AN INTRODUCTION TO THE n-IRREDUCIBLE SEQUENTS AND THE n-IRREDUCTIBLE NUMBER

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Abstract. In this work, we introduce the n-irreductible sequents and the n-irreductible numbers defined with the help of the second order logic. We give many concrete examples of n-irreductible numbers and n-irreductible sequents with the Peano's axioms and the axioms of the real numbers. Shortly, a sequent is n-irreductible iff the sequent is composed by some closed hypotheses and a n-irreductible formula (a close formula with one internal variable such that the formula is only true when we set that variable to the unique natural number n), and it does not exist some strict sub-sequent which are composed by some closed sub-hypotheses and some sub-m-irreductible formula with $m > 1$. The definition is motivated by the intuition that the "Nature's hypotheses"do not carry natural numbers or "hidden natural numbers" except for the numbers 0 and 1, i.e., they can be used in a n-irreductible sequent. Moreover, we postulate at second order of logic that the "Nature's hypotheses"are not chosen randomly: the "Nature's hypotheses"are the only hypotheses which give the largest n-irreductible number $N_Z < 10^{3.026\times10^7} \cong 2^{1.005\times10^8}$. The Goldbach's conjecture, the Polignac's conjecture, the Firoozbakht's conjecture, the Oppermann's conjecture, the Agoh-Giuga conjecture, the generalized Fermat's conjecture and the Schinzel's hypothesis H are reviewed with this new (second order logic) n-irreductible axiom. Finally, two open questions remain: Can we prove that a natural number is not n-irreductible? If a n-irreductible number n is found with a function symbol f where its outputs values are only 0 and 1, can we always replace the function symbol f by a another function symbol $\tilde{f}$ such that $\tilde{f} = 1 - f$ and the new sequent is still n-irreductible?

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Contents

1. Introduction

The present paper is motivated by the consequences (in number theory and in fundamental physics) of the definition of a n-irreductible number, the definition of a n-irreductible sequent, the (second order logic) n-irreductible axiom which states the existence of the largest n-irreductible number N_Z and the (second order logic) n-irreductible hypothesis on the “Nature's hypotheses”(the required hypotheses to explain the physical measurements). The first set of consequences are: the Goldbach's conjecture, the Polignac's conjecture, the Firoozbakht's conjecture, the Oppermann's conjecture, the Agoh-Giuga conjecture, the generalized Fermat's conjecture (which requires computational resources which are not reached today even for checking the simplest case: $a = 2$ and $b = 1$) and the Schinzel's hypothesis H (which requires computational resources which are not reached today for checking about $\left(\pi\left(N_Z\right)2^{N_Z}\right)^{N_Z}$ cases) are solved by the (second order logic) n-irreductible axiom. The second set of consequences are the “Nature's hypotheses”generated by the n-irreductible hypothesis on the “Nature's hypotheses”.

From researches in fundamental physics, the n-irreductible numbers and the n-irreductible sequents definitions arise from the intuition that the “Nature's hypotheses”do not carry natural numbers or "hidden natural numbers” except for 0 and 1, i.e. the “Nature's hypotheses”can be used in a n-irreductible sequent. Shortly, a sequent is n-irreductible iff the sequent is composed by some closed hypotheses and a n-irreductible formula (a close formula with one internal variable such that the formula is only true when we set that variable to the unique natural number n), and it does not exist some strict sub-sequent which are composed by some closed sub-hypotheses and some sub-m-irreductible formula with $m > 1$. Moreover, we postulate (at second order of logic) that the “Nature's hypotheses”are not chosen randomly: the “Nature's hypotheses”are the only hypotheses which give the largest n-irreductible number $N_Z < 10^{3.026\times10^7} \cong 2^{1.005\times10^8}$.

The paper is organized as follow: firstly, we present the notations used throughout this paper. Secondly, we define what is an explicit sub-formula in order to define what is a n-irreductible number and a n-irreductible sequent. Thirdly, we present some n-irreductible number examples. Fourthly, we present the Goldbach's conjecture, the Polignac's conjecture, the Firoozbakht's conjecture, the Oppermann's conjecture, the Agoh-Giuga conjecture, the generalized Fermat's conjecture and the Schinzel's hypothesis H as n-irreductible sequents. Fifthly, we present the (second order logic) n-irreductible axiom, the (second order logic) hypothesis on the “Nature's hypotheses”and their consequences. Sixthly, we ask ourselves two open questions about the n-irreductible numbers. Seventhly, we present some larger n-irreductible number examples. Eighthly, we present some n-irreductible number examples with the axioms of the real numbers and ninethly, the conclusion and the acknowledgment.

2. Notations

In the present paper:

1- In general, the notation $f(...)$ should be read ${}^{f}(...)^{f}$ when we look at the explicit sub-formulas of a formula ϕ.

2- We omit some parentheses and parenthesis labels to improve the readability but they are necessary for writing the related explicit formulas and explicit sub-formulas.

3- We use the formula $\phi \to \psi$ instead of the formula $\neg\phi \vee \psi$ to improve the readability. However the strict explicit sub-formulas of the formula $\phi \to \psi$ are ϕ, ψ, $\neg\phi$, and $\phi \vee \psi$ instead of only ϕ and ψ.

4- We use the formula $t_1 = t_2$ instead of the formula $\mathcal{R}_= (t_1, t_2)$ to improve the readability. However the strict explicit sub-formulas of the formula $t_1 = t_2$ are only t_1 and t_2 instead of t_1, t_2, $t_1 =$ and $= t_2$. In general, if define the function f_{abc}: f_a, f_b, f_c, f_{bc}, f_{ac} or f_{ab} can not be obtain form a sub-formulas of f_{abc}

5- If the formula ϕ is previously defined, the formula $\phi_{[y/x]}$ is a shortcut for the formula ϕ written with the variable y instead of the variable x with respect to the explicit sub-formulas of $\phi_{[y/x]}$.

3. Definitions

Let consider a language $\mathcal{L} \cup \mathcal{L}_{Peano}$ of first order logic which contains the language needed for the Peano hypotheses (except the recursion hypothesis).

Let introduce the necessary preliminary definitions and lemmas:

1- Preliminary definitions and lemmas about the explicit sub-formulas of a formula ϕ:

a- A formula ϕ containing l pair of parentheses is an explicit formula iff the i^{th} opening parenthesis and the corresponding i^{th} closing parenthesis are labeled unambiguously with respect to the other parentheses with an injection $f : \{1, ..., l\} \subset \mathbb{N} \longrightarrow \mathbb{N}$ such that: $\ldots \underset{f(i)}{\Big(} \ldots\ldots \underset{f(i)}{\Big)} \ldots$.

b- Preliminary lemma:

Every formula ϕ can be written as an explicit formula.

c- An explicit formula ψ is an explicit sub-formula of a formula ϕ iff the formula ψ is an explicit formula and ψ is a sub-sequence of the symbol sequence of the formula ϕ written as an explicit formula.

Remark: an explicit sub-formula ψ of a formula ϕ may contain a function symbol f of arity strictly smaller than the function symbol f in the formula ϕ. Roughly speaking, an explicit sub-formula can be written by removing the same number of argument for each function symbol f of the original formula.

d- Preliminary lemma about the explicit sub-formulas of a formula ϕ:

An explicit sub-formula of an explicit sub-formula of a formula ϕ is an explicit sub-formula of the formula ϕ.

