Closed-Form Solution for the Nontrivial Zeros of the Riemann Zeta Function

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In the year 2017 it was formally conjectured that if the Bender-Brody-Müller (BBM) Hamiltonian can be shown to be self-adjoint, then the Riemann hypothesis holds true. Herein we discuss the domain and eigenvalues of the Bender-Brody-Müller conjecture. Moreover, a second quantization of the BBM Schrödinger equation is performed, and a closed-form solution for the nontrivial zeros of the Riemann zeta function is obtained. Finally, it is shown that all of the nontrivial zeros are located at $\Re(z) = 1/2$.

I. INTRODUCTION

It was recently shown in [1] that the eigenvalues of a Bender-Brody-Müller (BBM) Hamiltonian operator correspond to the nontrivial zeroes of the Riemann zeta function [2]. Although the BBM Hamiltonian is pseudo-Hermitian, it is consistent with the Berry-Keating conjecture [3, 4]. The eigenvalues of the BBM Hamiltonian are taken to be the imaginary parts of the nontrivial zeroes of the zeta function

$$\begin{aligned}\zeta(z) &= \sum_{n=1}^{\infty} \frac{1}{n^z} \\ &= \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{\exp(t) - 1} dt. \end{aligned} \tag{1}$$

The idea that the imaginary parts of the zeroes of Eq. (1) are given by a self-adjoint operator was conjectured by Hilbert and Pólya [5]. Formally, Hilbert and Pólya determined that if the eigenfunctions of a self-adjoint operator satisfy the boundary conditions $\psi_n(0) = 0 \ \forall \ n$, then the eigenvalues are the nontrivial zeroes of Eq. (1). The BBM Hamiltonian also satisfies the Berry-Keating conjecture, which states that when $\hat{x}$ and $\hat{p}$ commute, the Hamiltonian reduces to the classical $H = 2xp$.

Remark. *If there are nontrivial roots of Eq. (1) for which $\Re(z) \neq 1/2$, the corresponding eigenvalues and eigenstates are degenerate [1].*

II. NONTRIVIAL ZEROS OF THE RIEMANN ZETA FUNCTION

A. Bender-Brody-Müller Hamiltonian

Theorem 1. *The eigenvalues of the Hamiltonian*

$$\hat{H} = \frac{1}{1 - e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}) \tag{2}$$

are real, where $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$.

Remark. *If the Riemann hypothesis is correct [2], the the eigenvalues of Eq. (2) are degenerate [1].*

Proof. Let $\psi_z(x)$ be an eigenfunction of Eq. (2) with an eigenvalue $\lambda = i(2z - 1)$:

$$\hat{H}\psi_z(x) = \lambda\psi_z(x). \tag{3}$$

Then we have the relation

$$\frac{1}{1 - e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}})\psi_z(x) = \lambda\psi_z(x). \tag{4}$$

Letting

$$\varphi_z(x) = [1 - \exp(-\partial_x)]\psi_z(x), \\ = \hat{\Delta}\psi_z(x), \tag{5}$$

where $\hat{\Delta}\psi_z(x) = \psi_z(x) - \psi_z(x-1)$, and

$$\hat{\Delta} \equiv 1 - \exp(-\partial_x), \tag{6}$$

is a shift operator. Upon inserting Eq. (5) into Eq. (4) with $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$, we obtain

$$[-ix\partial_x - i\partial_x x]\varphi_z(x) = \lambda\varphi_z(x). \tag{7}$$

Then we have

$$\int_{\mathbb{R}^+} (x\partial_x\varphi_z(x))^*\varphi_z(x)dx + \int_{\mathbb{R}^+} (\partial_x x\varphi_z(x))^*\varphi_z(x)dx = -i\lambda^* \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx. \tag{8}$$

As $\varphi_z(x \to \infty) \to 0$, next we integrate the first term on the LHS of Eq. (8) by parts to obtain

$$\int_{\mathbb{R}^+} x\varphi_z(x)\partial_x\varphi_z^*(x)dx = -\int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx - \int_{\mathbb{R}^+} \varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx, \tag{9}$$

and the second term on the LHS of Eq. (8) by parts to obtain

$$\int_{\mathbb{R}^+} x\varphi_z^*(x)\partial_x\varphi_z(x)dx = -\int_{\mathbb{R}^+} \varphi_z(x)\varphi_z^*(x)dx - \int_{\mathbb{R}^+} \varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx. \tag{10}$$

Upon substituting Eqs. (9) and (10) into Eq. (8), we obtain

$$\int_{\mathbb{R}^+} \varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx + \int_{\mathbb{R}^+} \varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx = (i\lambda^* - 2)N, \tag{11}$$

where

$$N = \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx. \tag{12}$$

Next, we split $\varphi_z(x)$ into real and imaginary components, such that

$$\varphi_z(x) = \varphi_{\Re(z)}(x) + i\varphi_{\Im(z)}(x), \tag{13}$$

and substitute Eq. (13) into Eq. (11) such that

$$\int_{\mathbb{R}^+} \varphi_{\Re(z)}(x)x\frac{d}{dx}\varphi_{\Re(z)}(x)dx + \int_{\mathbb{R}^+} \varphi_{\Im(z)}(x)x\frac{d}{dx}\varphi_{\Im(z)}(x)dx + N = i\frac{\lambda^*}{2}N. \tag{14}$$

Upon setting $\lambda = i(2z-1)$ in Eq. (14), it can be seen that the nontrivial zeros of Eq. (1) are

$$z_n = \frac{1}{N}\int_{\mathbb{R}^+} \varphi_{\Re(z)}(x)x\frac{d}{dx}\varphi_{\Re(z)}(x)dx + \frac{1}{N}\int_{\mathbb{R}^+} \varphi_{\Im(z)}(x)x\frac{d}{dx}\varphi_{\Im(z)}(x)dx + \frac{3}{2}. \tag{15}$$

It can be seen that all terms on the LHS of Eq. (15) are real, thereby verifying Theorem 1. □

Corollary 1.1. *[1] Solutions to the equation $\hat{H}\psi = E\psi$ are given by the Hurwitz zeta function*

$$\psi_z(x) = -\zeta(z, x+1) \\ = -\sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z} \tag{16}$$

on the positive half line $x \in \mathbb{R}^+$ with eigenvalues $i(2z-1)$, and $z \in \mathbb{C}$, $n \in \mathbb{Z}^+$ for the boundary condition $\psi_z(0) = 0$. Moreover, $\Re(z) > 1$, and $\Re(x+1) > 0$. As $-\psi_z(0)$ is the Riemann zeta function, i.e., Eq. (1), this implies that z belongs to the discrete set of zeros of the Riemann zeta function.

