The formulation and interpretation of the Lorentz transformation

Per Hokstad, phokstad@gmail.com

Abstract. The Lorentz transformation of the special theory of relativity (STR) describes the time dilation between two reference frames moving relative to each other at a constant speed. The present work focuses on the fact that the observations of time will depend on the location of the clocks used for time registrations. Primarily, the paper presents a unified framework for the various ‘observational principles’, *i.e.* specification of the position of the clocks being applied. We also investigate the time dilation for time intervals; specifying how these depend on the various observational principles. A thorough discussion of the travelling twin example is included. Throughout we restrict to consider a single space coordinate.

Key words: Lorentz transformation; time dilation; symmetry; observational principle; positional time, travelling twin example.

1 Introduction

The Lorentz transformation provides the mathematical description of space-time for two reference frames moving relative to each other at constant speed; *i.e.* the situation described in the special theory of relativity (STR). The present work strives to explore the Lorentz transformation, and the interpretation of time dilation. The first part of the paper (Ch. 2-3) presents an abridged and modified version of [1]. This includes some critical remarks to the current narrative on time dilation.

The basis for the discussions is the standard theoretical experiment, two co-ordinate systems (reference frames), K and K' moving relative to each other at speed, v. We investigate the relation between space and time parameters, (x, t) on system K and the corresponding parameters (x', t') on the system K'; thus, restricting to have just one space coordinate.

A vast amount of literature exists on this topic. As a background we consider a small sample, authored by experienced scientists in the field: books by Bridgman,[2], Giulini, [3] and Mermin, [4]; further, some web pages; Andrew Hamilton, [5] and Pössel, (‘Einstein Online’), [6]. These references mainly address non-experts. But it is of interest to see how the main ideas of the STR are presented.

Definition of *simultaneity* becomes crucial when clocks are moving relative to each other. However, we will in this paper restrict to consider events which occur at the same location *and* time. We will assume that each reference frame has a set of calibrated clocks, located at virtually any position. So in principle we can at any position compare the clocks of the two reference frames. Thus, any convention to define simultaneity across reference frames by use of light rays is not considered in the present work.

The question of *symmetry* is interesting. The STR essentially describes a symmetric situation for the two systems/observers moving relative to each other. And for instance the reference [5] specifies an experiment of complete symmetry, referring to two spaceships moving relative to each other. Other references, however, are not found that explicit, and describe situations apparently involving some asymmetry.

Further, I find the sources somewhat ambiguous regarding the very interpretation of *time dilation*. In what sense – and under which precise conditions – is time dilation to be considered a true physical phenomenon? How should we interpret the common statement: 'Moving clock goes slower'? Many authors apply the expression 'as seen' by the observer on the other reference system, perhaps indicating that it is an apparent effect, not a physical reality, without elaborating on the interpretation of 'as seen'. However, others stress that 'everything goes slower' on the 'moving system', not only the clocks; truly stating that the time dilation represents a physical reality also under the conditions of STR, (*i.e.* no gravitation *etc*.). On the other side Giulini [3] in Section 3.3 of his book states: ‘Moving clocks slow down' is 'potentially misleading and should not be taken too literally'. However, I do not find the expression 'not be taken too literally' to be very precise.

Further, I miss a more thorough discussion of the multitude of (time) solutions offered by the Lorentz transformation. As pointed out *e.g.* by Pössel [6] the phenomenon of time dilation is related to the fact that clocks of the two systems have to be compared at least twice, so it cannot be the same two clocks being compared. Mermin [4] states that what 'moves' is decided by which clocks are chosen to be synchronized. This seems to be is in line with the views of the present work: the procedure of clock synchronization and clock comparison decides which reference system has the time moving faster/slower.

In the present work we specify the various expressions for the time dilation following from the Lorentz transformation. In doing so, we introduce the concept of observational principle; that is, the specification of which clocks to apply for the required time comparisons. A unified framework for these observational principles is given, stressing that time measured for 'the other system' is given by the locations where the time readings/comparisons are carried out.

Further, in Chapter 4 we investigate the length of time intervals on the two reference frames. Again the magnitude of the time dilation depends heavily on *observational principle*, now given by the speed of the observer.

The considerations of the present work are essentially of mathematical nature, exploring the model suggested by the Lorentz transformation. However, we include a chapter with a thorough discussion of the travelling twin case. This also leads to the concept of 'simultaneity by symmetry'.

The previous version of the paper included a chapter, discussing transformations of the (time, space)-variable; this is removed in the present version.

Note that the present work essentially discusses well-known results. However, I believe that the presentation deviates from the main narratives on the topic. In particular, rather than focusing on a specific time dilation formula – which is typically based on a somewhat arbitrary definition of simultaneity – we will in the present work look at the total picture of *all* expressions for time dilation. And based on the resulting narrative (interpretation of the Lorentz transformation), we are – under the stated conditions – lead to consider time dilation as an observational phenomenon rather than a physical reality.

