A Children's Primer on Bell's Inequality and Quantum Entanglement

William O. Straub
Pasadena, California 91104
March 30, 2017

Introduction

Elementary particles such as electrons and photons can be *entangled* in pairs, meaning that while they appear to have separate lives they share a quantum-level interaction that defies a straightforward physical interpretation. In the case of electrons, this entanglement can manifest itself in spin states describing two particles that may be separated by enormous distances, yet somehow remain together in the same state. Consequently, a measurement performed on one electron's spin appears to instantaneously determine the spin of its partner, even if it's on the other side of the universe. This strange phenomenon, which has been verified many times in carefully conducted laboratory experiments, appears to violate the notions of objective reality and *locality*—the classical belief that nothing can travel faster than the speed of light.

Two unentangled electrons can exhibit any spin, up or down, completely independently of one another. If you measure one particle's spin as $+1/2$ in some direction, the other's measured spin will be completely random in any direction you choose (but will always be $\pm 1/2$). Entangled electrons, on the other hand, invariably exhibit a kind of collaboration with regard to their spins. If you measure the spin of the first electron as $\pm 1/2$ in the direction of, say, the North Star, then a measurement of the other's will *always* be $\mp 1/2$ in that same direction. There's nothing terribly mysterious about this—if one has two marbles colored red and blue, observing the color of one will always ensure knowledge of the other's color, no matter how far apart they are.

The spins of electrons, however, are like colors that can vary according to the directions their spins are measured in. This opens up a vast possibility of measurement options, since the choice of spin directions is infinite. When the spins of a large assemblage of disentangled electron pairs are measured, the observed spin values are seen to be completely random, regardless of which directions the spins are measured. On the other hand, pairs of electrons can be prepared in the laboratory whose spins exhibit a probabilistic correlation that is unexplained classically. This correlation is called *entanglement*.

1. Bell's Inequality

Like many others, the late Irish physicist John Bell (1928-1990) was fascinated by a famous paper that Albert Einstein, Boris Podolsky and Nathan Rosen published in May 1935, entitled "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" Despite the questionable grammar of the paper's title, it presented a problem for quantum physics (now known as the EPR paradox) in the form of a philosophical option: either objective reality of the physical world exists (meaning that physical objects and quantities can exist without being directly observed), or reality itself is not what it seems to be. Einstein, ever critical of quantum theory, questioned Niels Bohr's assertion that nothing can be known (or even surmised) about a quantum system until an observation is made. He furthermore questioned the notion that the wave functions of two particles, far removed from one another, could be made to collapse simultaneously upon measurement of only one particle. To Einstein, this smacked of "non-locality"—the idea that the consequences of physical measurement could be transmitted instantaneously across space from one particle to another, in apparent violation of special relativity.

Bell initially supported Einstein's argument in the EPR paper that quantum mechanics was either wrong or incomplete, since it made predictions that seemed to contradict reality, and he believed there might be "hidden variables" in quantum theory that would restore sensible notions of objective reality and the finite speed of signals sent across space. But as he pondered the problem, he began to realize that Einstein had been wrong. Locality does seem to be violated by quantum mechanics while strict objective reality likely does not exist, at least at the quantum level of the world.

In a paper published in late 1964, Bell proposed a way to test objective realism in the form of a simple mathematical inequality. While this inequality clearly held classically, it was seen to be violated by quantum

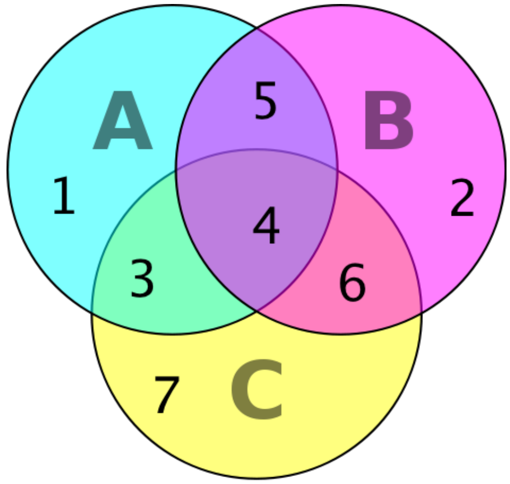

Figure 1: Venn Diagram

mechanics. A few decades later, physicists had developed methods to experimentally test both Bell's inequality and Einstein's notion of locality using polarized photons and electron pairs. Since that time many different kinds of tests have been conducted, and all have confirmed the violation of Bell's inequality by quantum mechanics.