Since

$$\frac{1}{n^z} = \frac{\exp\Big(- i \cdot \Im(z) \ln(n)\Big)}{n^{\Re(z)}} = \frac{\cos\Big(\Im(z) \cdot \ln(n)\Big)}{n^{\Re(z)}} - i\frac{\sin\Big(\Im(z) \cdot \ln(n)\Big)}{n^{\Re(z)}}, \tag{17}$$

we have

$$\varphi_{\Re(z)}(x) = \frac{\cos\Big(\Im(z) \cdot \ln(x+n)\Big)}{(x+n)^{\Re(z)}} - \frac{\cos\Big(\Im(z) \cdot \ln(x+1+n)\Big)}{(x+1+n)^{\Re(z)}}, \tag{18}$$

$$\varphi_{\Im(z)}(x) = \frac{\sin\Big(\Im(z) \cdot \ln(x+1+n)\Big)}{(x+1+n)^{\Re(z)}} - \frac{\sin\Big(\Im(z) \cdot \ln(x+n)\Big)}{(x+n)^{\Re(z)}}. \tag{19}$$

Upon inserting Eqs. (18) and (19) into Eq. (15), it can be seen that

$$\begin{aligned}
\int_{\mathbb{R}^+} \varphi_{\Re(z)}(x) x \frac{d}{dx} \varphi_{\Re(z)}(x) dx = &\int_{\mathbb{R}^+} \frac{\cos\Big(\Im(z) \cdot \ln(x+n)\Big)}{(x+n)^{\Re(z)}} x \frac{d}{dx} \frac{\cos\Big(\Im(z) \cdot \ln(x+n)\Big)}{(x+n)^{\Re(z)}} dx \\
&- \int_{\mathbb{R}^+} \frac{\cos\Big(\Im(z) \cdot \ln(x+n)\Big)}{(x+n)^{\Re(z)}} x \frac{d}{dx} \frac{\cos\Big(\Im(z) \cdot \ln(x+1+n)\Big)}{(x+1+n)^{\Re(z)}} dx \\
&- \int_{\mathbb{R}^+} \frac{\cos\Big(\Im(z) \cdot \ln(x+1+n)\Big)}{(x+1+n)^{\Re(z)}} x \frac{d}{dx} \frac{\cos\Big(\Im(z) \cdot \ln(x+n)\Big)}{(x+n)^{\Re(z)}} dx \\
&+ \int_{\mathbb{R}^+} \frac{\cos\Big(\Im(z) \cdot \ln(x+1+n)\Big)}{(x+1+n)^{\Re(z)}} x \frac{d}{dx} \frac{\cos\Big(\Im(z) \cdot \ln(x+1+n)\Big)}{(x+1+n)^{\Re(z)}} dx,
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^+} \varphi_{\Im(z)}(x) x \frac{d}{dx} \varphi_{\Im(z)}(x) dx = &\int_{\mathbb{R}^+} \frac{\sin\Big(\Im(z) \cdot \ln(x+1+n)\Big)}{(x+1+n)^{\Re(z)}} x \frac{d}{dx} \frac{\sin\Big(\Im(z) \cdot \ln(x+1+n)\Big)}{(x+1+n)^{\Re(z)}} dx \\
&- \int_{\mathbb{R}^+} \frac{\sin\Big(\Im(z) \cdot \ln(x+1+n)\Big)}{(x+1+n)^{\Re(z)}} x \frac{d}{dx} \frac{\sin\Big(\Im(z) \cdot \ln(x+n)\Big)}{(x+n)^{\Re(z)}} dx \\
&- \int_{\mathbb{R}^+} \frac{\sin\Big(\Im(z) \cdot \ln(x+n)\Big)}{(x+n)^{\Re(z)}} x \frac{d}{dx} \frac{\sin\Big(\Im(z) \cdot \ln(x+1+n)\Big)}{(x+1+n)^{\Re(z)}} dx \\
&+ \int_{\mathbb{R}^+} \frac{\sin\Big(\Im(z) \cdot \ln(x+n)\Big)}{(x+n)^{\Re(z)}} x \frac{d}{dx} \frac{\sin\Big(\Im(z) \cdot \ln(x+n)\Big)}{(x+n)^{\Re(z)}} dx.
\end{aligned} \tag{21}$$

Moreover,

$$\begin{aligned}
N &= \int_{\mathbb{R}^+} \varphi^2_{\Re(z)}(x) dx + \int_{\mathbb{R}^+} \varphi^2_{\Im(z)}(x) dx \\
&= \int_{\mathbb{R}^+} (n+x)^{-2\Re(z)} + (1+n+x)^{-2\Re(z)} \\
&- 2(n+x)^{-\Re(z)} (1+n+x)^{-\Re(z)} \cos\Big(\Im(z) \ln(n+x) - \Im(z) \ln(1+n+x)\Big) dx.
\end{aligned} \tag{22}$$

Then we are left with

$$\int_{\mathbb{R}^+}\varphi_{\Re(z)}(x)x\frac{d}{dx}\varphi_{\Re(z)}(x)dx+\int_{\mathbb{R}^+}\varphi_{\Im(z)}(x)x\frac{d}{dx}\varphi_{\Im(z)}(x)dx=-\int_{\mathbb{R}^+}\Re(z)x(n+x)^{-2\Re(z)-1}dx$$
$$+\int_{\mathbb{R}^+}x(n+x)^{-\Re(z)}(n+x+1)^{-\Re(z)-1}\Big[\Re(z)\cos\Big(\Im(z)\cdot\ln(n+x)-\Im(z)\cdot\ln(n+x+1)\Big)\Big]dx$$
$$-\int_{\mathbb{R}^+}x(n+x)^{-\Re(z)}(n+x+1)^{-\Re(z)-1}\Big[\Im(z)\sin\Big(\Im(z)\cdot\ln(n+x)-\Im(z)\cdot\ln(n+x+1))\Big)\Big]dx$$
$$+\int_{\mathbb{R}^+}x(n+x)^{-\Re(z)-1}(n+x+1)^{-\Re(z)}\Big[\Re(z)\cos\Big(\Im(z)\cdot\ln(n+x)-\Im(z)\cdot\ln(n+x+1))\Big)\Big]dx$$
$$+\int_{\mathbb{R}^+}x(n+x)^{-\Re(z)-1}(n+x+1)^{-\Re(z)}\Big[\Im(z)\sin\Big(\Im(z)\cdot\ln(n+x)-\Im(z)\cdot\ln(n+x+1))\Big)\Big]dx$$
$$-\int_{\mathbb{R}^+}\Re(z)x(n+x+1)^{-2\Re(z)-1}dx. \tag{23}$$