2- Preliminary definition about the n-irreductible formulas:

A formula $\phi_{n-irreductible}$ is a n-irreductible formula iff $\phi_{n-irreductible}$ is a closed formula and a formula ϕ exists such that:

$$\phi_{n-irreductible} \equiv \phi_{[\underbrace{f_s(\ldots f_s(}_{n \text{ times}} c_0 \underbrace{)\ldots)}_{n \text{ times}} /x]} \wedge \exists! y \left(\phi_{[y/x]}\right) \ . \tag{3.1}$$

We rewrite the previous equation without the shortcut symbol $\exists!$:

$$\phi_{n-irreductible} \equiv \phi_{[\underbrace{f_s(\ldots f_s(}_{n \text{ times}} c_0 \underbrace{)\ldots)}_{n \text{ times}} /x]} \wedge \neg\exists y \exists y' \left(\neg y = y' \wedge \phi_{[y/x]} \wedge \phi_{[y'/x]}\right) \ . \tag{3.2}$$

The main definition: the following sequent ()

$$\Gamma \vdash \phi_{n-irreductible} \tag{3.3}$$

where $\vdash$ means it exists a model such that

the n-irreductible formula $\phi_{n-irreductible}$ is verified under the hypotheses Γ

is a n-irreductible sequent and n is a n-irreductible number iff:

1- the hypotheses Γ are closed formulas and the formula $\phi_{n-irreductible}$ is a n-irreductible formula (see the equation 3.1),

2- and for every closed and explicit sub-formula Δ of the hypotheses Γ and for every m-irreductible formula $\psi_{m-irreductible}$ where ψ is an explicit sub-formula of the formula ϕ, we have the following relation:

$$\Delta \vdash \psi_{m-irreductible} \text{ and } m = c_0 \tag{3.4}$$

or

$$\Delta \vdash \psi_{m-irreductible} \text{ and } m = f_s\left(c_0\right) \tag{3.5}$$

or

$$\Delta \equiv \Gamma \text{ and } \psi_{m-irreductible} \equiv \phi_{n-irreductible} \tag{3.6}$$

or it exists a model such that,

$$\Delta \nvdash \psi_{m-irreductible} \tag{3.7}$$

Important remark: for all n-irreductible numbers in the present article, we include the Peano hypotheses inside the n-irreductible hypotheses Γ or either the following Peano sub-hypotheses:

(3.8) $\forall x \exists y \left(x = f_s\left(y\right)\right)$

$\forall x_1 \ldots \forall x_n \left(\left(\phi_{[c_0/x_0,x_1,\ldots,x_n]} \wedge \forall x_0 \left(\phi_{[x_0,x_1,\ldots,x_n]} \rightarrow \phi_{[f_s(x_0),x_1,\ldots,x_n]}\right)\right) \rightarrow \left(\forall x_0 \phi_{[x,x_1,\ldots,x_n]}\right)\right)$

with $\phi = \tilde{\phi} \vee \neg x_0 < 0$, from the following Peano hypotheses:

(3.9)

$\forall x \left(\neg f_s\left(x\right) = c_0\right)$

$\forall x \exists y \left(\neg x = c_0 \rightarrow x = f_s\left(y\right)\right)$

$\forall x_1 \ldots \forall x_n \left(\left(\phi_{[c_0/x_0,x_1,\ldots x_n]} \wedge \forall x_0 \left(\phi_{[x_0,x,\ldots,x_n]} \rightarrow \phi_{[f_s(x_0),x_1,\ldots,x_n]}\right)\right) \rightarrow \left(\forall x_0 \phi_{[x_0,x_1,\ldots,x_n]}\right)\right) \ .$

If we use the Peano sub-hypotheses above, we use the integers rather than the natural numbers. The use of Peano hypotheses in a n-irreductible sequent with respect to the

Peano sub-hypotheses should be motivated by other definitions like the prime number function definition.

4. Some n-irreductible number examples

We give in this section some examples of n-irreductible numbers. Firstly, we write the preliminary formulas satisfied by the following function symbols:

0- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the ordering function symbol $\mathcal{R}_<$:

$$\begin{aligned} &\forall x \forall y \left(\exists z \left(x = f_+ \left(y, z\right) \wedge \neg z = c_0\right) \rightarrow \mathcal{R}_< \left(x, y\right) = True\right) \\ &\forall x \forall y \left(\neg\exists z \left(x = f_+ \left(y, z\right) \wedge \neg z = c_0\right) \rightarrow \mathcal{R}_< \left(x, y\right) = False\right) \end{aligned} \tag{4.1}$$

1- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the prime function symbol f_{Prime}:

$$\begin{aligned} &\forall x \left(\exists y \exists z \left(y < x \wedge z < x \wedge x = f_\times \left(y, z\right)\right) \rightarrow f_{Prime} \left(x\right) = c_0\right) \\ &\forall x \left(\neg\exists y \exists z \left(y < x \wedge z < x \wedge x = f_\times \left(y, z\right)\right) \rightarrow f_{Prime} \left(x\right) = f_s \left(c_0\right)\right) \ . \end{aligned} \tag{4.2}$$

Important remark: If we use the prime function f_{prime} in a n-irreducible sequent, we should use the Peano hypotheses (natural numbers) rather than the Peano sub-hypotheses (integers). Using the above prime function f_{prime} with the Peano sub-hypotheses implies that all strictly positive integers are prime numbers and all negative integers are not prime numbers.

2- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the sub-function symbol $g_{\sigma_1 - id}$ and the function symbol $f_{\sigma_1 - id}$ which gives the sum of proper divisors:

$$\begin{aligned} &\forall x \left(g_{\sigma_1 - id}(x, f_s\left(c_0\right)) = c_0\right) \\ &\forall x \forall y \left(\exists z \left(x = f_\times \left(y, z\right)\right) \rightarrow g_{\sigma_1 - id}\left(x, f_s\left(y\right)\right) = f_+ \left(g_{\sigma_1 - id}(x, y), y\right)\right) \\ &\forall x \forall y \left(\neg\exists z \left(x = f_\times \left(y, z\right)\right) \rightarrow g_{\sigma_1 - id}\left(x, f_s\left(y\right)\right) = g_{\sigma_1 - id}(x, y)\right) \\ &\forall x \left(f_{\sigma_1 - id}(x) = g_{\sigma_1 - id}(x, x)\right) \ . \end{aligned} \tag{4.3}$$

Trivially, 0 and 1 are n-irreductible numbers with the following formulas ϕ:

$$\phi \equiv x = c_0 \text{ and } \phi \equiv x = f_s\left(c_0\right) \ . \tag{4.4}$$

2 is a n-irreductible number with the following formula ϕ:

$$\phi \equiv x = f_+ \left(f_s\left(c_0\right), f_s\left(c_0\right)\right) \tag{4.5}$$

or for instance, the following formula ϕ:

$$\phi \equiv \forall y \left(y < x \rightarrow \left(y = c_0 \vee y = f_s\left(c_0\right)\right)\right) \wedge \neg\exists x' \left(x < x' \wedge \forall y \left(y < x' \rightarrow \left(y = c_0 \vee y = f_s\left(c_0\right)\right)\right)\right) \ . \tag{4.6}$$

If we would like to include the Peano hypotheses (except the recursion hypothesis) for the n-irreductible number 2, we should look at the following formula ϕ which requires the multiplication hypothesis:

$$\begin{aligned} \phi = &\exists y \exists z \left(x = f_\times \left(y, z\right) \wedge \left(f_s(c_0) < y \vee f_s(c_0) < z\right)\right) \wedge \\ &\neg\exists x' \left(x < x' \wedge \exists y \exists z \left(x' = f_\times \left(y, z\right) \wedge \left(f_s(c_0) < y \vee f_s(c_0) < z\right)\right)\right) \ . \end{aligned} \tag{4.7}$$

In order to prove that some other natural numbers are n-irreductible, we use the prime function f_{Prime} (see 4.2).