2 Foundation

2.1 Basic assumptions

The main focus of this paper is the Lorentz transformation, describing two reference frames, K and K' passing each other at a relative speed, v. We restrict to consider just one space co-ordinate, (x-axis), with speed of light (c) observed to be constant for all reference frames. We apply 'Newtonian/classical' arguments for events relative to a specific coordinate system, and the model represents a completely idealized situation. Further, the discussions are based on the following specifications:

- There is a complete *symmetry* between the two co-ordinate systems, K and K'; the systems being identical in all respects.
- On both reference frames there is an arbitrary number of identical, synchronized clocks, located at various positions where it is required to measure time. Thus, if we restricting to consider events on a single, specific reference frame, simultaneity does not represent any problem..
- When we consider two different reference system, simultaneity of events will mean that they occur at the same time *and* at the same location; *i.e.* we only compare clocks on different reference frames if they are at the same location.
- At time $t = t' = 0$, clocks at the location $x = 0$ on K and location $x' = 0'$ on K' are synchronized. This represents the definite starting point, from which all events are measured; it is the 'point of initiation'.

- We will choose the *perspective* of one of the systems, say *K*, and refer to this as the *primary* system. We may think of this as the reference frame where the 'experiment' (chain of events) is carried out. In particular, the time on this 'primary' system is at any position, x given as a constant, $t(x) \equiv t$, independent of x, (all clocks being synchronized). In contrast, at a certain time, t on the primary system, the observed time, t' on the other ('secondary') system(s) will depend on the location where the time reading is carried out.

2.2 The Lorentz transformation

The Lorentz transformation represents the fundament for our discussion of time dilation. The so-called length contraction along the x-axis equals,

$$k_x = \sqrt{1 - (\frac{v}{c})^2} \quad (1)$$

The Lorentz transformation for time and one space co-ordinate equals

$$x' = \frac{x - vt}{\sqrt{1 - (\frac{v}{c})^2}} \quad (2)$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - (\frac{v}{c})^2}} \quad (3)$$

This transformation relates simultaneous time readings, t and t' performed at identical locations x on K and x' on K'. We note that an observer at K and an observer at K', who at an instant in time are at the same location - actually passing each other at the moment in question - will agree both on the time t at K and on the time t' at K'; these observed values being specified by the above Lorentz transformation. They will, however, (usually) observe $t \neq t'$.

3 Observational principles and time dilation

3.1 Two main observational principle

We now point to some direct consequences of the Lorentz transformation. At time, $t = t' = 0$ a first comparison of clocks at the two systems is carried out at the origin, $x = 0$, $x' = 0'$. The reference frames are moving relative to each other at speed, v, and for the next clock comparison we must take a crucial decision: Either to compare the clock at $x' = 0'$ with a passing clock on K, or to compare the clock at $x = 0$ with a passing clock on K'.

In case 1, the clock at $x' \equiv 0'$ on K' passes a position, x (on K) at time t. This actually equals the location, $x = vt$. Inserting this value for x in the Lorentz transformation, (3), we get the well-known relation between t and t' at this location:

$$t' = t\sqrt{1 - (\frac{v}{c})^2} \quad (4)$$

This is the 'standard' time dilation formula, and relative to K we will refer to this as the observational principle A; (we follow one fixed clock on K', moving relative to K).

In case 2 we follow a clock at $x \equiv 0$ on K, and at this position we make comparisons with clocks on K' as they pass along. Inserting this value, x=0 in the Lorentz transformation, it directly follows that at time t on K, then the time t' observed on K' at this location is given as

$$t' = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}} t \quad (5)$$

Relative to K we refer to this as observational principle B, (all observations on K made from one fixed position, $x = 0$). In a way, this *eq*. (5) gives the 'opposite' result of principle A, and due to the symmetry this should be no surprise.

To summarize, when the clock at $x' = 0'$ on K' is compared with various clocks on K, these clocks must have position $x = vt$, and we get the relation $t' = t\sqrt{1-(v/c)^2}$. However, when a specific clock at position $x = 0$, on K is used for comparisons with clocks on K', we get $t' = t/\sqrt{1-(v/c)^2}$ as the relation between t and t', (also giving $x' = -vt/\sqrt{1-(v/c)^2} = -vt'$). I think this duality provides a key to better understand time dilation.

Actually, these symmetric results, (4), (5) could be presented in a compact form. Let A be the reference frame where there are used two clocks for the time comparisons, and let x_A and t_A be the position and time for measurements on this system. Thus, there are two clocks on A, located at $x_A = 0$ and $x_A = vt_A$, respectively. The reference frame, B has time, t_B, and we utilizes just one clock on its system, located at $x_B = 0$. Then, the clock readings (at same location and same time), will be related by the formula:

$$t_B = t_A\sqrt{1-(v/c)^2} \tag{6}$$

a result which combines *eqs.* (4) and (5). Some comments are relevant here.

First we stress that observers on both reference frames agree on this result (6). Thus, I find it rather misleading here to apply the phrase 'as seen' regarding the clock reading on 'the other' system; which is a formulation used by some authors. The time readings are objective, and all observers (observational equipment) on the location in question will 'see' the same thing. The point is rather that observers at *different reference frames* will not agree regarding simultaneity of events.