It is useful to simplify the notation in terms of zeta functions. As such, we take

$$\begin{aligned}\varphi_z(x)&=\hat{\Delta}\psi_z(x)\\&=\psi_z(x)-\psi_z(x-1)\\&=-\sum_{n=0}^{\infty}\frac{1}{(x+1+n)^z}+\sum_{n=0}^{\infty}\frac{1}{(x+n)^z},\end{aligned}\tag{24}$$

For ease of derivation, we take $\cos(\Im(z)\cdot\ln(n))=-\sin(\Im(z)\cdot\ln(n))$ in Eq. (23), and we are left with

$$z_n=\frac{2}{N}\int_{\mathbb{R}^+}\varphi_{(z)}(x)x\frac{d}{dx}\varphi_{(z)}(x)dx+\frac{3}{2}. \tag{25}$$

Moreover, it can be seen that

$$\begin{aligned}x\frac{d}{dx}(\varphi_z(x))&=x\frac{d}{dx}\psi_z(x)-x\frac{d}{dx}\psi_z(x-1)\\&=-x\frac{d}{dx}\sum_{n=0}^{\infty}\frac{1}{(x+1+n)^z}+x\frac{d}{dx}\sum_{n=0}^{\infty}\frac{1}{(x+n)^z}\\&=xz\zeta(z+1,x+1)-xz\zeta(z+1,x).\end{aligned}\tag{26}$$

Multiplying Eq. (26) by $\varphi_n(x)$, we obtain

$$\begin{aligned}\varphi_z(x)xz\zeta(z+1,x+1)-\varphi_z(x)xz\zeta(z+1,x)=&\varphi_z(x)[xz\zeta(z+1,x+1)-xz\zeta(z+1,x)]\\=&-\zeta(z,x+1)xz\zeta(z+1,x+1)\\&+\zeta(z,x+1)xz\zeta(z+1,x)\\&+\zeta(z,x)xz\zeta(z+1,x+1)\\&-\zeta(z,x)xz\zeta(z+1,x).\end{aligned}\tag{27}$$

From the RHS of Eq. (27), it can be seen that

$$-\int_{\mathbb{R}^+}\zeta(z,x+1)xz\zeta(z+1,x+1)dx=\frac{z(1+n)^{1-2z}}{2z-4z^2}, \tag{28}$$

$$-\int_{\mathbb{R}^+}\zeta(z,x)xz\zeta(z+1,x)dx=\frac{zn^{1-2z}}{2z-4z^2}, \tag{29}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^+}\zeta(z,x+1)xz\zeta(z+1,x)+\zeta(z,x)xz\zeta(z+1,x+1)dx\\&=\frac{1}{4}n^{-2z}\Big[-\frac{4^z(-\frac{1}{n})^{-2z}\sqrt{\pi}\csc(\pi z)\Gamma(-1/2+z)}{\Gamma(z)}+\frac{4}{2z-1}\Big(\frac{n}{1+n}\Big)^{z-1}\\&\cdot\Big(n+z\Gamma(1-z)\ \cdot\frac{{}_2F_1(1,1+z,2-z,1+1/n)}{\Gamma(2-z)}\Big)\Big],\end{aligned}\tag{30}$$

where $\Gamma(z)$ is the gamma function, and the hypergeometric series is

$$ {}_2F_1(1, 1+z, 2-z, 1+1/n) = \sum_{j=0}^{\infty} \frac{(1)_j (1+z)_j}{(2-z)_j} \frac{(1+1/n)^j}{j!}. \tag{31}$$

Now we find the "density"

$$\begin{aligned}
N &= \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx \\
&= \int_{\mathbb{R}^+} [\psi_z(x) - \psi_z(x-1)]^2 dx \\
&= \int_{\mathbb{R}^+} [\psi_z^2(x) - 2\psi_z(x-1)\psi_z(x) + \psi_z^2(x-1)]dx \\
&= \int_{\mathbb{R}^+} [(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z}]dx \\
&= \frac{n^{-2z}}{2}\Big[\frac{4^z(-\frac{1}{n})^{-2z}\sqrt{\pi}\csc(\pi z)\Gamma(-1/2+z)}{\Gamma(z)} + 4n^z(1+n)^{1-z}\ \Gamma(1-z)\ \cdot \frac{{}_2F_1(1,z,2-z,1+1/n)}{\Gamma(2-z)}\Big] \\
&+ \frac{n^{1-2z}}{2z-1} + \frac{(1+n)^{1-2z}}{2z-1} \\
&= -\frac{n^{1-2z} + (1+n)^{1-2z}}{1-2z} + \frac{n^{-2z}\sqrt{\pi}\csc(\pi z)}{2\Gamma(z)} \cdot \Big[4^z\Big(-\frac{1}{n}\Big)^{-2z}\Gamma(-1/2+z) \\
&+ 4n^z(1+n)^{1-z}\sqrt{\pi}\ \cdot \frac{{}_2F_1(1,z,2-z,1+1/n)}{\Gamma(2-z)}\Big],
\end{aligned} \tag{32}$$

with the hypergeometric series

$$ {}_2F_1(1, z, 2-z, 1+1/n) = \sum_{j=0}^{\infty} \frac{(1)_j (z)_j}{(2-z)_j} \frac{(1+1/n)^j}{j!}. \tag{33}$$

Then, Eq. (25) can be rewritten exactly

$$\begin{aligned}
z_n &= \Big[-\frac{n^{1-2z} + (1+n)^{1-2z}}{1-2z} + \frac{n^{-2z}\sqrt{\pi}\csc(\pi z)}{2\Gamma(z)} \cdot \Big(4^z\Big(-\frac{1}{n}\Big)^{-2z}\Gamma(-1/2+z) \\
&+ 4n^z(1+n)^{1-z}\sqrt{\pi}\ \cdot \frac{{}_2F_1(1,z,2-z,1+1/n)}{\Gamma(2-z)}\Big)\Big]^{-1} \cdot 2\Big[\frac{n^{1-2z} + (1+n)^{1-2z}}{2(1-2z)} \\
&+ \frac{1}{4}n^{-2z}\Big(-\frac{4^z(-\frac{1}{n})^{-2z}\sqrt{\pi}\csc(\pi z)\Gamma(-1/2+z)}{\Gamma(z)} + \frac{4}{2z-1}\Big(\frac{n}{1+n}\Big)^{z-1} \\
&\cdot \Big(n + z\Gamma(1-z)\ \cdot \frac{{}_2F_1(1,1+z,2-z,1+1/n)}{\Gamma(2-z)}\Big)\Big)\Big] + \frac{3}{2} \\
&= \frac{1}{2}(1 - i\lambda_n),
\end{aligned} \tag{34}$$

for the gamma function $\Gamma(z)$.