3 is a n-irreductible number with the following formula ϕ (see 4.2):

$$\phi \equiv \forall y \left(f_{Prime}\left(y\right) = c_0 \rightarrow x < y\right) \wedge \neg \exists x' \left(x < x' \wedge \forall y \left(f_{Prime}\left(y\right) = c_0 \rightarrow x' < y\right)\right) \ . \tag{4.8}$$

4 is a n-irreductible number with the following formula ϕ (see 4.2):

$$\phi \equiv f_{Prime}\left(x\right) = c_0 \wedge \neg \exists x' \left(x' < x \wedge f_{Prime}\left(x'\right) = c_0\right) \ . \tag{4.9}$$

In order to prove that some other natural numbers are also n-irreductible, we use the function $f_{\sigma_1 - id}$ (see 4.3):

6 is a n-irreductible number with the following formula ϕ (see 4.3):

$$\phi \equiv \neg x = c_0 \wedge f_{\sigma_1 - id}(x) = x \wedge \neg \exists x' \left(x' < x \wedge \neg x' = c_0 \wedge f_{\sigma_1 - id}(x') = x'\right) \ . \tag{4.10}$$

5. Conjectures which induce monster n-irreductible numbers if counterexamples exist

In the previous section, we introduced some n-irreductible numbers that are small and easy to find. In this section, we examine how some monster n-irreductible numbers can be produced if some conjectures are false. Firstly, we write the preliminary formulas satisfied by the following function symbols:

1- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the unary inverse function symbol f_s^{-1}:

$$\begin{aligned} &\forall x \left(\neg x = c_0 \rightarrow f_s\left(f_s^{-1}\left(x\right)\right) = x\right) \\ &\forall x \left(x = c_0 \rightarrow f_s^{-1}\left(x\right) = c_0\right) \ . \end{aligned} \tag{5.1}$$

2- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the subtraction function symbol f_-:

$$\begin{aligned} &\forall x \forall y \left(y < x \rightarrow f_+\left(f_-\left(x, y\right), y\right) = x\right) \\ &\forall x \forall y \left(\neg y < x \rightarrow f_-\left(x, y\right) = c_0\right) \ . \end{aligned} \tag{5.2}$$

3- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the twin prime function symbol f_{Twin} (see 4.2 and 5.1):

$$\begin{aligned} &\forall x \left(\left(f_{Prime}\left(f_s^{-1}\left(x\right)\right) = f_s\left(c_0\right) \wedge f_{Prime}\left(f_s\left(x\right)\right) = f_s\left(c_0\right)\right) \rightarrow f_{Twin}\left(x\right) = f_s\left(c_0\right)\right) \\ &\forall x \left(\neg \left(f_{Prime}\left(f_s^{-1}\left(x\right)\right) = f_s\left(c_0\right) \wedge f_{Prime}\left(f_s\left(x\right)\right) = f_s\left(c_0\right)\right) \rightarrow f_{Twin}\left(x\right) = c_0\right) \ . \end{aligned} \tag{5.3}$$

4- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the ceiling prime function symbol f_{CP} (see 4.2):

$$\forall x \exists y \left(f_{CP}\left(x\right) = y \wedge f_{Prime}\left(y\right) = f_s\left(c_0\right) \wedge \neg \exists z \left(x < z \wedge z < y \wedge f_{Prime}\left(z\right) = f_s\left(c_0\right)\right)\right) \ . \tag{5.4}$$

5- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the function symbol $f_{p-Prime}$ which give the n^{th} prime number (see 4.2)

$$\begin{aligned} & f_{p-Prime}\left(f_s\left(c_0\right)\right)=c_0 \\ & \forall x\left(f_{p-Prime}\left(f_s\left(x\right)\right)=f_{CP}\left(f_{p-Prime}\left(x\right)\right)\right) \ . \end{aligned} \tag{5.5}$$

6- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the coprime function symbol $f_{Coprime}$:

$$\begin{aligned} & \forall x \forall x'\left(\exists y \exists z \exists z'\left(\neg y=f_s\left(c_0\right) \wedge x=f_\times\left(y, z\right) \wedge x'=f_\times\left(y, z'\right)\right) \rightarrow f_{Coprime}\left(x, x'\right)=c_0\right) \\ & \forall x \forall x'\left(\neg \exists y \exists z \exists z'\left(\neg y=f_s\left(c_0\right) \wedge x=f_\times\left(y, z\right) \wedge x'=f_\times\left(y, z'\right)\right) \rightarrow f_{Coprime}\left(x, x'\right)=f_s\left(c_0\right)\right) \ . \end{aligned} \tag{5.6}$$

7- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the power function symbol $f_\wedge$:

$$\begin{aligned} & \forall x \forall y\left(y=c_0 \rightarrow f_\wedge\left(x, y\right)=f_s\left(c_0\right)\right) \\ & \forall x \forall y\left(\neg y=c_0 \rightarrow f_\wedge\left(x, f_s\left(y\right)\right)=f_\times\left(f_\wedge\left(x, y\right), x\right)\right) \ . \end{aligned} \tag{5.7}$$

5.1. *Goldbach's conjecture.* If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the sub-function symbol $g_{Goldbach-1}$ and the function symbol $f_{Goldbach-1}$ which gives the minimal number of prime numbers necessary to express a natural number as a sum of prime number minus one (see 4.2 and 5.2):

$$\begin{aligned} & \forall x \forall y\left(f_{Prime}\left(y\right)=f_s\left(c_0\right) \rightarrow g_{Goldbach-1}(x, y)=c_0\right) \\ & \forall x \forall y\Bigg(f_{Prime}\left(y\right)=c_0 \rightarrow \exists z \\ & \left(z<y \wedge f_{Prime}\left(z\right)=f_s\left(c_0\right) \wedge g_{Goldbach-1}(x, y)=f_s\left(g_{Goldbach-1}(x, f_-\left(y, z\right))\right) \wedge \neg \exists z'\right. \\ & \left.\left(z'<y \wedge f_{Prime}\left(z'\right)=f_s\left(c_0\right) \wedge g_{Goldbach-1}(x, f_-\left(y, z'\right))<g_{Goldbach-1}(x, f_-\left(y, z\right))\right)\right)\Bigg) \\ & \forall x\left(f_{Goldbach-1}(x)=g_{Goldbach-1}(x, x)\right) \ . \end{aligned} \tag{5.8}$$

If a first counterexample m_Z exists for the Goldbach's conjecture [Hel13], we can show that m_Z is a n-irreductible number with the following formula ϕ (see the previous equation):

$$\begin{aligned} \phi \equiv & \neg x=c_0 \wedge \neg x=f_s\left(c_0\right) \wedge \forall y\left(y<x \rightarrow f_{Goldbach-1}(y)<f_{Goldbach-1}(x)\right) \wedge \\ & \forall y\left(x<y \rightarrow \neg f_{Goldbach-1}(x)<f_{Goldbach-1}(y)\right) \ . \end{aligned} \tag{5.9}$$

5.2. *Polignac's conjecture.* If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the Polignac function symbol $f_{Polignac}$ which gives the difference between the two next prime numbers of a natural number (see 5.2 and 5.4):

$$\forall x\left(f_{Polignac}\left(x\right)=f_-\left(f_{CP}\left(f_{CP}\left(x\right)\right), f_{CP}\left(x\right)\right)\right) \ . \tag{5.10}$$

If a first counterexample m_Z exists for the Polignac's conjecture [dP51], we can show that m_Z is a n-irreductible number with following formula ϕ (see the previous equation):

$$\begin{aligned} \phi \equiv & \neg x = c_0 \wedge \neg x = f_s(c_0) \wedge \\ & \exists y \left(f_{Polignac}(x) = y \wedge \neg\exists z \left(x < z \wedge f_{Polignac}(x) = f_{Polignac}(z)\right)\right) \wedge \\ & \neg\exists x' \left(x' < x \wedge f_{Polignac}(x') = y \wedge \neg\exists z \left(x' < z \wedge f_{Polignac}(x') = f_{Polignac}(z)\right)\right) \ . \end{aligned} \tag{5.11}$$

Since the set of prime numbers is infinite, the following explicit sub-formula will not work (see 5.2 and 5.4):

$$\forall x \left(f_{Polignac}(x) = f_{-}\left(f_{CP}(x), x\right)\right) \ . \tag{5.12}$$

5.3. *Firoozbakht's conjecture.* If a first counterexample m_Z exists for the Firoozbakht's conjecture [20004], we can show that m_Z is a n-irreductible number with the following formula ϕ (see 5.5):

$$\begin{aligned} \phi \equiv & \neg x = c_0 \wedge \neg f_{\wedge}\left(f_{p-Prime}(f_s(x)), x\right) < f_{\wedge}\left(f_{p-Prime}(x), f_s(x)\right) \wedge \\ & \forall x' \left(x' < x \rightarrow f_{\wedge}\left(f_{p-Prime}\left(f_s(x')\right), x'\right) < f_{\wedge}\left(f_{p-Prime}(x'), f_s(x')\right)\right) \ . \end{aligned} \tag{5.13}$$

5.4. *Oppermann's conjecture.* We define the first variant of the Oppermann's conjecture [vsOFS83]:

For all natural numbers x such that $x > 1$, there is at least one prime number between $x(x-1)$ and x^2.