We can reproduce one version of Bell's inequality using a simple Venn diagram. Take three bins of any size and shape, labeled A, B, C having open tops (we'll use circular bins of the same size for simplicity). Imagine that these bins have walls that can overlap one another, so that seven individual compartments are formed (Figure 1). We now drop N marbles into the bins so that they fall randomly into the compartments. Each compartment will have some number of marbles, which we designate as $N_1, N_2, \ldots N_7$. The number of marbles in Bin A that is *excluded* from those in Bin B is clearly $N_1 + N_3$. We designate this quantity as $A \backslash B$, so that $A \backslash B = N_1 + N_3$. Similarly, we have $A \backslash C = N_1 + N_5$ and $B \backslash C = N_2 + N_5$. Now, since

$$A \backslash B + B \backslash C = N_1 + N_3 + N_2 + N_5 = A \backslash C + N_2 + N_3$$

we have the obvious inequality

$$A \backslash B + B \backslash C \geq A \backslash C$$

the equality holding only when $N_2 = N_3 = 0$. If we divide each of these quantities by the total marble number N, we have instead the probability statement

$$P(A \backslash B) + P(B \backslash C) \geq P(A \backslash C) \tag{1.1}$$

where $P(A \backslash B) = (A \backslash B)/N$ represents the classical probability that $A \backslash B$ marbles will be found, etc. Equation (1.1) is one form of Bell's inequality. The proof is trivial, based on the apparently unassailable logic underlying classical probability and elementary sets and subsets. Indeed, the inequality is obviously true even if we don't bother to actually perform any observations with marbles. Nevertheless, we will see that the quantum-mechanical counterpart of Bell's inequality is violated. This violation can be shown to be a consequence of particle entanglement, which involves quantum states and substates and not ordinary sets of countable numbers.

2. State Descriptions of Electron Pairs

Instead of counting marbles in three connected bins, we'll use the spins of electrons in three arbitrary directions to derive a straightforward version of Bell's inequality. First a note about what might be called independent and dependent (entangled) electron pairs.

Assume that we have two electrons A and B whose spin states are independent of one another. Since electrons can only be found with spin-up (u) and spin-down (d) as measured in some direction, we can write

$$\begin{aligned} |A\rangle &= \alpha_A|\mathrm{u}\rangle + \beta_A|\mathrm{d}\rangle \\ |B\rangle &= \alpha_B|\mathrm{u}\rangle + \beta_B|\mathrm{d}\rangle \end{aligned}$$

where the coefficients are complex numbers. Since a complex number has two real parts, there are a total of four numbers involved for each ket. The normalization conditions

$$\alpha_A^*\alpha_A + \beta_A^*\beta_A = 1,$$
$$\alpha_B^*\alpha_B + \beta_B^*\beta_B = 1$$

reduce this number by one each and, because there is an overall phase invariance associated with each ket, there are a total of just two real numbers for each ket. This brings the total number to four, so that the product state

$$|AB\rangle = |A\rangle|B\rangle = \big(\alpha_A|\mathrm{u}\rangle + \beta_A|\mathrm{d}\rangle\big)\cdot\big(\alpha_B|\mathrm{u}\rangle + \beta_B|\mathrm{d}\rangle\big)$$

can be described with just four real coefficients.

Now let us consider an electron-pair system in which the electrons are tied to one another in the combined state described by

$$|AB\rangle = \alpha|\mathrm{uu}\rangle + \beta|\mathrm{ud}\rangle + \gamma|\mathrm{du}\rangle + \delta|\mathrm{dd}\rangle$$

The normalization condition

$$\alpha^*\alpha + \beta^*\beta + \gamma^*\gamma + \delta^*\delta = 1$$

coupled with phase invariance results in a total of six real numbers in the coefficients required to describe the state. Such a state is called an *entangled state*, and it is apparent that its description is fundamentally different from and richer than a product state. This is due to the fact that electrons in a product state are really independent of one another—a measurement performed on one has no effect on the other—while a measurement conducted on one member of an entangled pair is really a measurement conducted on the *entire* state. One of the truly bizarre consequences of this is the fact that a measurement performed on one member of an entangled electron pair will *instantaneously* affect the other electron, even if the electrons have been separated by an enormous distance.