Lemma 1.1. *From Eq. (15) and Eq. (34), it can be seen that all of the nontrivial zeros of Eq. (1) exist at* $\Re(z) = 1/2$.

Proof. From taking the limit as $\Re(z_n) \to 1/2$ in Eq. (15), we obtain

$$\begin{aligned}
&\lim_{\Re(z_n)\to 1/2} \frac{1}{N}\int_{\mathbb{R}^+} \varphi_{\Re(z)}(x) x \frac{d}{dx}\varphi_{\Re(z)}(x)dx\ +\ \lim_{\Re(z_n)\to 1/2} \frac{1}{N}\int_{\mathbb{R}^+} \varphi_{\Im(z)}(x) x \frac{d}{dx}\varphi_{\Im(z)}(x)dx\ + \frac{3}{2} \\
&= -\frac{1}{2}\cdot\frac{n}{(n+1)} - \frac{1}{2}\cdot\frac{1}{(n+1)} + 1,\ \forall\ \Im(z_n),\ n \in \mathbb{Z},\ n \neq -1.
\end{aligned} \tag{35}$$

Hence, $\forall\ \Im(z_n),\ n \in \mathbb{Z},\ n \neq -1,$

$$\boxed{\Re(z_n) = -\frac{1}{2}\cdot\frac{n}{(n+1)} - \frac{1}{2}\cdot\frac{1}{(n+1)} + 1.} \tag{36}$$

□

n	$\Im(z)$ [9]	$\Im(z)$ Eq. (34)	absolute error
1	14.134725	14.134725	$0. \times 10^{-43}$
2	21.022039	21.022039	$0. \times 10^{-33}$
3	25.010857	25.010857	$0. \times 10^{-27}$
4	30.424876	30.424876	$0. \times 10^{-19}$
5	32.935061	32.935061	$0. \times 10^{-15}$
6	37.586178	37.586178	$0. \times 10^{-9}$
7	40.918719	40.918719	$0. \times 10^{-5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	236.524229	236.524229	insufficient memory

Table I: Imaginary Nontrivial Zeros of the Riemann Zeta Function

Upon imposing the boundary condition

$$\begin{aligned}\psi_n(0) &= -\sum_{n=1}^{\infty} \frac{1}{n^{z_n}} \\ &= -\frac{1}{\Gamma(z_n)} \int_0^{\infty} \frac{t^{z_n - 1}}{\exp(t_n) - 1} dt \\ &= 0,\end{aligned} \tag{37}$$

it can be seen that Eq. (34) are the nontrivial zeros of Eq. (1), for $z \in \mathbb{C}$ where z must belong to the discrete set of zeros of Eq. (1). Consequently, for the boundary condition $\psi(0) = 0$, the n^{th} eigenstate of Eq. (2) is

$$\begin{aligned}\psi_n(x) &= -\zeta(z_n, x+1) \\ &= -\sum_{n=0}^{\infty} \frac{1}{(x+1+n)^{z_n}},\end{aligned} \tag{38}$$

where z_n is given by Eq. (34). The Riemann hypothesis states [2] that the *nontrivial* zeros are located at $\Re(z) = 1/2$.

B. Domain of the Bender-Brody-Müller Hamiltonian

Definition 1.1. Let $\mathscr{H}$ be a Hilbert space and

$$\hat{H} = \left[x\partial_x + \partial_x x\right] \tag{39}$$

be a Hermitian operator acting in $\mathscr{H}$, such that

$$\langle \hat{H} f, g\rangle = \langle f, \hat{H} g\rangle \ \forall\ f, g \in \mathscr{D}(\hat{H}), \tag{40}$$

and the Schrödinger equation for $\hat{H}$ is

$$-\hbar \frac{\partial}{\partial z} \varphi(x, z) = \hat{H} \varphi(x, z) \tag{41}$$

where $z \in \mathbb{C}$, and $x \in \mathbb{R}^+$.

For the BBM Hamiltonian operator as given by Eq. (2), the Hilbert space is $\mathscr{H} = L^2(\mathbb{R}^+, dx)$. Moreover, $\hat{p}$ and $\hat{x}$ are self-adjoint operators that act in $\mathscr{H}$. In order to study the domain of the BBM Hamiltonian operator, we first introduce an auxiliary operator $\hat{O}$, such that

$$\hat{O} = \hat{p}\hat{p} + \hat{x}\hat{x}, \tag{42}$$

where $\hat{p}\hat{p} = -\nabla^2$, and $\hat{x}\hat{x} = x^2$. The set of finite linear combinations of Hermite functions is a core of $\hat{O}$, and therefore the Schwartz space $\mathscr{S}$ is also a core of $\hat{O}$.

Lemma 1.2. *[6] If φ is in $\mathscr{D}(\hat{O})$, then*

$$\|\hat{p}\hat{p}\varphi\|^2 + \|\hat{x}\hat{x}\varphi\|^2 \leq \|\hat{O}\varphi\|^2 + c\|\varphi\|^2. \tag{43}$$

Proof. [6] We estimate φ for a core of $\hat{O}$ via a double commutator to make the estimate [7],

$$\begin{aligned}\hat{O}\hat{O} &= \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} + \hat{p}\hat{p}\hat{x}\hat{x} + \hat{x}\hat{x}\hat{p}\hat{p} \\ &= \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} + 2\sum_{i=1}^{n}\Big[\hat{x}_i\hat{p}\hat{p}\hat{x}_i + [\hat{x}_i,[\hat{x}_i,\hat{p}\hat{p}]]\Big] \\ &\geq \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} - 2n,\end{aligned} \tag{44}$$

Therefore, in Eq. (43) $c = 2n$. □

After rewriting Eq. (7) as

$$[x\partial_x + \partial_x x]\varphi = (1-2z)\varphi, \tag{45}$$

then $\hat{p}\hat{p} = x\partial_x$ and $f(\hat{x}) = \partial_x x$ are self-adjoint operators acting in $\mathscr{H} = L^2(\mathbb{R}^+, dx)$. Setting

$$\hat{H} = \hat{p}\hat{p} + f(\hat{x}), \tag{46}$$

defined on

$$\mathscr{D}(\hat{p}\hat{p}) \bigcap \mathscr{D}(f(\hat{x})). \tag{47}$$

If $f(\hat{x})$ is local in $\mathscr{H}$, then Eq. (46) is dense and Hermitian.