If a first counterexample m_Z exists for the Oppermann's conjecture [vsOFS83], we can show that m_Z is a n-irreductible number with the following formula ϕ (see 4.2 and 5.1):

$$\begin{aligned} \phi \equiv & \neg x = c_0 \wedge \neg x = f_s(c_0) \wedge \neg\exists y \left(f_{\times}\left(x, f_s^{-1}(x)\right) < y \wedge y < f_{\times}(x, x) \wedge f_{Prime}(y) = f_s(c_0)\right) \\ & \wedge \neg\exists x' \neg\exists y \left(x' < x \wedge f_{\times}\left(x', f_s^{-1}(x')\right) < y \wedge y < f_{\times}(x', x') \wedge f_{Prime}(y) = f_s(c_0)\right) \ . \end{aligned} \tag{5.14}$$

If a first counterexample m_Z exists for the Oppermann's conjecture [vsOFS83] and the first variant of the Oppermann's conjecture [vsOFS83] is true, we can show that m_Z is a n-irreductible number with the following formula ϕ (see 4.2 and 5.1):

$$\begin{aligned} & \phi \equiv \neg x = c_0 \wedge \neg x = f_s(c_0) \wedge \\ & \neg\exists y \exists y' \left(f_{\times}\left(x, f_s^{-1}(x)\right) < y \wedge y < f_{\times}(x, x) \wedge f_{\times}(x, x) < y' \wedge y' < f_{\times}(x, f_s(x)) \wedge \right. \\ & \left. f_{Prime}(y) = f_s(c_0)\right) \wedge \neg\exists x' \neg\exists y \exists y' \\ & \left(x' < x \wedge f_{\times}\left(x', f_s^{-1}(x')\right) < y \wedge y < f_{\times}(x', x') \wedge f_{\times}(x', x') < y' \wedge y' < f_{\times}\left(x', f_s(x')\right) \wedge \right. \\ & \left. f_{Prime}(y) = f_s(c_0)\right)\Big) \ . \end{aligned} \tag{5.15}$$

5.5. *Agoh-Giuga conjecture.* If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the Giuga sub-function symbol g_{Giuga} and the

Giuga function symbol f_{Giuga}(see 4.2, 5.1 and 5.7):

$$
\begin{aligned}
&\forall x \left(g_{Giuga}\left(x, c_0\right) = f_s\left(c_0\right)\right)\\
&\forall x \forall y \left(g_{Giuga}\left(x, f_s\left(y\right)\right) = f_+\left(g_{Giuga}\left(x, y\right), f_\wedge\left(y, f_s^{-1}\left(x\right)\right)\right)\right)\\
&\forall x \left(\left(f_{Prime}\left(x\right) = f_s\left(c_0\right) \rightarrow \exists y \left(f_s\left(g_{Giuga}\left(x, x\right)\right) = f_\times\left(x, y\right)\right)\right) \rightarrow f_{Giuga} = f_s\left(c_0\right) \right)\\
&\forall x \left(\neg \left(f_{Prime}\left(x\right) = f_s\left(c_0\right) \rightarrow \exists y \left(f_s\left(g_{Giuga}\left(x, x\right)\right) = f_\times\left(x, y\right)\right)\right) \rightarrow f_{Giuga} = c_0 \right) .
\end{aligned}
\tag{5.16}
$$

If m_Z is the last natural number where the Agoh-Giuga conjecture [Giu51] is true, we can show that m_Z is a n-irreductible number with the following formula ϕ (see the previous equation):

$$
\phi \equiv f_{Giuga}\left(x\right) = c_0 \wedge \neg \exists x' \left(x' < x \wedge f_{Giuga}\left(x'\right) = c_0\right) . \tag{5.17}
$$

5.6. *Generalized Fermat's conjecture.* We define the generalized Fermat's conjecture [Rie11]:

Let be some natural number a and c, there is an infinite number of natural numbers b such that $a^b + c^b$ is a prime number.

If m_Z is the last number where the generalized Fermat's conjecture [Rie11] for some fixed natural number a and c is true and every explicit sub-formulas which are equivalent to the Generalized Fermat's conjecture [Rie11] with the fixed natural number a' and c' are true , we can show that m_Z is a monster n-irreductible number with the following formula ϕ (see 4.2 and 5.7):

$$
\begin{aligned}
\phi \equiv\, & f_{Prime}\left(f_+\left(f_\wedge\left(n_a, x\right), f_\wedge\left(n_c, x\right)\right)\right) = f_s\left(c_0\right) \wedge\\
& \neg \exists x' \left(x < x' \wedge f_{Prime}\left(f_+\left(f_\wedge\left(n_a, x'\right), f_\wedge\left(n_c, x'\right)\right)\right) = f_s\left(c_0\right)\right) \wedge\\
& \neg \exists x'' \Big(x'' < x \wedge f_{Prime}\left(f_+\left(f_\wedge\left(n_a, x''\right), f_\wedge\left(n_c, x''\right)\right)\right) = f_s\left(c_0\right) \wedge\\
& \neg \exists x' \left(x'' < x' \wedge f_{Prime}\left(f_+\left(f_\wedge\left(n_a, x''\right), f_\wedge\left(n_c, x''\right)\right)\right) = f_s\left(c_0\right)\right) \Big) ,
\end{aligned}
\tag{5.18}
$$

where $n_a = \underbrace{f_s(...f_s(}_{a \text{ times}} c_0 \underbrace{)...)}_{a \text{ times}}$ and $n_c = \underbrace{f_s(...f_s(}_{c \text{ times}} c_0 \underbrace{)...)}_{c \text{ times}}$.

If we can show that the generalized Fermat's conjecture is true for many fixed natural numbers a and c, we can show that m_Z is a n-irreductible number with the following formula

ϕ (see 4.2 and 5.7):

$$\begin{aligned}\phi \equiv \exists y \Bigg(& f_{Prime}\left(f_{+}\left(f_{\wedge}\left(n_a, y\right), f_{\wedge}\left(x, y\right)\right)\right) = f_s\left(c_0\right) \wedge \\ & \neg\exists x'\left(x < x' \wedge f_{Prime}\left(f_{+}\left(f_{\wedge}\left(n_a, y\right), f_{\wedge}\left(x', y\right)\right)\right) = f_s\left(c_0\right)\right)\Bigg) \wedge \\ & \neg\exists x''\exists y\Bigg(x'' < x \wedge f_{Prime}\left(f_{+}\left(f_{\wedge}\left(n_a, y\right), f_{\wedge}\left(x'', y\right)\right)\right) = f_s\left(c_0\right) \wedge \\ & \neg\exists x'\left(x'' < x' \wedge f_{Prime}\left(f_{+}\left(f_{\wedge}\left(n_a, y\right), f_{\wedge}\left(x', y\right)\right)\right) = f_s\left(c_0\right)\right)\Bigg) . \end{aligned} \tag{5.19}$$

If we can show that the generalized Fermat's conjecture is true for many fixed natural numbers a, we can show that m_Z is a n-irreductible number with the following formula ϕ (see 4.2 and 5.7):

$$\begin{aligned}\phi \equiv \exists y\exists z \Bigg(& f_{Prime}\left(f_{+}\left(f_{\wedge}\left(x, y\right), f_{\wedge}\left(z, y\right)\right)\right) = f_s\left(c_0\right) \wedge \\ & \neg\exists x'\left(x < x' \wedge f_{Prime}\left(f_{+}\left(f_{\wedge}\left(x', y\right), f_{\wedge}\left(z, y\right)\right)\right) = f_s\left(c_0\right)\right)\Bigg) \wedge \\ & \neg\exists x''\exists y\exists z\Bigg(x'' < x \wedge f_{Prime}\left(f_{+}\left(f_{\wedge}\left(x'', y\right), f_{\wedge}\left(z, y\right)\right)\right) = f_s\left(c_0\right) \wedge \\ & \neg\exists x'\left(x'' < x' \wedge f_{Prime}\left(f_{+}\left(f_{\wedge}\left(x', y\right), f_{\wedge}\left(z, y\right)\right)\right) = f_s\left(c_0\right)\right)\Bigg) . \end{aligned} \tag{5.20}$$

