum.

Cantor was right to explore the infinite as a meaningful mathematical concept. But who could resist the weight of history, the opinions of such giants as Aristotle and Gauss or the logic itself. In this context, there was only one solution: the never-ending sequence without maximum was to be maintained and placed inside a single fixed collection, the set. And as a new principle of generation, this single object was to be made the first transfinite ordinal. It seemed like a good idea at the time. Indeed, it was a great idea. But the Set Totality theorem is proof that this infinite set is inconsistent. That in the fullness of its identity the set of all natural numbers must itself contain at least one element that is not a natural number. The transfinite makes its appearance in the set by the act of set formation itself ... and in a sense, it is placed where all intuition always said it would be when imagining a list; juxtaposed with but beyond the great chasm at the transfinite end of the endless natural sequence.
There remains today an attitude concerning the infinite that:

Infinite totalities do not exist in any sense of the word (i.e. either really or ideally). More precisely, any mention, or purported mention, of infinite totalities is, literally, meaningless. Nevertheless, we should act as if infinite totalities really existed.

These words are from Abraham Robinson [2, P230], a student of Abraham Fraenkel. They express a typically modern depending-on-the-company-one-keeps nominalism. It is a duplicitous, somewhat imprecise and certainly far from bold understanding of the mathematical infinite.

And it is a nonsensical understanding that does mathematics no service. To the realist, for whom the infinite holds no automatic terrors, the actually infinite is neither more nor less meaningful or real than the tangents to a circle and vanishing points, negative numbers and their roots or the existence of the real $\frac{1}{2\pi}$ and its commensurate expression as a sum of discrete finitely defined rationals.

The difficulty here is not with the idea of the set, or even the actual infinite. The problem is with the contradictory schizomorphic actually infinite set constrained to be forever finite.

Whether he saw it or not, this hybrid infinite set is surely a candidate for Hermann Weyl’s inner instability of the foundations when he wrote [3] in 1920:

The antinomies of set theory are usually regarded as border skirmishes that concern only the remotest provinces of the mathematical empire … (but) every earnest and honest reflection must lead to the realization that the troubles … (are) symptoms (of an) inner instability of the foundations upon which the structure of the empire rests.

The very existence of the Set Totality theorem should give us pause, because it uses simple every-day logic. A logic happily used and accepted in other contexts as transparent and definitive. Given the implications that rest upon the theorem’s efficacy – and because it is so simple – one would hope and expect that a refutation is not argued merely on the grounds that it is simple, challenges definition or just cannot be right.
This paper, *The Simple Infinite Set* is an edited extract of a larger paper of 42 pages, which in turn is from an unpublished book. The author of these works can be contacted at kenseton@gmail.com