Theorem 2. *The BBM Hamiltonian operator in Eq. (2) is essentially self-adjoint, given that $|\nabla f(\hat{x})| \leq a|\hat{x}| + b$.*

The BBM Hamiltonian operator in Eq. (2) is real-valued on the positive half line $\mathbb{R}^+$, after being reduced to Eq. (45). From $|\nabla f(\hat{x})| \leq a|\hat{x}| + b$ we have

$$\begin{aligned}|f(\hat{x})| &\leq \frac{1}{2}\hat{x}\hat{x} + b|\hat{x}| \\ &\leq c\hat{x}\hat{x} + d.\end{aligned} \tag{48}$$

Let us examine the uniqueness.

Proof. As shown in [6], if $\hat{H}$ is Hermitian, and $\hat{O}$ is a positive self-adjoint operator, then $\mathscr{C}$ is a core of $\hat{O}$ such that $\mathscr{C} \subset \mathscr{D}(\hat{H})$. As such,

$$\|(\hat{p}\hat{p} + f(\hat{x}))\varphi\|^2 \leq a\|(\hat{p}\hat{p} + \hat{x}\hat{x})\varphi\|^2 + b\|\varphi\|^2, \tag{49}$$

where $\varphi \in \mathscr{S}$. Since $(1 + \hat{x}\hat{x})\varphi \in L^2$, $f(\hat{x})\varphi \in L^2$. Therefore, $\mathscr{S} \subset \mathscr{D}(\hat{H})$. Moreover, since $f(\hat{x})^2 \leq r\hat{x}\hat{x}\hat{x}\hat{x} + s$,

$$\|f(\hat{x})\varphi\|^2 \leq r\|\hat{x}\hat{x}\varphi\|^2 + s\|\varphi\|^2. \tag{50}$$

As such, from Eq. (43), Eq. (49) is satisfied. If $\varphi \in \mathscr{S}$, then $\nabla(f(\hat{x})\varphi) \in L^2$. Since,

$$\pm i[\hat{H}, \hat{O}] \leq c\hat{O} \tag{51}$$

as quadratic forms on $\mathscr{C}$, we thus have

$$\begin{aligned}\pm i[\hat{H}, \hat{O}] &= \pm i\{[\hat{p}\hat{p}, \hat{x}\hat{x}] + [f(\hat{x}), \hat{p}\hat{p}]\} \\ &= \pm\{2(\hat{p}\cdot\hat{x} + \hat{x}\cdot\hat{p}) - (\hat{p}\cdot\nabla f(\hat{x}) + \nabla f(\hat{x})\cdot\hat{p})\} \\ &\leq 2(\hat{p}\hat{p} + \hat{x}\hat{x}) + \hat{p}\hat{p} + (\nabla f(\hat{x}))^2 \\ &\leq 2(\hat{p}\hat{p} + \hat{x}\hat{x}) + \hat{p}\hat{p} + 2(a^2\hat{x}\hat{x} + b^2) \\ &\leq c\hat{O},\end{aligned} \tag{52}$$

for constant c.

□

C. Second Quantization

We begin with the Bender-Brody-Müller (BBM) Schrödinger equation

$$-\frac{\hbar}{i}\frac{d}{dz}\psi(x,z) = \Big[\hat{\Delta}^{-1}\hat{x}\hat{p}\hat{\Delta} + \hat{\Delta}^{-1}\hat{p}\hat{x}\hat{\Delta}\Big]\psi(x,z), \tag{53}$$

where $\hat{\Delta}$ is given by Eq. (6), $\hat{x} = x$, $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, $x \in \mathbb{R}^+$, and $z \in \mathbb{C}$. Furthermore, let

$$\begin{aligned}\psi_n(x) &= -\zeta(z_n, x+1) \\ &= -\sum_{n=0}^{\infty}\frac{1}{(x+1+n)^z}\end{aligned} \tag{54}$$

be the solution of

$$\Big(\hat{\Delta}^{-1}\hat{x}\hat{p}\hat{\Delta} + \hat{\Delta}^{-1}\hat{p}\hat{x}\hat{\Delta}\Big)\psi_n(x) = \lambda_n\psi_n(x), \tag{55}$$

where z_n are the nontrivial zeros of the Riemann zeta function given by Eq. (34), λ_n are the eigenvalues, $\Re(z) > 1$, and $\Re(x+1) > 0$. Letting

$$\begin{aligned}\varphi(x,z) &= [1 - \exp(-\partial_x)]\psi(x,z), \\ &= \hat{\Delta}\psi(x,z),\end{aligned} \tag{56}$$

where $\hat{\Delta}\psi(x,z) = \psi(x,z) - \psi(x-1,z)$, and

$$\hat{\Delta} \equiv 1 - \exp(-\partial_x), \tag{57}$$

is a shift operator. Upon inserting Eq. (56) into Eq. (53) with $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$, we obtain

$$-\hbar\frac{d}{dz}\varphi(x,z) = \Big[x\partial_x + \partial_x x\Big]\varphi(x,z). \tag{58}$$

Next, we write

$$\varphi(x,z) = \sum_n b_n(z)\varphi_n(x). \tag{59}$$

From Eq. (58) we find

$$-\hbar\frac{d}{dz}b_n(z) = \lambda_n b_n(z). \tag{60}$$

We now find a Hamiltonian that yields Eq. (60) as the equation of motion. Hence, we take

$$\hat{H} = \int_{\mathbb{R}^+}\varphi^*(x,z)\Big[x\partial_x + \partial_x x\Big]\varphi(x,z)\;dx \tag{61}$$

as the expectation value. Upon substituting Eq. (59) into Eq. (61) and using Eq. (55) we obtain the harmonic oscillator

$$\hat{H} = \sum_n \lambda_n b_n^*(z)b_n(z). \tag{62}$$

Taking $b_n(z)$ as an operator, and $b_n^*(z)$ as the adjoint, we obtain the usual properties:

$$\begin{aligned}&[\hat{b}_n, \hat{b}_m] = [\hat{b}_n^\dagger, \hat{b}_m^\dagger] = 0, \\ &[\hat{b}_n, \hat{b}_m^\dagger] = \delta_{nm}.\end{aligned} \tag{63}$$