5.7. *Schinzel's hypothesis H.* If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the r polynomials function symbol $f_{i,Schinzel}$ of maximal degree d (see 5.7):

$$\begin{aligned} & \forall x\,(\\ & f_{1,Schinzel}\left(x\right) = f_{+}\left(f_{+}\left(...f_{+}\left(f_{\times}\left(a_{10}, f_{\wedge}\left(x, b_0\right)\right), f_{\times}\left(a_{11}, f_{\wedge}\left(x, b_1\right)\right)\right)...\right), f_{\times}\left(a_{1d}, f_{\wedge}\left(x, b_d\right)\right)\right) \\ & \qquad \vdots \\ & f_{r,Schinzel}\left(x\right) = f_{+}\left(f_{+}\left(...f_{+}\left(f_{\times}\left(a_{r0}, f_{\wedge}\left(x, b_0\right)\right), f_{\times}\left(a_{r1}, f_{\wedge}\left(x, b_1\right)\right)\right)...\right), f_{\times}\left(a_{rd}, f_{\wedge}\left(x, b_d\right)\right)\right) \\ &) \end{aligned} \tag{5.21}$$

where $a_{ij} = \underbrace{f_s(...f_s(}_{a(i,j)\text{ times}}\ c_0\ \underbrace{)...)}_{a(i,j)\text{ times}}$ and $b_i = \underbrace{f_s(...f_s(}_{i\text{ times}}c_0\ \underbrace{)...)}_{i\text{ times}}$.

Since the r polynomials $f_{i,Schinzel}$ are irreducible, the polynomial coefficients a_{ij} satisfy the first following constraint (see the previous equation):

$$\left(\nexists x f_{1,Schinzel}\left(x\right) = c_0\right) \wedge ... \wedge \left(\nexists x f_{r,Schinzel}\left(x\right) = c_0\right) . \tag{5.22}$$

Since the product of the r polynomials $f_{i,Schinzel}$ has not a fixed prime divisor, the polynomial coefficients a_{ij} satisfy the second following constraint (see 5.22):

$$\nexists x \forall y \exists z \Bigg(f_{Prime}\left(x\right) = f_s\left(c_0\right) \wedge$$

$$f_\times\left(f_\times\left(...f_\times\left(f_{1,Schinzel}\left(y\right), f_{2,Schinzel}\left(y\right)\right)...\right), f_{r,Schinzel}\left(y\right)\right) = f_\times\left(x, z\right) \Bigg) . \tag{5.23}$$

If m_Z is the last number where the Schinzel's hypothesis H [Guy04] for some fixed polynomial is true and every explicit sub-formulas which are equivalent to the Schinzel's hypothesis H [Guy04] for some fixed polynomials are true, we can show that m_Z is a monster n-irreductible number with the following formula ϕ (see 4.2 and 5.22):

$$\begin{aligned} \phi \equiv & f_{Prime}\left(f_{1,Schinzel}\left(x\right)\right) = f_s\left(c_0\right) \wedge ... \wedge f_{Prime}\left(f_{r,Schinzel}\left(x\right)\right) = f_s\left(c_0\right) \wedge \\ & \nexists x' \left(x < x' \wedge f_{Prime}\left(f_{1,Schinzel}\left(x'\right)\right) = f_s\left(c_0\right) \wedge ... \wedge f_{Prime}\left(f_{r,Schinzel}\left(x'\right)\right) = f_s\left(c_0\right)\right) . \end{aligned} \tag{5.24}$$

We build new formulas for new monster n-irreductible numbers like in the generalized Fermat's conjecture:

If we can show that the Schinzel's hypothesis H [Guy04] is true for a fixed number of polynomial r, a fixed maximal degree d, many fixed polynomial coefficients a_{ij} and some running polynomial coefficients a_{ij}, we can show that m_Z is a monster n-irreductible number with a formula ϕ.

We build new formulas for new monster n-irreductible numbers like in the generalized Fermat's conjecture:

If we can show that the Schinzel's hypothesis H [Guy04] is true for a fixed number of polynomial r, many maximal degrees d and the running polynomial coefficients a_{ij}, we can show that m_Z is a monster n-irreductible number with a formula ϕ.

Finally, we build new formulas for new monster n-irreductible numbers like in the generalized Fermat's conjecture:

If we can show that the Schinzel's hypothesis H [Guy04] is true for many numbers of polynomial r, the running maximal degree d and the running polynomial coefficients a_{ij}, we can show that m_Z is a monster n-irreductible number with a formula ϕ.

6. The second order logic n-irreductible axiom and the second order logic hypothesis on the "Nature's hypotheses"

We introduce one important axiom on n-irreductible numbers and one important hypothesis on the "Nature's hypotheses"at second order logic for both of them:

The (second order logic) n-irreductible axiom:

Any n-irreductible number is smaller or equal to:

$$N_Z < 10^{3.026 \times 10^7} \cong 2^{1.005 \times 10^8} \tag{6.1}$$

. The (second order logic) hypothesis on the "Nature's hypotheses":

The hypotheses of any N_Z-irreductible sequent are the “Nature’s hypotheses”which explain the physical measurements.

Some consequences:

1- The physical measurements confirm but do not prove that the “Nature’s hypotheses”, the mathematical explorations over the n-irreductible numbers confirm but do not prove that N_Z is the largest n-irreductible number and they do not prove but confirm that the hypotheses of any N_Z-irreductible sequent are the “Nature’s hypotheses”which explain the physical measurements.

2- The Goldbach’s conjecture, the Polignac’s conjecture, the Firoozbakht conjecture’s, the Oppermann’s conjecture, the Agoh-Giuga conjecture can be checked with quantum computers with $2^{1.005\times10^8}$ qubits or less since the computation can be fully parallel [LS14], the generalized Fermat’s conjecture (which requires computational resources which are far from what we can imagine technically even for the simplest case: $a = 2$ and $b = 1$) and the Schinzel’s hypothesis H (which requires monster computational resources for checking about $\left(\pi\left(N_Z\right)2^{N_Z}\right)^{N_Z}$ cases). . Moreover, 27 is a n-irreductible number if the Goldbach’s conjecture is true.

3- A paper is under preparation in order to present the theory of everything where its hypotheses are the hypotheses of a N_Z-irreductible sequent (N_Z would be the number of Lagrangian terms but experimentally, we can only access to a very small fraction of that terms) and to show that any obvious variant of the theory of everything requires some hypotheses which give a n-irreductible number strictly smaller than N_Z.

7. Some open questions about the n-irreductible sequents and the n-irreductible numbers

1- Can we show that a natural number n is not n-irreductible? The difficulty is to prove that there is no n-irreductible sequent among an infinite set of possible sequents which give the n-irreductible number n.

2- If a n-irreductible number n is found with the help of a function symbol f where its output values are only 0 and 1, can we replace the function symbol f by a function symbol $\tilde{f}$ such that $\tilde{f} = 1 - f$ and the new sequent is still n-irreductible?

3- In quantum field field theories with gauge fields, the number of space-time dimensions should be larger or equal to 4 in order to have a ”renormalizable” theory. Therefore the Nature’s hypotheses are n-irreducible only if there are 4 space-time coordinates. The Nature’s hypotheses with 5 space-time coordinates are not n-irreducible and one specific spatial coordinate may be skipped at the multiple places where it is written in the theory. From that example, we conclude that every sub-formulas should be considered when we study some n-irreducible sequent. It confirms that the required number of sub-sequents to explore in order check that a sequent is a n-irreducible sequent is roughly exponential to its number of symbols in the general case. However, for a specific n-irreducible sequent, some specific mathematical tools may be developed in order to explore a set of sub-sequents in one time.