Secondly, we have the formulation 'moving clock goes slower'. It is true that an observer on a reference frame (here A), observing a *specific clock* passing by (on B), will see this clock going slower when it is compared to his own clocks. So in a certain sense this confirms the standard phrase 'moving clock goes slower'. However, we could equally well take the perspective of the single clock (here on B), considering this to be at rest, implying that the clocks on A are moving. The point is definitely not that clocks on B are moving and clocks on A are not. Rather, we could focus on the symmetry of the situation: We are starting out with two clocks at origin, $x_A = x_B = 0$, moving relative to each other. And the decision to either compare the clock at $x_A = 0$ *or* the clock at $x_B = 0$ with clocks on the other system, (a decision that can be interchanged at random!), will decide which of the two clocks comes out as the fast one!

So, first of all, none of the clocks are more moving than the other. Further, it is the observational principle that decides which of the two clocks initially at the origin, which we see to move slower. Therefore I find the talk about the 'moving clock' rather misleading.

This choice to either follow the single clock on A or the single clock on B is obviously crucial, and it introduces an asymmetry between the reference frames

3.2 Unidirectional flashes. An observational principle based on light rays

Now consider the situation that at time $t = t' = 0$ there is emitted a flash of light at location $x = 0$ (and/or $x' = 0'$) along the positive x-axis. At a later time, t on K we know that position, $x = ct$ coincides with position, $x' = ct'$ (due to constancy of speed of light). Inserting $x = ct$ in (3) we directly get:

$$t' = \frac{1-v/c}{\sqrt{1-(\frac{v}{c})^2}}\,t = \frac{\sqrt{1-v/c}}{\sqrt{1+v/c}}\,t; \quad (\text{for } x = ct) \tag{7}$$

For observing this we apply two clocks on both system: one at $x = 0$ and one at $x = ct$ on K; and similarly, one at $x' = 0'$ and one at $x' = ct'$ on K'. We may refer to this approach for time comparison as observational principle C.

So eq. (7) is valid when the light ray is emitted in the positive direction ($x > 0$); *i.e.* c having the same direction as the velocity v, as seen from K. In the negative direction, (choosing $x = -ct$) we similarly get another well-known result:

$$t' = \frac{1+v/c}{\sqrt{1-(\frac{v}{c})^2}}t = \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}t; \quad (\text{for } x = -ct) \tag{8}$$

We refer to this approach as observational principle C*. So the difference between the observational principles C and C* is not so much the different use of clocks, but rather the direction of the light flash. Note that *eqs.* (7)-(8) give quite other time dilation formulas than those obtained by observational principles A and B discussed above. – We finally mention that also use of bidirectional rays is relevant, see [1], but does not seem to give essential new insight and will not be discussed here.

3.3 A generalization

We have seen that the observational principles, A, B and C give different relations between t and t'. At an instant when the time all over K is found to equal t, one will at different locations on K observe different times (clock readings), t' on K'.

Now consider a generalization of the principles A, B and C. Taking the perspective of K, we may at time t choose an 'observational position' equal to $x = wt$, (for an arbitrarily chosen w). By inserting $x = wt$ in (3) we directly get that time on K' at this position equals:

$$t' = \frac{1-\frac{vw}{c^2}}{\sqrt{1-(\frac{v}{c})^2}}t \tag{9}$$

Further, we specify a w' so that position $x' = w't'$, at this time corresponds to (has the same location as) $x = wt$. Now, also inserting $x' = w't'$ in (2), we will after some manipulations obtain

$$w' = \frac{w-v}{1-\frac{w}{c}\frac{v}{c}} \tag{10}$$

So equations (9), (10) represent the version of Lorentz transformation, expressed by (t, w) rather than (t, x). Here we see that the results for the observational principles A, B and C directly follows from (9) by choosing $w = 0$, $w = v$ and $w = c$, respectively. Actually, (see (9)), we could consider

$$\gamma_{v,w} = (1 - \frac{vw}{c^2}) \,/ \sqrt{1 - (\frac{v}{c})^2} \tag{11}$$

to be the generalized time dilation factor, valid for any observational principle, (any $w=x/t$). Note that we do not here think of w as a velocity, rather a way to specify a certain position $x=wt$, representing the location of the clocks being applied at time t; (also see Section 4.2 on the interpretation of w).

In addition, now having these general expressions, (9), (10), we could ask which value of w (and thus w') would results in $t = t'$. It is easily derived that this equality is obtained by choosing

$$w = w_0 = \frac{c^2}{v}\left(1 - \sqrt{1 - \left(\frac{v}{c}\right)^2}\right) = \frac{v}{1+\sqrt{1-(v/c)^2}} \tag{12}$$

Further $w' = -w_0$. This means that if we consistently consider the positions where simultaneously $x = w_0 t$ and $x' = -w_0 t' = -w_0 t$, then no time dilation will be observed at these positions. Since here $x' = -x$, we consider it to be the midpoint between the origins of the two reference frames; in total providing a nice symmetry.