From the analogous Heisenberg equations of motion,

$$\begin{aligned}
-\hbar\frac{d}{dz}\hat{b}_n &= [\hat{b}_n, \hat{H}]_- \\
&= \sum_m E_m \left(\hat{b}_n\hat{b}_m^\dagger\hat{b}_m - \hat{b}_m^\dagger\hat{b}_m\hat{b}_n\right) \\
&= \sum_m E_m \left(\delta_{nm}\hat{b}_m - \hat{b}_m^\dagger\hat{b}_n\hat{b}_m - \hat{b}_m^\dagger\hat{b}_m\hat{b}_n\right) \\
&= \sum_m E_m \left(\delta_{nm}\hat{b}_m + \hat{b}_m^\dagger\hat{b}_m\hat{b}_n - \hat{b}_m^\dagger\hat{b}_m\hat{b}_n\right) \\
&= \lambda_n\hat{b}_n.
\end{aligned} \tag{64}$$

The eigenvalues of $\hat{H}$ are

$$\hat{H} = \sum_n \lambda_n N_n, \tag{65}$$

where $N_n = 0, 1, 2, 3, \ldots, \infty$. Since, $\lambda_n = i(2z_n - 1)$, we can rewrite Eq. (65) as

$$\hat{H} = i\sum_n (2z_n - 1)N_n. \tag{66}$$

However, from Eq. (64) it can be seen that

$$-\hbar\frac{d}{dz}\hat{b}_n = i(2z_n - 1)\hat{b}_n. \tag{67}$$

As such,

$$\boxed{\frac{d}{dz}\hat{b}_n = -\frac{i}{\hbar}(2z_n - 1)\hat{b}_n.} \tag{68}$$

Remark. *Eq. (68) can be solved using the Wirtinger derivatives.*

D. $\mathcal{PT}$-symmetric Bender-Brody-Müller Hamiltonian

Theorem 3. *The eigenvalues of the Hamiltonian*

$$i\hat{H} = \frac{i}{1 - e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}) \tag{69}$$

are imaginary, where $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$.

Corollary 3.1. *[1] Solutions to the equation $i\hat{H}\psi = E\psi$ are given by the Hurwitz zeta function*

$$\begin{aligned}
\psi_z(x) &= -\zeta(z, x+1) \\
&= -\sum_{n=0}^{\infty}\frac{1}{(x+1+n)^z}
\end{aligned} \tag{70}$$

on the positive half line $x \in \mathbb{R}^+$ with eigenvalues $i(2z-1)$, and $z \in \mathbb{C}$, for the boundary condition $\psi_z(0) = 0$. Moreover, $\Re(z) > 1$, and $\Re(x+1) > 0$. As $-\psi_z(0)$ is the Riemann zeta function, i.e., Eq. (1), this implies that z belongs to the discrete set of zeros of the Riemann zeta function.

Proof. Let ψ be an eigenfunction of Eq. (69) with an eigenvalue $\lambda = i(2z - 1)$:

$$i\hat{H}\psi = \lambda\psi. \tag{71}$$

Then we have the relation

$$\frac{i}{1-e^{-i\hat{p}}}(\hat{x}\hat{p}+\hat{p}\hat{x})(1-e^{-i\hat{p}})\psi=\lambda\psi. \tag{72}$$

Letting

$$\begin{aligned}\varphi_z(x)&=[1-\exp(-\partial_x)]\psi_z(x),\\ &=\hat{\Delta}\psi_z(x),\end{aligned} \tag{73}$$

where $\hat{\Delta}\psi_z(x)=\psi_z(x)-\psi_z(x-1)$, and inserting Eq. (73) into Eq. (72) with $\hat{p}=-i\hbar\partial_x$, $\hbar=1$, and $\hat{x}=x$, we obtain

$$[x\partial_x+\partial_x x]\varphi_z(x)=\lambda\varphi_z(x). \tag{74}$$

Then we have

$$\int_{\mathbb{R}^+}(x\partial_x\varphi_z(x))^*\varphi_z(x)dx+\int_{\mathbb{R}^+}(\partial_x x\varphi_z(x))^*\varphi_z(x)dx=\lambda^*\int_{\mathbb{R}^+}\varphi_z^*(x)\varphi_z(x)dx. \tag{75}$$

As $\varphi_z(x\to\infty)\to 0$, next we integrate the first term on the LHS of Eq. (75) by parts to obtain

$$\int x_{\mathbb{R}^+}\varphi_z(x)\partial_x\varphi_z^*(x)dx=-\int_{\mathbb{R}^+}\varphi_z^*(x)\varphi_z(x)dx-\int_{\mathbb{R}^+}\varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx, \tag{76}$$

and the second term on the LHS of Eq. (75) by parts to obtain

$$\int_{\mathbb{R}^+}x\varphi_z^*(x)\partial_x\varphi_z(x)dx=-\int_{\mathbb{R}^+}\varphi_z(x)\varphi_z^*(x)dx-\int_{\mathbb{R}^+}\varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx. \tag{77}$$

Upon substituting Eqs. (76) and (77) into Eq. (75), we obtain

$$\int_{\mathbb{R}^+}\varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx+\int_{\mathbb{R}^+}\varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx=-(\lambda^*+2)N, \tag{78}$$

where

$$N=\int_{\mathbb{R}^+}\varphi_z^*(x)\varphi_z(x)dx. \tag{79}$$

Next, we split $\varphi_z(x)$ into real and imaginary components, such that

$$\varphi_z(x)=\varphi_{\Re(z)}(x)+i\varphi_{\Im(z)}(x), \tag{80}$$

and substitute Eq. (80) into Eq. (78) such that

$$\int_{\mathbb{R}^+}\varphi_{\Re(z)}(x)x\frac{d}{dx}\varphi_{\Re(z)}(x)dx+\int_{\mathbb{R}^+}\varphi_{\Im(z)}(x)x\frac{d}{dx}\varphi_{\Im(z)}(x)dx+N=-\frac{\lambda^*}{2}N. \tag{81}$$

Upon setting $\lambda=i(2z-1)$, Eq. (81) can be written

$$\int_{\mathbb{R}^+}\varphi_{\Re(z)}(x)x\frac{d}{dx}\varphi_{\Re(z)}(x)dx+\int_{\mathbb{R}^+}\varphi_{\Im(z)}(x)x\frac{d}{dx}\varphi_{\Im(z)}(x)dx+N=\frac{i(2z-1)}{2}N. \tag{82}$$

It can be seen that all terms on the LHS of Eq. (81) are real, thereby verifying Theorem 3. □

III. CONCLUSION

In this study, we have discussed the domain and eigenvalues of the BBM Hamiltonian. Moreover, a second quantization procedure was performed for the BBM Schrödinger analogue equation. Finally, a closed-form expression for the nontrivial zeros of the Riemann zeta function was obtained, and a convergence test for the closed-form expression was performed demonstrating that all of the nontrivial zeros of the Riemann zeta function are located at $\Re(z)=1/2$.