3. Bell's Inequality for Entangled Electron Pairs

Consider that we now prepare a large number of entangled electron pairs produced by the decays of stationary, hypothetical spin-zero particles (alternatively, they could be electron-positron pairs that result from similar stationary spin-zero systems). The electrons all fly rapidly away from their partners in opposite directions, each carrying their own linear and spin-1/2 angular momenta but with the requirement that the total linear and angular spin momenta be zero for each entangled pair. A spin measurement carried out on one electron would then instantly affect its partner, as predicted, but the directions in which we choose to measure the spin are completely arbitrary. Let us follow one group of electrons (call it Group 1) with the intention of measuring the electrons' spins in any of three particular directions that we'll label with the unit vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Meanwhile, the electrons in Group 2 move further and further away, unmolested.

If we were to measure the spins of a representative sample of Group 1 electrons in the $\vec{a}$ direction, we'd get some mixture of $+1/2$ and $-1/2$ values. Alternatively, measurements conducted in the $\vec{b}$ direction would give a different set of results. From such data, we could assign a probability to the situation in which the spin of an electron in Group 1 is the same in directions $\vec{a}$ and $\vec{b}$. We could also assign a probability to the situation where the spins are measured with *opposing* signs. If the spin of a particular electron in Group 1 in the direction $\vec{b}$ were *different* than that in direction $\vec{a}$, then we could say with *absolute certainty* that the spin of its partner over in Group 2 would have the same value in the direction $\vec{a}$. Thus, the statement "$a\backslash b$" shall represent the situation in which a Group 1 electron's spin in direction $\vec{a}$ is the same as a Group 2 electron's spin in the $\vec{b}$ direction. Given a sufficiently large number of electron pairs, we could than calculate the associated probability $P(a\backslash b)$. You should carefully review and confirm this logic, which is based solely upon classical statistics and the fact that entangled electrons will invariably exhibit opposing spins when measured in the same direction.

We can use the exact same argument to calculate the probabilities $P(b\backslash c)$ and $P(a\backslash c)$. Since everything is based on observational outcomes and the statistics of large sample sets, Bell's inequality (1.1) must also hold for entangled electrons. The only problem is that it does not, and in the following we will prove why Bell's inequality is violated in quantum mechanics.

4. Mathematical Approach

In quantum mechanics, physical measurements are represented by Hermitian *operators* operating on state kets. Spin-1/2 systems are characterized by two-component kets operated upon by the 2×2 Pauli matrices,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Entangled electron pairs can be represented by the normalized state ket

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big(|\mathrm{ud}\rangle \pm |\mathrm{du}\rangle \big) \tag{4.1}$$

where the notation ud is taken to mean that a Group 1 electron's spin is up (u) while that of the Group 2 electron is down (d), etc. The alternating up-down convention in (4.1) guarantees that the measured signs of the paired electron spins will always oppose one another. Such a state ket is considered to be in the *singlet state* when the minus sign is used, but we will leave the sign between the kets undetermined for the time being. Later we'll see that the minus sign must be assumed, a condition that ensures that the total spin of any entangled pair is zero.

Lastly, note that the various eignkets are orthogonal to one another, so that we have conditions like

$$\langle \mathrm{ud}|\mathrm{ud}\rangle = 1, \quad \langle \mathrm{ud}|\mathrm{du}\rangle = 0, \;\; \text{etc.}$$

We'll also encounter the conditions

$$\langle \mathrm{uu}|\mathrm{uu}\rangle = 1, \quad \langle \mathrm{uu}|\mathrm{ud}\rangle = 0, \;\; \text{etc.}$$

4.1. Spin Eigenvectors and Density Operators

We'll want to extract probabilities from the state ket (4.1), and to do this we need the spin eigenvectors associated with the arbitrary directions $\vec{a}, \vec{b}$ and $\vec{c}$. From these eigenvectors we'll construct a specific kind of 2×2 matrix called a *density operator*, which will give us the probabilities directly.