8. Extra: some larger n-irreductible number examples

In this section, with the help of the formulas satisfied by the symbol function f_{Prime} joined to the hypotheses of some n-irreductible sequents, we try to reach the closest n-irreductible number (1024 in the present section) to the largest one $N_Z < 10^{3.026\times 10^7} \cong 2^{1.005\times 10^8}$. Firstly, we write the preliminary formulas satisfied by the following function symbols:

1- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the sub-function symbol $g_{\sigma_0 - 1}$ and the function symbol $f_{\sigma_0 - 1}$ which gives the number of proper divisor of a natural number:

$$\begin{aligned}
&\forall x\,(g_{\sigma_0-1}(x, f_s\,(c_0)) = c_0)\\
&\forall x\forall y\,(\exists z\,(x = f_\times\,(y,z)) \to g_{\sigma_0-1}\,(x, f_s\,(y)) = f_s\,(g_{\sigma_0-1}(x,y)))\\
&\forall x\forall y\,(\neg\exists z\,(x = f_\times\,(y,z)) \to g_{\sigma_0-1}\,(x, f_s\,(y)) = g_{\sigma_0-1}(x,y))\\
&\forall x\,(f_{\sigma_0-1}(x) = g_{\sigma_0-1}(x,x))\ .
\end{aligned}\tag{8.1}$$

2- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the highly composite function symbol f_{HC} (see the previous equation):

$$\begin{aligned}
&\forall x\,(\forall y\,((\neg y = c_0 \wedge y < x) \to f_{\sigma_0-1}\,(y) < f_{\sigma_0-1}\,(x)) \to f_{HC}(x) = f_s\,(c_0))\\
&\forall x\,(\neg\,(\forall y\,(\neg y = c_0 \wedge y < x) \to f_{\sigma_0-1}\,(y) < f_{\sigma_0-1}\,(x)) \to f_{HC}(x) = c_0)\ .
\end{aligned}\tag{8.2}$$

3- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the Euler's totient sub-function symbol g_Φ and the Euler's totient function symbol f_Φ which gives the number of coprime numbers below it (see 5.6):

$$\begin{aligned}
&\forall x\,(g_\Phi(x, f_s\,(c_0)) = c_0)\\
&\forall x\forall y\,(f_{Coprime}\,(x,y) = f_s\,(c_0) \to g_\Phi\,(x, f_s\,(y)) = f_s\,(g_\Phi(x,y)))\\
&\forall x\forall y\,(f_{Coprime}\,(x,y) = c_0 \to g_\Phi\,(x, f_s\,(y)) = g_\Phi(x,y))\\
&\forall x\,(f_\Phi(x) = g_\Phi(x,x))
\end{aligned}\tag{8.3}$$

4- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the highly coprime function symbol f_{HCP}(see the previous equation):

$$\begin{aligned}
&\forall x\,(\forall y\,((\neg y = c_0 \wedge y < x) \to f_\Phi\,(y) < f_\Phi\,(x)) \to f_{HCP}(x) = f_s\,(c_0))\\
&\forall x\,(\neg\forall y\,((\neg y = c_0 \wedge y < x) \to f_\Phi\,(y) < f_\Phi\,(x)) \to f_{HCP}(x) = c_0)\ .
\end{aligned}\tag{8.4}$$

5- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the sub-function symbol g_{σ_1-1} and the function symbol f_{σ_1-1} which gives the sum of divisors minus one of a natural number (see 5.1):

$$\begin{aligned}
&\forall x\,(g_{\sigma_1-1}(x, f_s\,(c_0)) = c_0)\\
&\forall x\forall y\,\big((\neg y = f_s\,(c_0) \wedge \exists z\,(x = f_\times\,(y,z))) \to g_{\sigma_1-1}\,(x,y) = f_+\,\big(g_{\sigma_1-1}(x, f_s^{-1}\,(y)), y\big)\big)\\
&\forall x\forall y\,\big(\neg\,(\neg y = f_s\,(c_0) \wedge \exists z\,(x = f_\times\,(y,z))) \to g_{\sigma_1-1}\,(x,y) = g_{\sigma_1-1}(x, f_s^{-1}\,(y))\big)\\
&\forall x\,(f_{\sigma_1-1}(x) = g_{\sigma_1-1}(x,x))\ .
\end{aligned}\tag{8.5}$$

With the concept of complement, we can show that 10 is a n-irreductible number with the following formula ϕ (see 8.2 and 8.4):

(8.6)
$$\phi \equiv f_{HC}(x) = c_0 \wedge f_{HCP}(x) = f_s(c_0) \wedge \neg\exists y\,(y < x \wedge f_{HC}(x) = c_0 \wedge f_{HCP}(x) = f_s(c_0))$$

and we can show that 24 is a n-irreductible number with the following formula ϕ (see 8.2 and 8.4):

(8.7)
$$\phi \equiv f_{HC}(x) = f_s(c_0) \wedge f_{HCP}(x) = c_0 \wedge \neg\exists y\,(y < x \wedge f_{HC}(x) = f_s(c_0) \wedge f_{HCP}(x) = c_0)\ .$$

In order to find much larger n-irreductible number, we use the concept of amicable numbers:

1- 220 is a n-irreductible number with the following formula ϕ (see 8.5):

(8.8)
$$\phi \equiv \exists y\,(x < y \wedge f_{\sigma_1 - 1}(x) = f_{\sigma_1 - 1}(y)) \wedge \forall z\,(z < x \to \neg\exists y\,(z < y \wedge f_{\sigma_1 - 1}(z) = f_{\sigma_1 - 1}(y)))\ .$$

2- 284 is a n-irreductible number with the following formula ϕ (see 8.5):

(8.9)
$$\phi \equiv \exists y\,(y < x \wedge f_{\sigma_1 - 1}(x) = f_{\sigma_1 - 1}(y)) \wedge \forall z\,(z < x \to \neg\exists y\,(y < z \wedge f_{\sigma_1 - 1}(z) = f_{\sigma_1 - 1}(y)))\ .$$

3- 503 is a n-irreductible number with the following formula ϕ (see 8.5):

$\phi \equiv \exists y \exists z$

(8.10)
$$(x = f_{\sigma_1 - 1}(y) \wedge x = f_{\sigma_1 - 1}(z)) \wedge \forall w\,(w < x \to \neg\exists y \exists z\,(w = f_{\sigma_1 - 1}(y) \wedge w = f_{\sigma_1 - 1}(z)))\ .$$

9. Extra bis: some n-irreductible number examples with the axioms of the real numbers

In this section, with the help of the formulas satisfied by the axioms of the real numbers joined to the hypotheses of some n-irreductible sequents, we try to reach the closest n-irreductible number (1024 in the present section) to the largest one $N_Z < 10^{3.026 \times 10^7} \cong 2^{1.005 \times 10^8}$. Firstly, we write the preliminary formulas satisfied by the following function symbols:

1- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the natural number function symbol $f_{\mathbb{N}}$:

$$f_{\mathbb{N}}(c_0) = f_s(c_0)$$
$$\forall x\,(f_{\mathbb{N}}(x) = f_s(c_0) \to f_{\mathbb{N}}(f_s(x)) = f_s(c_0))$$
$$\forall x\,(f_{\mathbb{N}}(x) = c_0 \to f_{\mathbb{N}}(f_s(x)) = c_0)\ . \tag{9.1}$$

2- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the integer part function symbol f_{IP} (see the previous equation):

$$\forall x\,(\exists n\,(f_{\mathbb{N}}(n) = f_s(c_0) \wedge \neg n < x \wedge x < f_s(n) \wedge f_{IP}(x) = n))\ . \tag{9.2}$$

3- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the ceiling function symbol $f_{Ceiling}$ (see 9.1):

$$\forall x\,(\exists n\,(f_{\mathbb{N}}(n) = f_s(c_0) \wedge x < n \wedge \neg f_s(n) < x \wedge f_{Ceiling}(x) = n))\ . \tag{9.3}$$

4- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the negative function symbol f_-

$$\begin{aligned} &\forall x \exists y \left(f_+\left(x,y\right) = c_0 \wedge f_-\left(x\right) = y\right) \\ &\forall x \neg \exists y \exists y' \left(\neg y = y' \wedge f_+\left(x,y\right) = c_0 \wedge f_+\left(x,y'\right) = c_0\right) \ . \end{aligned} \tag{9.4}$$