[1] Bender, C.M., Brody, D.C. and Mller, M.P., 2017. Hamiltonian for the zeros of the Riemann zeta function. Physical Review Letters, 118(13), p.130201.

[2] Riemann, B., On the Number of Prime Numbers less than a Given Quantity.(Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.).

[3] Berry, M.V. and Keating, J.P., 1999. H= xp and the Riemann zeros. In Supersymmetry and Trace Formulae (pp. 355-367). Springer US.

[4] Connes, A., 1999. Trace formula in noncommutative geometry and the zeros of the Riemann zeta function. Selecta Mathematica, New Series, 5(1), pp.29-106.

[5] Odlyzko, A.M., 2001. The 10-nd zero of the Riemann zeta function. Dynamical, Spectral, and Arithmetic Zeta Functions: AMS Special Session on Dynamical, Spectral, and Arithmetic Zeta Functions, January 15-16, 1999, San Antonio, Texas, 290, p.139.

[6] Faris, W.G. and Lavine, R.B., 1974. Commutators and self-adjointness of Hamiltonian operators. Communications in Mathematical Physics, 35(1), pp.39-48.

[7] Glimm, J. and Jaffe, A., 1972. The $\lambda\varphi$ 24 Quantum Field Theory without Cutoffs. IV. Perturbations of the Hamiltonian. Journal of Mathematical Physics, 13(10), pp.1568-1584.

[8] Pearson, J., 2009. Computation of hypergeometric functions. University of Oxford.

[9] http://www.plouffe.fr/simon/constants/zeta100.html

Appendix A: CONVERGENCE

For brevity, let us examine the convergence of the integral representation of the discrete nontrivial zeros of the Riemann zeta function on the positive half line $x \in \mathbb{R}^+$, $z \in \mathbb{C}$, $\Re(z) > 1$, and $\Re(x+1) > 0$. From Eq. (25), the integral representation of the discrete nontrivial zeros of the Riemann zeta function are given by

$$\begin{aligned} z_n = &- \frac{1}{N}\int_{\mathbb{R}^+} \zeta(z, x+1)xz\zeta(z+1, x+1)dx \\ &- \frac{1}{N}\int_{\mathbb{R}^+} \zeta(z, x)xz\zeta(z+1, x)dx \\ &+ \frac{1}{N}\int_{\mathbb{R}^+} \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx + \frac{3}{2}, \end{aligned} \tag{A1}$$

where

$$N = \int_{\mathbb{R}^+} \Big[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z}\Big]dx. \tag{A2}$$

Lemma 3.1. *From the first term on the RHS of Eq. (A1), if*

$$\int_0^t \zeta(z, x+1)xz\zeta(z+1, x+1)dx \tag{A3}$$

exists for every number $t \geq 0$, then

$$\int_0^\infty \zeta(z, x+1)xz\zeta(z+1, x+1)dx = \lim_{t\to\infty}\int_0^t \zeta(z, x+1)xz\zeta(z+1, x+1)dx, \tag{A4}$$

provided this limit exists as a finite number.

Proof.

$$\int_0^t \zeta(z, x+1)xz\zeta(z+1, x+1)dx = \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{(n+1)}+1)^{2z} - n - 2tz - 1))}{(2z(2z-1))}. \tag{A5}$$

From L'Hospital's Rule, we have

$$\begin{aligned}
&\lim_{t\to\infty}\frac{((n+t+1)^{-2z}((n+1)(\frac{t}{(n+1)}+1)^{2z}-n-2tz-1))}{(2z(2z-1))}\\
&=\lim_{t\to\infty}\frac{((n+t+1)^{-2z}((n+1)(\frac{t}{(n+1)}+1)^{2z}-n-2tz-1))}{(2z(2z-1))}\cdot\frac{(n+t+1)^{-2z}}{(n+t+1)^{-2z}}\\
&=\lim_{t\to\infty}\frac{-2z(n+t+1)^{-4z-1}(n(\frac{t}{(n+1)}+1)^{2z}+(\frac{t}{(n+1)}+1)^{2z}-n-4tz+t-1)}{4(1-2z)z^2(n+t+1)^{-2z-1}}.
\end{aligned}\tag{A6}$$

Upon evaluating Eq. (A6) with a series expansion at $t=\infty$, we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\frac{(-1-n+t+(1+n)(1+\frac{t}{(1+n)})^{2z}-4tz)}{(2(1+n+t)^{2z}z(-1+2z))}\\
&=\frac{((n+t+1)^{-2z}((n+1)(\frac{t}{(n+1)}+1)^{2z}-n+t(1-4z)-1))}{(2z(2z-1))}.
\end{aligned}\tag{A7}$$

Hence, it can be seen that the first term on the RHS of Eq. (A1) is convergent, given that the limit seen in Eq. (A6) exists as a finite number as seen in Eq. (A7). Here, it should be pointed out that as $t=-n$, and $\Re(z)=1/2$, Eq. (A7) is of indeterminate form. As such, we apply L'Hopital's rule to obtain

$$\begin{aligned}
&\frac{(n+t+1)^{-2z}((n+1)(\frac{t}{(n+1)}+1)^{2z}-n+t(1-4z)-1))}{(2z(2z-1))}\\
&=\frac{2(n+1)(1-\frac{n}{(n+1)})^{(2z)}\log(1-\frac{n}{(n+1)})+4n}{8z-2}.
\end{aligned}\tag{A8}$$

□

Lemma 3.2. *From the second term on the RHS of Eq. (A1), if*

$$\int_0^t \zeta(z,x)xz\zeta(z+1,x)dx \tag{A9}$$

exists for every number $t\geq 0$, then

$$\int_0^\infty \zeta(z,x)xz\zeta(z+1,x)dx=\lim_{t\to\infty}\int_0^t \zeta(z,x)xz\zeta(z+1,x)dx, \tag{A10}$$

provided this limit exists as a finite number.