We refer to this choice, $x = w_0 t$, as observational principle D. Observe that when we choose this observational principle - which is symmetric with respect to the two reference frames - then absolutely everything is symmetric, and so $t' = t$. Thus, it is tempting to state that in cases when we get $t' \neq t$ this is caused by applying a non-symmetric observational principle, as everything, except the observational principle is symmetric.

Figure 1 below illustrates eq. (3) - or equivalently (9) - giving t' as a linear, decreasing function of x, for t fixed. It presents a total picture of the relation between t and t', as obtained by the Lorentz transformation. Thus, the time t' is given by the position x on K where it is observed, and we might thus refer to *positional* (*i.e.* location specific) time. The observational principles A-D are indicated along this line.

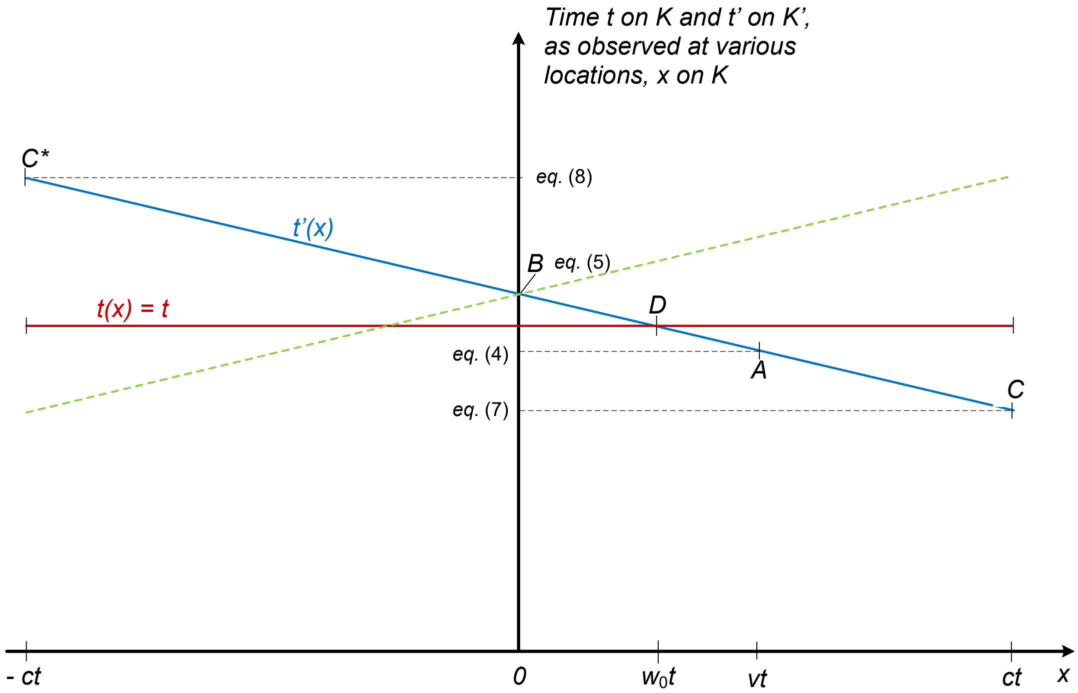

Figure 1 Time, $t' = t'(x)$, (blue) on K' at different locations, $x = wt$, on K, when the time on K equals t, (red); *i.e.* perspective of K. Various observational principles specified along the blue line, $t'(x)$. The dotted green line gives values of, $t' = t'(x)$, when v is replaced by $-v$.

As a further illustration we have in the figure included a dotted green line, representing time, t' - as measured at a position x on K - on a system moving at speed, $-v$. We see that the blue and green lines, corresponding to v and $-v$, respectively, form a 'bow tie' with a knot in observational principle B. This midpoint of the 'bow tie' is shifted upwards relative to the red line, t, with a factor $1/\sqrt{1-\left(\frac{v}{c}\right)^2}$, *cf. eq* (5). Actually, if we from this figure should suggest an 'overall' (average) time dilation factor for the 'moving system', then this factor could seem quite appropriate; but actually, this would rather tell us that *on the average* 'time on the moving system goes *faster*'. As previously noted we actually prefer not to use the term 'moving system'.

In summary, the relations presented in Figure 1 are rather fundamental for the interpretation of time dilation. We now go on to consider *time differences*; *i.e.* time intervals, (*e.g.* of length t).

4 Time intervals

In this chapter we consider time differences rather than 'absolute times' considered in the previous chapters.