The Pauli matrix σ_x acts on an eigenvector for the $x-$direction, giving back the same eigenvector multiplied by an eigenvalue. Since the squares of the Pauli matrices are all unity, all the eigenvalues will be ± 1. However, we do not want to limit ourselves to the $x-$direction, but instead want an arbitrary direction that we'll specify as $\vec{a}$. To do this we need to construct the quantity $\vec{\sigma} \cdot \vec{a}$, which is

$$\vec{\sigma} \cdot \vec{a} = \sigma_x a_x + \sigma_y a_y + \sigma_z a_z$$

Using the identities of the Pauli matrices given earlier, this is

$$\vec{\sigma} \cdot \vec{a} = \begin{bmatrix} a_z & a_- \\ a_+ & -a_z \end{bmatrix} \tag{4.1.1}$$

where $a_\pm = a_x \pm i a_y$ is a convenient shorthand. We now need an eigenvector associated with (4.1.1). This can be derived from the expression

$$\begin{bmatrix} a_z & a_- \\ a_+ & -a_z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = +1 \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

where α, β are arbitary complex numbers satisfying the normalization condition $\alpha^* \alpha + \beta^* \beta = 1$. There are any number of suitable values for these coefficients, and any will do the job. We'll use

$$\alpha = \sqrt{\frac{1 + a_z}{2}}, \quad \beta = \frac{a_+}{\sqrt{2(1 + a_z)}}$$

We thus have the eigenket associated with the direction $\vec{a}$:

$$|a\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1+a_z}{2}} \\ \frac{a_+}{\sqrt{2(1+a_z)}} \end{bmatrix}$$

The *density operator* associated with this eigenket is the outer product $|a\rangle\langle a|$, which we define as $\rho(a)$:

$$\rho(a) = |a\rangle\langle a|$$

This works out to be the simple quantity

$$|a\rangle\langle a| = \frac{1 + \vec{\sigma}\cdot\vec{a}}{2} \tag{4.1.2}$$

We thus have, for the three directions in question,

$$\rho(a) = |a\rangle\langle a| = \frac{1 + \vec{\sigma}\cdot\vec{a}}{2},$$

$$\rho(b) = |b\rangle\langle b| = \frac{1 + \vec{\sigma}\cdot\vec{b}}{2},$$

$$\rho(c) = |c\rangle\langle c| = \frac{1 + \vec{\sigma}\cdot\vec{c}}{2}$$

4.2. Probabilities

A nice property of density operators is that their expectation values give probabilities directly. For example, the expectation value of the operator $\rho(a)$ is defined by

$$\langle \rho(a)\rangle = \langle \Psi|\rho(a)|\Psi\rangle \tag{4.2.1}$$

But this is just

$$\langle \rho(a)\rangle = \langle \Psi|a\rangle\langle a|\Psi|\rangle = |\langle a|\Psi|\rangle|^2 \tag{4.2.2}$$

which is the probability that the eigenket will be found pointing in the direction $\vec{a}$.

The state ket we want to use in calculating the probabilities is that given by (4.1). This presents a complication, because each term in the ket involves two electrons, not one, each with its own spin direction. The way around this is to use a *double* density operator that consists of two parts, one that operates only on Electron 1 with the other confined to operate on Electron 2. To understand this, consider the operator

$$\rho(a,b) = \Big(\frac{1 + \vec{\sigma}_1\cdot\vec{a}}{2}\Big)\Big(\frac{1 + \vec{\sigma}_2\cdot\vec{b}}{2}\Big) \tag{4.2.3}$$

in which $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are just the Pauli matrices but with the following condition: $\vec{\sigma}_1$ operates only on Electron 1, while $\vec{\sigma}_2$ operates only on Electron 2. For example, in the eigenket $|\mathrm{ud}\rangle$ the first term (u) will always refer to Electron 1, while the second (d) will always refer to Electron 2.