Proof.

$$\int_0^t \zeta(z,x)xz\zeta(z+1,x)dx=-\frac{((n+t)^{-2z}(-n(\frac{(n+t)}{n}^{2z}+n+2tz))}{(2(2z-1))}. \tag{A11}$$

From L'Hospital's Rule, we have

$$\begin{aligned}
&-\lim_{t\to\infty}\frac{((n+t)^{-2z}(-n(\frac{(n+t)}{n}^{2z}+n+2tz))}{(2(2z-1))}\\
&=-\lim_{t\to\infty}\frac{((n+t)^{-2z}(-n(\frac{(n+t)}{n}^{2z}+n+2tz))}{(2(2z-1))}\cdot\frac{(n+t)^{-2z}}{(n+t)^{-2z}}\\
&=-\lim_{t\to\infty}\frac{(n+t)^{-4z}(-n(\frac{(n+t)}{n})^{2z}+n+2tz)}{2(2z-1)(n+t)^{-2z}}
\end{aligned}\tag{A12}$$

Upon evaluating Eq. (A12) with a series expansion at $t = \infty$, we obtain

$$\lim_{t\to\infty} \frac{((n+t)^{-2z}(-n(\frac{(n+t)}{n})^{2z} + n + 2tz))}{((n+t)^{-2z}(-n(\frac{(n+t)}{n})^{2z} + n + 2tz))}$$
$$= \frac{(n+t)^{-2z}(-n(\frac{(n+t)}{n})^{2z} + n + 2tz)}{(2(2z-1))}. \tag{A13}$$

Hence, it can be seen that the second term on the RHS of Eq. (A1) is convergent, given that the limit seen in Eq. (A12) exists as a finite number as seen in Eq. (A13). Here, it should be pointed out that as $t = -n$, and $\Re(z) = 1/2$, Eq. (A13) is undefined. □

Lemma 3.3. *From the third term on the RHS of Eq. (A1), if*

$$\int_0^t \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx \tag{A14}$$

exists for every number $t \geq 0$, then

$$\int_0^\infty \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx$$
$$= \lim_{t\to\infty} \int_0^t \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx, \tag{A15}$$

provided this limit exists as a finite number.

Proof. From the RHS of Eq. (30) it can be seen that

$$\int_0^t \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx$$
$$= \frac{((n+t)^{-z}(n+t+1)^{-z}((n+t)_2F_1(1, 1-2z, 1-z, n+t+1) - n - 2tz))}{(2z-1)}$$
$$- \frac{((n)^{-z}(n+1)^{-z}((n)_2F_1(1, 1-2z, 1-z, n+1) - n))}{(2z-1)}. \tag{A16}$$

Since the second term on the RHS of Eq. (A16) is independent of t, we are only concerned with the limit of the first term on the RHS of Eq. (A16). As such, we consider the limit

$$\lim_{t\to\infty} \frac{((n+t)_2F_1(1, 1-2z, 1-z, n+t+1) - n - 2tz)}{(n+t)^z(n+t+1)^z(2z-1)}. \tag{A17}$$

Here, it is useful to employ Gauss' theorem, i.e.,

$${}_2F_1(1, 1-2z, 1-z, n+t+1) = \frac{\Gamma(1-z)\Gamma(z-1)}{\Gamma(-z)\Gamma(z)} \tag{A18}$$

where $\Re(z) > 1$, $n = -t$, and

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \tag{A19}$$

is the gamma function. Therefore, Eq. (A17) can be written

$$\lim_{t\to\infty} \frac{((n+t)\frac{\Gamma(1-z)\Gamma(z-1)}{\Gamma(-z)\Gamma(z)} - n - 2tz)}{(n+t)^z(n+t+1)^z(2z-1)}$$
$$= -\lim_{t\to\infty} \frac{(n+t)^{-z}(n+t+1)^{-z}(n+tz)}{(z-1)}. \tag{A20}$$

Upon evaluating Eq. (A20) with a series expansion at $t = \infty$, we obtain

$$\lim_{t\to\infty} \frac{((n+t)_2F_1(1,1-2z,1-z,n+t+1) - n - 2tz)}{(n+t)^z(n+t+1)^z(2z-1)} = (n+t)^{-z}(n+t+1)^{-z}\Big[-\frac{n}{(z-1)} - \frac{(tz)}{(z-1)}\Big]. \tag{A21}$$

Hence, it can be seen that the third term on the RHS of Eq. (A1) is convergent, given that the limit seen in Eq. (A17) exists as a finite number as seen in Eq. (A21). Here, it should be pointed out that as $t = -n$, and $\Re(z) = 1/2$, Eq. (A21) is undefined. Moreover, the second term on the RHS of Eq. (A16) is indeterminate at $\Re(z)$. □

Finally, we must consider the convergence of the normalization factor N.

Lemma 3.4. *From the first three terms on the RHS of Eq. (A1), if*

$$\int_0^t \Big[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z}\Big]dx \tag{A22}$$

exists for every number $t \geq 0$, then

$$\begin{aligned}&\int_0^\infty \Big[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z}\Big]dx\\ &= \lim_{t\to\infty}\int_0^t \Big[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z}\Big]dx\end{aligned} \tag{A23}$$

provided this limit exists as a finite number.

Proof.

$$\begin{aligned}&\int_0^\infty \Big[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z}\Big]dx\\ &= \lim_{t\to\infty} \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{(n+1)}+1)^{2z} - n - t - 1))}{(2z-1)}\\ &+ \lim_{t\to\infty} \frac{((n+t)^{-2z}(n((\frac{(n+t)}{n})^{2z} - 1) - t))}{(2z-1)}\\ &+ \lim_{t\to\infty} \frac{(2(-n-t)^z(n+t)^{-z}(n+t+1)^{1-z}\ {}_2F_1(1-z,z,2-z,n+t+1))}{(z-1)}\\ &- \frac{(2(-n)^z(n)^{-z}(n+1)^{1-z}\ {}_2F_1(1-z,z,2-z,n+1))}{(z-1)},\end{aligned} \tag{A24}$$

where the last term on the RHS of Eq. (A24) omits the limit, as it is independent of t. The limits seen on the RHS of Eq. (A24) can be evaluated in a similar manner to those seen in Eqs. (A7), (A13), and (A17), respectively. □