4.1 Two formulas for time dilation of time differences

We now first introduce a change of notation. Rather than t' and x' we will write $t_v(t)$ and $x_v(t)$. Then the Lorentz transformation (2), (3) takes the form

$$x_v(t) = \frac{x - vt}{\sqrt{1-(\frac{v}{c})^2}} \tag{13}$$

$$t_v(t) = \frac{t - \frac{v}{c^2}x}{\sqrt{1-(\frac{v}{c})^2}} \tag{14}$$

(Observe that in this notation $t=t_0(t)$ and $x=x_0(t)$.) Further, we define

$$t_{v,diff}(t) = t_v(\tau + t) - t_v(\tau)$$

Which equals the length of the time interval on K' that correspond to the interval $(\tau, \tau+t)$ on K at position, x. Now e*q.* (14) immediately gives the following expression for this interval

$$t_{v,diff}(t) = \frac{t}{\sqrt{1-(v/c)^2}} \tag{15}$$

First we observe that this result is independent both of τ and of the location, x at which the time measurements are carried out. So at any fixed position, x on K the time difference on K' is actually given by (15), which equals (5). As previously noted, this differs from the standard time dilation formula.

However, we obtain an alternative formula if the time differences are rather observed on a fixed location $x_v(t)$ on K'. Combining (13) and (14) we easily derive the following alternative result

$$t_v(t) = t\sqrt{1-\left(\frac{v}{c}\right)^2} - \frac{v}{c}\cdot\frac{x_v(t)}{c} \tag{16}$$

Thus, we see that by using *eq.* (16) rather than (14), we get the following expression for $t_{v,diff}(t)$:

$$t_{v,diff}(t) = t_v(\tau + t) - t_v(\tau) = t\sqrt{1-(v/c)^2} \tag{17}$$

As we see, (17) equals the standard time dilation formula, (4).

Thus, we have two apparently contradictory results, (15) and (17). But from the above derivation we realize that these equations have different interpretations. *Eq.* (15) gives the time difference observed on K' when the clock measurements are carried out at a fixed (but arbitrary) positions, x on K. *Eq.* (17), similarly, gives the time difference observed on K' when the clock measurements are carried out at a fixed (but arbitrary) positions, $x' = x_v(t)$ on K'. Again it is worth noting that the result (15) is independent of x, and (17) is independent of $x_v(t)$.

This shows that time differences observed at any fixed location (either on K or K') will be independent of position. Further, it generalizes results in Chapter 3 on observational principles A and B. In Section 3.1 we obtained a result for the interval $(0, t)$. Now we have the same result for an arbitrary time interval $(\tau, \tau + t)$ on K.

4.2 A generalization

We could illustrate the above results in Figure 1. The line $t_v(0)$as given by (14) will go through origin, being parallel with the blue line. Thus, the time difference, (15), would be a horizontal line (parallel with the red line), and would go through the point B. Similarly, the time difference, (17), would also be a horizontal line, and would go through the point A. - Since the time differences are independent of positions, we can actually restrict to consider the t- axis.

Similar to the generalization given in Section 3.3, we now present a generalization of the results in Section 4.1. We use the notation introduced there, and also let $K' = K_v$ be any reference frame moving relative to K at speed v.

The main objective is to find a relation between $t_v(t)$ and t (for a fixed v), but the point of observation will be denoted $x_w(t) = x_w$, and will thus be the fixed position on a reference frame K_w at speed w relative to K. For any such w, we can write *eq.* (13) as

$$x = x_w \sqrt{1-(w/c)^2} + wt$$

We recall that here $x=x_0(t)$ equals the chosen position on K at time t. Now inserting this expression in (14) we get

$$t_v(t) = \frac{1-wv/c^2}{\sqrt{1-(v/c)^2}} t - \frac{x_w \sqrt{1-(w/c)^2}}{\sqrt{1-(v/c)^2}} \frac{v}{c^2}$$

and it directly follows that if we consider two time instants, τ and $\tau + t$, then the corresponding time difference, $t_v(t+\tau) - t_v(\tau)$ on K_v is independent of τ and equals

$$t_{v,diff}(t) = t_v(\tau + t) - t_v(\tau) = \frac{1-wv/c^2}{\sqrt{1-(v/c)^2}} t \tag{18}$$

We observe that the observational position, x_w on the reference frame K_w does not enter (18) and so is irrelevant for this result. The results (15) and (17) follow as special cases by inserting $w = 0$ and $w = v$, respectively in (18).

Observe that no clock is required on K_w. This reference frame is just used to obtain the location for simultaneous clock readings on $K = K_0$ and $K' = K_v$. So neither the position nor the time on K_w will affect this task, (only the relative speed, w of K_w).

We also observe that the factor $\frac{1-wv/c^2}{\sqrt{1-(v/c)^2}}$ of (18) equals what we previously referred to as the generalized time dilation factor, *cf.* eq. (11). But in the present formulation, w actually *do* refer to a speed, while in Chapter 3 we chose just to let w be a way to specify a specific location on K, through $x = wt$. But altogether, how the observation point $x = wt$ is chosen is immaterial; it could equally well be picked out by an (imaginary) reference frame K_w moving along K.

Now, of course, we will question which speed, w will result in $t_{v,diff}(t) = t$. We easily derive that this is obtained when w equals

$$w_0 = v/\left(1 + \sqrt{1-(v/c)^2}\right)$$

cf. eq. (12). As pointed out in Chapter 3 this represents a value of symmetry between $K = K_0$ and $K' = K_v$.