4.3. Bell's Inequality in Quantum Mechanics

We now remind ourselves how the Pauli matrices operate on ket states. Assuming the usual shorthand notation for spin vectors given by

$$|\mathrm{u}\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |\mathrm{d}\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

we then have the identities

$$\sigma_x|u\rangle = |d\rangle, \quad \sigma_y|u\rangle = i|d\rangle, \quad \sigma_z|u\rangle = |u\rangle$$

$$\sigma_x|d\rangle = |u\rangle, \quad \sigma_y|d\rangle = -i|u\rangle, \quad \sigma_z|d\rangle = -|d\rangle$$

Because the entangled state ket (4.1) involves two electrons, we then have

$$\sigma_{1x}|uu\rangle = |du\rangle, \quad \sigma_{1y}|uu\rangle = i|du\rangle, \quad \sigma_{1z}|uu\rangle = |uu\rangle$$

$$\sigma_{1x}|ud\rangle = |dd\rangle, \quad \sigma_{1y}|ud\rangle = i|dd\rangle, \quad \sigma_{1z}|ud\rangle = |ud\rangle$$

$$\sigma_{1x}|du\rangle = |uu\rangle, \quad \sigma_{1y}|du\rangle = -i|uu\rangle, \quad \sigma_{1z}|du\rangle = -|du\rangle$$

$$\sigma_{1x}|dd\rangle = |ud\rangle, \quad \sigma_{1y}|dd\rangle = -i|ud\rangle, \quad \sigma_{1z}|dd\rangle = -|dd\rangle$$
$$\sigma_{2x}|uu\rangle = |ud\rangle, \quad \sigma_{2y}|uu\rangle = i|ud\rangle, \quad \sigma_{2z}|uu\rangle = |uu\rangle$$
$$\sigma_{2x}|ud\rangle = |uu\rangle, \quad \sigma_{2y}|ud\rangle = -i|uu\rangle, \quad \sigma_{2z}|ud\rangle = -|ud\rangle$$
$$\sigma_{2x}|du\rangle = |dd\rangle, \quad \sigma_{2y}|du\rangle = i|dd\rangle, \quad \sigma_{2z}|du\rangle = |du\rangle$$
$$\sigma_{2x}|dd\rangle = |du\rangle, \quad \sigma_{2y}|dd\rangle = -i|du\rangle, \quad \sigma_{2z}|dd\rangle = -|dd\rangle$$

Using (4.1) as our normalized eigenket along with (4.2.3), the probability that Electron 1 will be found pointing in the $\vec{a}$ direction with Electron 2 pointing in the $\vec{b}$ direction is

$$P(a\backslash b) = \frac{1}{2}\Big(\langle ud| \pm \langle du|\Big)|a+\rangle\langle a+|b+\rangle\langle b+|\Big(|ud\rangle \pm |du\rangle\Big)$$

or

$$P(a\backslash b) = \frac{1}{2}\Big(\langle ud| \pm \langle du|\Big)\frac{1+\vec{\sigma}_1\cdot\vec{a}}{2}\,\frac{1+\vec{\sigma}_2\cdot\vec{b}}{2}\Big(|ud\rangle \pm |du\rangle\Big) \tag{4.3.1}$$

Using the orthogonality of the various bras and kets, the above Pauli operations and some careful algebra, you should be able to show that the $a\backslash b$ case works out to be

$$P(a\backslash b) = \frac{1}{4}\left(1 \pm a_1 b_1 \pm a_2 b_2 - a_3 b_3\right) \tag{4.3.2}$$

We now see that this expression can be meaningful only if we had assumed a minus sign between the original two kets in (4.1) from the beginning. With this in mind, we have the simple expression

$$P(a\backslash b) = \frac{1}{4}\left(1 - \vec{a}\cdot\vec{b}\right) \tag{4.3.3}$$