5 If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the factorial function symbol $f_!$ (see 5.1 and 9.1):

$$\begin{aligned} &\forall n \left(f_{\mathbb{N}}\left(n\right) = f_s\left(c_0\right) \wedge n = c_0 \rightarrow f_!\left(n\right) = f_s\left(c_0\right)\right) \\ &\forall n \left(f_{\mathbb{N}}\left(n\right) = f_s\left(c_0\right) \wedge \neg n = c_0 \rightarrow f_!\left(n\right) = f_\times\left(n, f_!\left(f_s^{-1}\left(n\right)\right)\right) \right) \ . \end{aligned} \tag{9.5}$$

6 If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the exponential series function symbol g_{Exp} (see 5.1 and 5.7):

$$\begin{aligned} &\forall x \left(g_{Exp}\left(x,c_0\right) = c_0\right) \\ &\forall x \forall y \left(g_{Exp}\left(x, f_s\left(y\right)\right) = f_+\left(g_{Exp}\left(x,y\right), f_\times\left(f_\wedge\left(x,y\right), f_{-1}\left(f_!\left(y\right)\right)\right) \right) \right) \ . \end{aligned} \tag{9.6}$$

7 If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the exponential function symbol f_{Exp} (see 9.1 and the previous equation):

$$\begin{aligned} &\forall \epsilon \exists N \forall n \Big(\left(0 < \epsilon \wedge N < n \wedge f_{\mathbb{N}}\left(N\right) = f_s\left(c_0\right) \wedge f_{\mathbb{N}}\left(n\right) = f_s\left(c_0\right)\right) \\ &\qquad \rightarrow \left(g_{Exp}\left(x,n\right) < f_{Exp}\left(x\right) \wedge f_{Exp}\left(x\right) < f_+\left(\epsilon, g_{Exp}\left(x,n\right)\right)\right) \Big) \ . \end{aligned} \tag{9.7}$$

We suppose we can define the Lebesgue integral or the Riemann integral irreducibly in order to define the function $f_{\sqrt{\pi}/2}$ in a n-irreductible form (see 5.7, 9.1, 9.4 and 9.7):

$$\forall n \left(f_{\mathbb{N}}\left(n\right) = f_s\left(c_0\right) \wedge \neg n = c_0 \rightarrow f_{\sqrt{\pi}/2}\left(n\right) = \int_0^\infty f_{Exp}\left(f_-\left(f_\wedge\left(x,n\right)\right)\right) dx \right) \ . \tag{9.8}$$

We can define the real number $\sqrt{\pi}/2$ irreducibly with the following formula ϕ:

$$\phi \equiv \exists n \left(f_{\mathbb{N}}\left(n\right) = f_s\left(c_0\right) \wedge x = f_{\sqrt{\pi}/2}\left(n\right)\right) \wedge \neg \exists n \left(f_{\mathbb{N}}\left(n\right) = f_s\left(c_0\right) \wedge f_{\sqrt{\pi}/2}\left(n\right) < x\right) \ . \tag{9.9}$$

Sketch to prove that 5 and 7 are n-irreductible numbers: The $(n-1)$-sphere of radius R and center $\vec{r}$ can be defined irreducibly by imposing a maximum volume for a fixed surface in $\mathbb{R}^n$ or a minimum surface for a fixed volume. By defining a n-irreductible n-cube with vertex coordinates $\underbrace{(\pm 1, ..., \pm 1)}_{n \text{ times}}$ and taking the biggest $(n-1)$-sphere inside it, we can find the n which maximize the volume $V(n)$ or the surface $S(n)$ of the $(n-1)$-sphere: 5 or 7.

Therefore, we can also define irreducibly the real numbers $16\pi^3/15$ and $8\pi^2/15$ with the following formula ϕ:

$$\phi \equiv \exists n \left(f_{\mathbb{N}}\left(n\right) = f_s\left(c_0\right) \wedge x = S\left(n\right)\right) \wedge \neg \exists n' \left(f_{\mathbb{N}}\left(n'\right) = f_s\left(c_0\right) \wedge x < S(n')\right) \ , \tag{9.10}$$

and the following formula ϕ

$$\phi \equiv \exists n \left(f_{\mathbb{N}}(n) = f_s(c_0) \rightarrow x = V(n) \right) \wedge \neg \exists n' \left(f_{\mathbb{N}}(n') = f_s(c_0) \wedge x < V(n') \right) \ . \tag{9.11}$$

We can also define irreducibly the real number e with the following formula ϕ:

$$\phi \equiv x = f_{Exp}\left(f_s(c_0) \right) \ . \tag{9.12}$$

For some n-irreductible real numbers x and x', we can show that n is a n-irreductible number with the following formula ϕ:

$$\begin{gathered} \phi \equiv \exists m \exists p \exists q \left(f_{\mathbb{N}}(n) = f_s(c_0) \wedge f_{\mathbb{N}}(m) = f_s(c_0) \wedge f_{\mathbb{N}}(p) = f_s(c_0) \wedge f_{\mathbb{N}}(q) = f_s(c_0) \wedge \right. \\ \left. f_{\times}\left(p, f_{\wedge}(x, n)\right) = f_{\times}\left(q, f_{\wedge}\left(x', m\right)\right) \wedge f_{Coprime}(p, q) = f_s(c_0) \wedge f_{Coprime}(m, n) = f_s(c_0) \right) \ . \end{gathered} \tag{9.13}$$

For some n-irreductible real numbers x and x', we can show that p is a n-irreductible number with the following formula ϕ:

$$\begin{gathered} \phi \equiv \exists n \exists m \exists q \left(f_{\mathbb{N}}(n) = f_s(c_0) \wedge f_{\mathbb{N}}(m) = f_s(c_0) \wedge f_{\mathbb{N}}(p) = f_s(c_0) \wedge f_{\mathbb{N}}(q) = f_s(c_0) \wedge \right. \\ \left. f_{\times}\left(p, f_{\wedge}(x, n)\right) = f_{\times}\left(q, f_{\wedge}\left(x', m\right)\right) \wedge f_{Coprime}(p, q) = f_s(c_0) \wedge f_{Coprime}(m, n) = f_s(c_0) \right) \ . \end{gathered} \tag{9.14}$$

Therefore, from the n-irreductible real numbers, $\sqrt{\pi}/2$, $16\pi^3/15$, $8\pi^2/15$ and the help of the two last formulas, we deduce that 2, 3, 4, 6, 15, 128 and 1024 are n-irreductible numbers.

With the help of the integer part function f_{IP} and the ceiling function $f_{Ceiling}$ on the n-irreductible real number $8\pi^2/15$ and $16\pi^3/15$, we deduce that 5, 6, 33 and 34 are n-irreductible numbers.

Finally, we can derive that 8 and 9 are n-irreductible numbers with the help of the following prime exponential number function $f_{PrimeExpPI}$ and $f_{PrimeExpCeiling}$:

$$\forall n \left(f_{\mathbb{N}}(n) = f_s(c_0) \rightarrow f_{PrimeExpPI}(n) = f_{Prime}\left(f_{PI}\left(f_{Exp}(n) \right) \right) \right) \tag{9.15}$$

$$\forall n \left(f_{\mathbb{N}}(n) = f_s(c_0) \rightarrow f_{PrimeExpCeiling}(n) = f_{Prime}\left(f_{Ceiling}\left(f_{Exp}(n) \right) \right) \right) \tag{9.16}$$

7 is a n-irreductible number with the following formula ϕ:

$$\forall m \left(\neg n < m \rightarrow f_{PrimeExpPI}(m) = f_s(c_0) \right) \wedge \nexists n' \forall m \left(n < n' \wedge \neg n' < m \rightarrow f_{PrimeExpPI}(m) = f_s(c_0) \right) \tag{9.17}$$

20 is a n-irreductible number with the following formula ϕ:

$$\forall m \left(m < n \rightarrow f_{PrimeExpPI}(m) = f_s(c_0) \right) \wedge \nexists n' \forall m \left(n < n' \wedge m < n' \rightarrow f_{PrimeExpPI}(m) = f_s(c_0) \right) \tag{9.18}$$

8 is a n-irreductible number with the following formula ϕ:

$$\begin{gathered} \forall m \left(m < n \rightarrow f_{PrimeExpCeiling}(m) = f_s(c_0) \right) \wedge \\ \nexists n' \forall m \left(n < n' \wedge m < n' \rightarrow f_{PrimeExpCeiling}(m) = f_s(c_0) \right) \end{gathered} \tag{9.19}$$

10. A set of n-irreductible numbers

In this section, we insert every n-irreductible number derived in the present paper inside an only one set:

$$\mathcal{S} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 14, 15, 20, 24, 27, 31, 33, 34, 37, 128, 220, 284, 503, 1024\} \tag{10.1}$$

11. The Riemann hypothesis

In this section, with the help of the formulas satisfied by the axioms of the complex numbers and the new complex constant variable c_i $(c_0 = f_+(f_s(c_0), f_\times(c_i, c_i)))$ joined to the hypotheses of some n-irreductible sequent, we propose a method to prove the Riemann hypothesis.

1- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the log function symbol f_{Log}:

$$\forall x\,(0 < x \rightarrow f_{Exp}(f_{Log}(x)) = x) \tag{11.1}$$

2- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the real power function symbol $f_{\wedge\mathbb{R}}$ (see 11.1):

$$\forall x\,(0 < x \rightarrow f_{\wedge\mathbb{R}}(x, y) = f_{Exp}(f_\times(y, f_{Log}(x)))) \tag{11.2}$$

3- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the Riemann zeta series function symbol g_ζ (see 11.2)

$$\begin{aligned}&\forall x\,(x < 0 \rightarrow g_\zeta(x, c_0) = c_0)\\ &\forall x \forall y \left(x < 0 \rightarrow g_\zeta(x, f_s(y)) = f_+ \left(g_\zeta(x, y), f_{\wedge\mathbb{R}}(y, x) \right) \right).\end{aligned} \tag{11.3}$$

4- If required, we add to the hypotheses of the n-irreductible sequents the following formulas satisfied by the Riemann zeta function symbol f_ζ (see 11.3 and the previous equation):

$$\begin{aligned}&\forall x \forall \epsilon \exists N \forall n \Bigg((0 < \epsilon \wedge N < n \wedge f_{\mathbb{N}}(N) = f_s(c_0) \wedge f_{\mathbb{N}}(n) = f_s(c_0))\\ &\rightarrow (g_\zeta(x, n) < f_\zeta(x) \wedge f_\zeta(x) < f_+(\epsilon, g_\zeta(x, n))) \Bigg).\end{aligned} \tag{11.4}$$

We assume we can define irreducibly the homomorphic functions and the analytic continuation. Therefore, we can define the holomorphic Riemann zeta function $f_{\zeta\mathbb{C}}$.

From here, we define the function $f_{\zeta NTZ}$ which enumerates the imaginary part of the none trivial zeros of the holomorphic zeta function f_ζ:

$$\begin{aligned}&f_{\zeta NTZ}(c_0) = c_0\\ &\forall n\,(f_{\mathbb{N}}(n) = f_s(c_0) \rightarrow f_{\zeta NTZ}(n) < f_{\zeta NTZ}(f_s(n)))\\ &\forall n\,(f_{\mathbb{N}}(n) = f_s(c_0) \rightarrow f_{\zeta\mathbb{C}}(f_{\zeta NTZ}(n)) = c_0)\end{aligned}$$

(11.5)

$$\neg \exists a \exists b \exists n\,(f_{\mathbb{N}}(n) = f_s(c_0) \wedge f_{\zeta\mathbb{C}}(f_+(a, f_\times(c_i, b))) = c_0 \wedge f_{\zeta NTZ}(n) < b \wedge b < f_{\zeta NTZ}(f_s(n)))$$

With the help of the function $f_{\zeta NTZ}$, we can find four n-irreductible numbers (14,15,31,37) derived from four n-irreductible sequents:

(11.6)
$$\phi \equiv x = f_{IP}\left(f_{\zeta NTZ}\left(f_s\left(c_0\right)\right)\right)$$

(11.7)
$$\phi \equiv x = f_{Ceiling}\left(f_{\zeta NTZ}\left(f_s\left(c_0\right)\right)\right)$$

$$\phi \equiv \exists n\left(f_{\mathbb{N}}\left(n\right) = f_s\left(c_0\right) \wedge x = f_{Prime}\left(f_{Ceiling}\left(f_{\zeta NTZ}\left(f_s\left(n\right)\right)\right) = f_s(c_0)\right) \wedge\right.$$

(11.8)
$$\left.\neg\exists n'\left(f_{\mathbb{N}}\left(n'\right) = f_s\left(c_0\right) \wedge 0 < n' \wedge n' < n \wedge f_{Prime}\left(f_{Ceiling}\left(f_{\zeta NTZ}\left(f_s\left(n\right)\right)\right) = f_s(c_0)\right)\right)\right)$$

$$\phi \equiv \exists n\left(f_{\mathbb{N}}\left(n\right) = f_s\left(c_0\right) \wedge x = f_{Prime}\left(f_{IP}\left(f_{\zeta NTZ}\left(f_s\left(n\right)\right)\right) = f_s(c_0)\right) \wedge\right.$$

(11.9)
$$\left.\neg\exists n'\left(f_{\mathbb{N}}\left(n'\right) = f_s\left(c_0\right) \wedge c_0 < n' \wedge n' < n \wedge f_{Prime}\left(f_{IP}\left(f_{\zeta NTZ}\left(f_s\left(n\right)\right)\right) = f_s(c_0)\right)\right)\right)$$

From the holomorphic Riemann zeta function $f_{\zeta\mathbb{C}}$, if the Riemann hypothesis is false, we can define a monster n-irreductible number m_Z with the following n-irreductible sequent:

$$\phi \equiv \exists a\exists b\exists n\left(f_{\mathbb{N}}\left(n\right) = f_s\left(c_0\right) \wedge x = n \wedge \neg f_{\times}\left(f_s\left(f_s(c_0)\right), a\right) = f_s(c_0) \wedge f_{\zeta NTZ}\left(f_s\left(n\right)\right) = b \wedge\right.$$

(11.10)
$$\left.\neg\exists a'\exists b'\exists n'\left(f_{\mathbb{N}}\left(n'\right) = f_s\left(c_0\right) \wedge n' < n \wedge \neg f_{\times}\left(f_s\left(f_s(c_0)\right), a'\right) = f_s(c_0) \wedge f_{\zeta NTZ}\left(f_s\left(n'\right)\right) = b'\right)\right)$$

12. Conclusion

This paper may open a new area in second order logic with some important consequences in number theory and in fundamental physics if we do not notice contradictions between the (second order logic) n-irreductible axiom and other well known axioms, and we do not observe experimental contradictions between the hypotheses to produce the largest n-irreductible number found and the experimental measurements. It is the first paper which gives a hint to solve the Goldbach's conjecture, the Polignac's conjecture, the Firoozbakht's conjecture, the Oppermann's conjecture, the Agoh-Giuga conjecture with a quantum computer of $2^{1.005\times 10^8}$ qubits or less and with only one (second order logic) n-irreductible axiom. The generalized Fermat's conjecture requires computational resources which are far from what we can imagine technically even for the simplest case: $a = 2$ and $b = 1$ and the Schinzel's hypothesis H requires also monster computational resources for checking about $\left(\pi\left(N_Z\right)2^{N_Z}\right)^{N_Z}$ cases. It is also the first paper which gives a hint to generate the "Nature's hypotheses" with only one (second order logic) hypothesis.

Since I am not a mathematician and I am a lonely human, I may have overseen some mistakes (especially, I could miss an explicit sub-formula in the present paper since the number of sub-formulas is roughly 2^n for a formula with n symbols or I do not noticed that a sequent is not n-irreductible or my approach to the generalized Fermat's conjecture and the Schinzel's hypothesis H are sensitive to some mistakes since it is one more level of abstraction from the other prime conjectures). Moreover, N_Z may change after the publication of the next paper about the theory of everything. Please send me an email (see it below the references) for any mistake noticed in the present paper. Every ideas or comments related to the present paper are also very welcome.

13. Acknowledgement

Dimitri, Nicolas, Sophie, Cynthia Zaganidis, Alice Chau, Mikhail Shaposhnikov, Marc-André Dupertuis, Marco Drewes, Nitin Rughoonauth, Francois Morin and to the essential workers who feed me, heal me, educate me and connect me throughout my life.

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