Summary

Now sum up the interpretation of the result, (18). We start out with the two reference frames, $K = K_0$ and $K' = K_v$, moving relative to each other at speed, v. In addition, we introduce a reference frame, K_w, moving at speed, w relative to K, and we consider this to be the observational system. Actually, we choose an arbitrary but fixed position on K_w, and at two instants make simultaneous time readings both of the clocks on K and on K_v at this position x_w on K_w. The time difference of the clocks at K equals t, and the time difference of clocks at K_v equals $t_{v,diff}(t)$. Now *eq.* (18) gives the relation between these two time differences.

The special cases are as follows. If K_w is at rest with respect to K (w=0), then the time difference at K_v is given by (15). If K_w is at rest with respect to K_v, (w=v), then the time difference at K_v is given by (17). Further, if K_w is moving at the above speed, w_0 with respect to K, then the time differences are equal: $t_{v,diff}(t) = t$. Finally, if K_w is moving at speed, c, then we get the result (7); and if its speed equals -c, we get the result (8).

So, actually, Chapter 4 just provides a generalization of the result in Section 3.3. There we considered the time interval (0, t) on K; now we have a similar result for an arbitrary interval of length t.

5 The travelling twin revisited.

We follow up our discussions on (time, space) relations by taking a look at the so-called travelling twin paradox. As stated for instance in [4] this paradox illustrates that two identical clocks, initially in the same place and reading the same time, can end up with different readings if they move apart from each other and then back together.

Reference [4] gives the following numerical example, (Chapter 10): "If one twin goes to a star 3 light years away in a super rocket that travels at 3/5 the speed of light, the journeys out and back each takes 5 years in the frame of the earth. But since the slowing-down factor is $\sqrt{1-(3/5)^2} = 4/5$ the twin on the rocket will age only 4 years on the outward journey, and another 4 years on the return journey. When she gets back home, she will be 2 years younger than her stay-at-home sister, who has aged the full 10 years." So the claim is that the referred difference in ageing occurs during the periods of the journey with a constant speed; *i.e.* under the conditions of the STR, (ignoring the acceleration/deceleration periods). Thus, our discussion will focus on the periods of constant velocity.

The distance between earth and the 'star' equals x = 3 light years, and the rocket has speed, v = (3/5)c, giving $\sqrt{1-(v/c)^2} = 4/5$. It follows that in the reference frame of the earth/star, the rocket reaches the star at time, $t = x/v$ = 5 years. This would be verified if the twin on the earth had located a clock on this star, being synchronized with his own clock on the earth. In the reference frame of the rocket, however, the clock on the rocket is located at $x' = 0$, and the Lorentz transformation gives that this clock at the arrival reads $t' = t \cdot \sqrt{1-(v/c)^2} = 4$ years. Repeating this argument for the journey back, we have that the clock on the rocket then shows 8 years, while the clock on the earth shows 10 years; so obviously, $t'/t = 0{,}8$. (We ascribe this to the length contraction: Seen from the perspective of the travelling twin, the distance between earth and star equals 3·0.8 = 2.4 light years.)

Actually, this seems a rather convincing argument. From the Lorentz transformation it follows that the returning clock shows 8 years when the earthbound shows 10 years. However, recalling the discussion of Chapter 3 the case is perhaps not that straightforward. And as we have made no assumption of asymmetry for these periods of constant velocity, we seem to have a true paradox here.

We should start by discussing the observational principle. The above description has taken the perspective of the earth, and then we apply observational principle A, *cf.* Ch. 3. This may seem the only feasible principle here, but as we know, there are other possibilities of clock comparisons to explore.

Now we simply assume that there is also another reference frame, *i.e.* that of the travelling twin; he being equipped with rockets at appropriate and fixed distances from his own rocket, all moving with constant speed in the same direction as himself, and all equipped with a synchronized clock showing the same time, t'. Whether this is practically feasible is not relevant here. We are referring to the model of the STR, and point out what this theory tells about clock readings, *if* we provide such an arrangement.

So, what happens if we make this assumption, and apply observational principle B (in the perspective of the earth). As seen in Chapter 3, we then get the 'opposite' result, that is $t'/t = 1/0{,}8 = 1.25$, whatever instant we consider.

To be more specific, consider various instants at which we observe the clocks positioned at (passing by) the earth, (now referring to three 'perspectives'):

1. When the travelling twin arrives to the star, the clock on the rocket shows 4 years. So all clocks on the reference frame of the travelling twin show time, $t' = 4$ years. This is also the case for the clock which as this moment is passing the earth, *i.e.* at $x = 0$. The Lorentz transformation directly gives that at this moment the clock on the earth shows time $t = t' \cdot 0{,}8 = 3.2$ years.
2. Next take the perspective of the earth/star system. At the instant when the twin arrives at the star all the clocks of this system show time $t = 5$ years. Performing a clock comparison at $x = 0$ at this moment, gives that $t' = t/0{,}8 = 6.25$ years for the clock passing the earth at this moment.
3. The above two cases demonstrate that the two twins completely disagree about which event at the earth is simultaneous with the travelling twin's arrival at the star. We now consider a third case. There is a moment when the clock on the earth (x =0) shows 4 years and at the same time the passing 'travelling clock' shows 5 years. This obviously occurs in between the previous two moments, and represents a moment being 'symmetric' to the event of the twin's arrival at the star (regarding clock readings!). Due to the complete symmetry of these events, it is actually tempting to refer to a kind of 'simultaneity' of this event at the earth and the arrival of the travelling twin at the star.