Now, the dot product $\vec{a}\cdot\vec{b}$ is just $\cos\theta_{ab}$, where θ_{ab} is the angle between the two spin directions (remember that $\vec{a}$ and $\vec{b}$ are unit vectors). We can now use a simple trigonometric identity to write

$$1 - \cos\theta_{ab} = 2\sin^2\frac{1}{2}\theta_{ab}$$

so we have, finally, the three probabilities

$$P(a\backslash b) = \frac{1}{2}\sin^2\frac{1}{2}\theta_{ab}$$

$$P(b\backslash c) = \frac{1}{2}\sin^2\frac{1}{2}\theta_{bc}$$

$$P(a\backslash c) = \frac{1}{2}\sin^2\frac{1}{2}\theta_{ac}$$

From (1.1), Bell's theorem in quantum mechanics is thus given by the inequality

$$\sin^2\frac{1}{2}\theta_{ab} + \sin^2\frac{1}{2}\theta_{bc} \geq \sin^2\frac{1}{2}\theta_{ac} \tag{4.3.4}$$

As is easily shown, Bell's inequality is violated in quantum mechanics. A set of angles commonly used in the literature to demonstrate this is $\theta_{ab} = \pi/4, \theta_{bc} = \pi/4, \theta_{ac} = \pi/2$ which, using (4.3.4), gives

$$0.293 \overset{?}{\geq} 0.5$$

Although more commonly-seen sets of angles *will* obey Bell's inequality, the main point here is that, unlike (1.1), the inequality *can* be violated in quantum mechanics.

5. Comments

It was our intent here only to explain in some detail the conventionally understood physics behind quantum theory's violation of Bell's inequality. To date, this violation has been experimentally verified many times by many researchers and to very high degrees of precision. The interested student is encouraged to delve much more deeply into the subject than what is presented here, particularly with respect to the relevance and use of density operators in the now-burgeoning theory of quantum information. In addition, the consequences of quantum entanglement are now being actively explored in fields as diverse as thermodynamics (especially entropy) and cosmology (entangled black holes).

The upshot of the EPR paradox—that objective reality, locality and logical causal influence are incompatible with quantum mechanics—merely serves to show that familiar, common-sense notions of classical physics are simply wrong at the quantum level. Consequently, questions involving what constitutes quantum reality have passed into the realm of metaphysics and philosophy, although it seems highly doubtful that quantum reality is less fundamental than classical reality, regardless of how well the latter has served humanity over the millennia before the advent of quantum physics. It is entirely possible, if not probable, that the real mystery of the quantum lies not in Nature but in our intellectual inability (or unwillingness) to understand our world.

Many physicists have adopted the position of completely avoiding the philosophical aspects of quantum theory, as typified by the rallying cry "Shut up and calculate!" Others are not so dismissive of the theory's underlying philosophical aspects, believing instead that only when they are fully understood will we be able to develop a complete theory of Nature. In response to the strange and as-yet unexplained paradoxes of quantum mechanics, the noted Princeton physicist James Peebles has stated that

> However, it would be illogical, even ungrateful, to fault quantum physics for such bizarre predictions, because to the accuracy of available experimental tests the predictions agree with what is observed. The successes of quantum principles are convincing evidence that they are showing us a physical reality deeper than classical physics. On the other hand, there is no reason to believe the quantum paradigm is the ultimate truth, and there certainly will continue to be great interest in experimental tests of the standard theory and possible variants. And the search for variations on the standard theory will continue to be motivated by the question: What does the state vector $|\Psi\rangle$ really mean?

References

1. J.S. Bell, *On the Einstein Podolsky Rosen paradox*, Physics, Vol.1 No. 3, pp. 195-200, 1964.
2. A. Einstein, B. Podolsky and N. Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Phys. Rev. 47, 1935.
3. P.J.E. Peebles, *Quantum mechanics*, Princeton University Press, 1992.
4. J. Sakurai, *Modern quantum mechanics*, Addison-Wesley Publishing, 1985.
5. L. Susskind, *Quantum entanglements, part 1*, Stanford University Video Lectures, 2008. These video lectures, presented by the incomparably brilliant Stanford physicist Leonard Susskind, are available at both the Stanford website

   https://physics.stanford.edu/

   and YouTube

   https://www.youtube.com/watch?v=0Eeuqh9QfNI

   Comprising some 15 hours of detailed instruction, they arguably offer the clearest and most comprehensive presentations of quantum entanglement possible to undergraduates and beginning graduate students. A set of companion lecture notes, not affiliated with the university or Professor Susskind, is available at

   http://www.lecture-notes.co.uk/susskind/quantum-entanglements/