In summary, when we carry out the clock comparison at the earth, we always get $t'/t = 1.25$; so it is the clock at the earth that 'goes slower'. We summarize the findings in the below table.

Table 1. Various clock readings (light years) at/on the earth, potentially 'corresponding to' the arrival of the travelling twin at the star.

	Perspective		
Location	1.Travelling twin	2.Earthbound twin	3.Symmetric
Travelling twin system:	4	6.25	5
Earth/star system:	3.2	5	4

This makes the problem more complex. It is a true paradox, and the solution is hardly obvious. It seems hard to deny that if the travelling twin is in charge, and he decides to turn at the star then his clock shows 4 light years and the 'clock on the star' shows 5 years, then his clock on return shows only eight year. Actually, this is not very surprising, according to his speed (and length contraction) he observes the distance travelled as just 3·0.8 = 2.4 light years back and forth.

However, taking now the perspective of the earthbound twin, the picture becomes more complex, *cf.* Table 1. First of all we could also consider him as travelling back and forth along the reference frame of the travelling twin; giving exactly the same result for him; *i.e.* travelling time = 4 years.

Secondly, let us again focus on the travelling twin; now assuming the earthbound twin is in charge. He could control the twin's travel by sending a light signal to the star, which on arrival initiates a return of the travelling twin. One possibility is that he sends a signal that arrives at the star when all the clocks on his earthbound system shows 5 years (*i.e.* perspective 2). The 'problem' is that at this moment the clock on the travelling twin system passing the earth shows 6.25 years, and thus the travelling twin will have aged 12.5 years (and not 8) when he returns. Due to the length contraction observed by the travelling twin, such a strategy will infer that he travels a much longer distance than the intended 3 light years.

So if the earthbound twin should be in charge, I guess the following strategy should be the most ingenious. Knowing about the length contraction, he will know that the travelling twin will observe a travelling distance to the star that equals just 2.4 light years. So the earthbound twin will adopt strategy 3: he sends a signal ordering to turn, such that the travelling twin will receive this signal when the clocks on his own earthbound system shows 4 years, (and thus, the clock on the passing travelling twin system at the earth shows 5 years; *cf.* Table 1).

Consequently, at what they now both consider the turning of the rocket the twins will 'agree' on the following fact: Their own clock shows 4 years, and the other twin has at this moment apparently aged more than himself by a factor 1.25 (adjacent clock on the other system showing 5 years).

Following these arguments, we should actually conclude that by the reunion *both* clocks show 8 years. When we stick to the symmetry both twins also have a return travel of 3 light years as measured in the other twin's reference frame, *i.e.* 2.4 light years measured from their own frame due to length contraction. Thus, also the return travel to unite the two twins have a duration of 4 years on both the travelling and the earthbound clock.

So in my opinion, one should truly question the claim that the travelling twin actually ages more slowly. We have in our presentation focused on the full symmetry of the situation; and in this perspective it should actually be rather meaningless to claim that one ages faster than the other does. So if we ignore the effects of the acceleration/deceleration periods, it is hard to see the asymmetries here justifying a claim that there is a true difference in ageing.

We could actually give an idealized description without acceleration/deceleration as follows: One System B moves relative to a system A (earth) at a speed $v=0.6c$. At the origins of these, there are clocks, which we synchronize at time 0. The distance travelled for clock B, as measured at A equals 3 light years. On arrival, the local clocks show time 5 years on A and 4 years on B. At that instant there is also passing a third reference frame, C, moving relative to A at speed $v=0.6c$ but in the opposite direction. Further, there is a clock on C, which we synchronize with clock B; (thus, both showing 4 years). When this C clock arrives at the origin of A it will show 8 years, as also the local clock on A will do! To summarize: All three clocks move relative to another system, over a distance, which at rest is measured to equal 3 light years. But due to length contraction the 'moving clocks' will all observe the distances to equal just $3\cdot 0.8 = 2.4$ light years; so each way they all measure a time duration of 4 years. This argument is of course equally valid for the clock on A; irrespective of whether system B has an additional rocket/clock for time comparison with system A.

So sticking to such an idealized, symmetric description of the experiment, I find no convincing argument that could cause a true age difference to occur during periods of constant speed. Thus, my conclusion would be that we should explain any true difference in ageing by the presence of acceleration/deceleration periods.

6 'Simultaneity by symmetry'

We now pursue an interesting point in the argument of Chapter 5. We observed that 'perspective 3' above (Table 1) provides a nice symmetry between the clock readings at the earth and at the star. Applying this perspective 3, means that the moment at the earth when its clock shows 4 years and the passing clock of the travelling twin system reads 5 years 'corresponds to' the moment at the star where the twin turns, (and where also the clocks read 4 and 5 years). Neither in the perspective of the travelling twin, nor in the perspective of the earthbound twin, is this moment simultaneous with the instant when travelling twin makes his turn at the star. However, taking an overall perspective, these two instants are completely symmetric, and taking a holistic view, we could see these to represent some kind of 'simultaneity'. We choose to refer to this as 'simultaneity by symmetry', and consider this a potentially useful concept.

In the current case it means that the for the time space vector (t, x) on the earth the following two points are 'simultaneous by symmetry': $(t, x) = (4, 0)$ and $(t, x) = (5, 3)$. (The corresponding values for the travelling twin reference frame are (5, -3) and (4, 0).) Now, we can further argue that all points on the straight line between these two points should just as well be consider to be 'simultaneous'. This would imply that all (t, x) values on the line

$$t = (1/3)\cdot x + 4$$

on the reference frame of the earth are said to be 'simultaneous by symmetry' relative to the two chosen reference frames..

We could be generalize this further. By not just consider points that have moved apart by a speed v, and utilizing results of Chapter 4, we obtain a similar symmetry for any two points (t_1, x_1) and (t_2, x_2) in the (time, space) room satisfying

$$t_2 = \frac{t_1 - \frac{v}{c^2}x_1}{\sqrt{1-(\frac{v}{c})^2}}$$

$$x_2 = \frac{vt_1 - x_1}{\sqrt{1-(\frac{v}{c})^2}}$$

As we see, these relations are *almost* identical to the Lorentz transformation. We could also now draw a line between these two points to specify points of 'simultaneity by symmetry'. But the formulas are not particularly attractive (unless $x_1 = 0$).

However, the travelling twin example suggests that the concept could be fruitful in some applications, and such simultaneity considerations might supplement the pure comparison of clock readings.

7 Conclusions

We use the Lorentz transformation to discuss a number of standard results on time dilation between two reference frames moving relative to each other at constant speed v. Further assumptions are:

- There is a complete *symmetry* between the two reference frames, and we synchronize all clocks on the same reference frame.
- We do not utilize any definition of *simultaneity* across systems. The approach restricts to explore direct comparisons of clocks being at the same location at the same time. (Chapter 6 makes an extension.)
- We do not use the expression '*as seen'* (from the other reference frame). Observers (observational equipment) on both reference frames agree on the time readings; as they are carried out 'on location'.
- We specify the applied *observational principle*, *i.e.* the location of the clocks that are used for time comparisons. Thus, we focus on how observed time, t', on the 'other' '(secondary') system depends on the position, x on the 'primary' reference frame. It is argued that one should look at the total picture, taking all information into account, as provided by the Lorentz transformation
- We do not describe the phenomenon as 'moving clock goes slower'. It seems irrelevant which of the two reference frames we consider to be moving, as it is rather the observational principle that matters.

Thus, we stress the fact that at a given time, t on K, the time, t' observed on K' will depend on the position, x on K. The usual special cases are treated, *e.g.* the standard result, $t' = t\sqrt{1-(v/c)^2}$. Further, we discuss the result that if we at a time t choose the midpoint between $x = 0$ and $x' = 0'$ as the location for time comparison, then we will at this location observe $t' = t$. This choice represents an observational principle being symmetric with respect to the two reference frames. So if we observe $t \neq t'$, in an otherwise symmetric situation, we could claim that this is caused by the asymmetry of the chosen observational principle.

So an observer moving relative to a reference frame where the actual event takes place might be a rather 'unreliable' observer regarding time. The various observational principles will provide him with different results; so one should be careful to let such an 'outside' observer define the phenomenon, (without properly considering his position).

We obtain essentially the same result when we compare time *differences* (length of time intervals) on the two reference frames. Again, the observational principle is crucial, and provides a variety of expressions for time dilation.

Now, under the conditions of having strict symmetry in all respects it would be rather meaningless to claim a 'true' time dilation, causing different ageing on the two systems. So it should be interesting to identify the conditions – in particular departures from symmetry - that would cause time dilation to represent such a physical reality.

In view of this symmetry we include a separate discussion on the travelling twin case. The present paper claims that the observational principle is essential to explain the phenomenon. As the standard example goes, the travelling twin will – at a speed of $0.6{\cdot}c$ – age only 8 years during his trip, as opposed to the 10 years passed on the earth. By taking an overall view of the situation, we argue that due to the length contraction, the trip actually takes just 8 years, and the earthbound twin has aged the same number of years (8).

As part of this argumentation, we introduce the concept 'simultaneity by symmetry'. This applies for events that actually are not simultaneous in any reference frame, *i.e.* the clock readings of synchronized clocks are different. However, considering the interaction of two specific reference frames, these events exhibit such a symmetry that we might refer to a kind of 'simultaneity'.

The main results here are a direct consequence of the Lorentz transformation, and are probably well known. However, I believe that the presentation on time dilation, as provided here, has some distinct differences, as compared to the current narratives on this topic.

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