Triangle Inscribed-Triangle Picking

Arman Maesumi

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Abstract. Given a triangle $ABC$, the average area of an inscribed triangle $RST$ whose vertices are uniformly distributed on $BC, CA$ and $AB$, is proven to be one-fourth of the area of $ABC$. The average of the square of the area of $RST$ is shown to be one-twelfth of the square of the area of $ABC$, and the average of the cube of the ratio of the areas is $5/144$. A Monte Carlo simulation confirms the theoretical results, as well as a Maxima program which computes the exact averages.

Keywords. Geometric probability, computational geometry, triangle triangle picking.

1 Introduction

In 1865, Professor James Joseph Sylvester proved [1] that the average area of a random triangle, whose vertices are picked inside of a given triangle with area $A$, is equal to $A/12$. This problem, originally proposed by S. Watson, and known as Triangle Triangle Picking, is one of the earliest examples of Geometric Probability. Many similar problems have been proposed [2,3], including Sylvester’s own four-point problem which asks for the probability that the convex hull of four random points is a triangle. Here we study a sub-class of such problems, where the interior polygon has its vertices on the edges of the base polygon.

2 An Application of Barycentric Coordinates

Suppose the vertices of a triangle are denoted by the vectors $\vec{A}, \vec{B}, \vec{C}$. The barycentric coordinates [4] of a point $\vec{P}$, with respect to the triangle $ABC$, is $(\alpha, \beta, \gamma)$ if $\vec{P} = \alpha \vec{A} + \beta \vec{B} + \gamma \vec{C}$, and $\alpha + \beta + \gamma = 1$.

Bottema’s Theorem [5]: Assume the vertices $P_i$ of a triangle $P_1P_2P_3$ have barycentric coordinates $(x_i, y_i, z_i)$, with respect to the triangle $ABC$ then,

\[
\text{area}(P_1P_2P_3) = \det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \text{area}(ABC).
\]
Theorem: Given a triangle $ABC$, if three points $R, S$, and $T$ are chosen randomly, on the faces $AB, BC, CA$ then the average of area of $RST$ is one-fourth of the area of $ABC$.

Proof: Consider an inscribed triangle whose vertices $R, S, T$, are defined by

$$
\begin{align*}
\vec{T} &= \vec{A} + t \vec{AB} \\
\vec{R} &= \vec{B} + r \vec{BC} \\
\vec{S} &= \vec{C} + s \vec{CA}
\end{align*}
$$

(1)

where $r, s, t$ are random numbers in $[0, 1]$.

In this case, the points $R, S, T$ are respectively given by barycentric coordinates $(t, 1 - t, 0), (0, r, 1 - r)$, and $(1 - s, 0, s)$. Therefore, the corresponding determinant is,

$$f(r, s, t) = \det \begin{bmatrix} r & 1 - r & 0 \\ 0 & s & 1 - s \\ 1 - t & 0 & t \end{bmatrix} = rst + (1 - r)(1 - s)(1 - t).$$

Hence, by Bottema’s theorem, $\text{area}(RST) = f(r, s, t)\text{area}(ABC)$. The average value of $rst$, and $(1 - r)(1 - s)(1 - t)$ can be represented as the product of the averages of $r, s, t$ which is equivalent to $(\frac{1}{2})^3$. Therefore,

$$\langle f(r, s, t) \rangle = \langle (rst + (1 - r)(1 - s)(1 - t)) \rangle = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{1}{4};$$

As a result $\langle \text{area}(RST) \rangle = \text{area}(ABC)/4$.

2.1 Computation of the 2nd and 3rd Moment

To calculate the second moment of $f(r, s, t)$, we expand $f^2$ as,

$$f^2(r, s, t) = [rst + (1 - r)(1 - s)(1 - t)]^2
= r^2s^2t^2 + (1 - r)^2(1 - s)^2(1 - t)^2 + 2r(1 - r)s(1 - s)t(1 - t).$$

(2)

Using $\langle r^2 \rangle = \frac{1}{3}, \langle (1 - r)^2 \rangle = \frac{1}{3}$, and $\langle r(1 - r) \rangle = \frac{1}{6}$, the mean value of $f^2(r, s, t)$ is equal to,

$$\langle f^2(r, s, t) \rangle = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{6}\right)^3 = \frac{1}{12}.$$

Therefore, the average square of the area of triangle $RST$ is equal to one-twelfth of the square of the area of $ABC$.

Similarly, to calculate the third moment of $f$,

$$f^3(r, s, t) = r^3s^3t^3 + 3r(1 - r)^2s(1 - s)^2t(1 - t)^2 + 3r^2(1 - r)s^2(1 - s)t^2(1 - t) + (1 - s)^3(1 - r)^3(1 - t)^3$$

(3)
\[
\langle f^3(r,s,t) \rangle = \left(\frac{1}{4}\right)^3 + 3 \left(\frac{1}{3} - \frac{1}{4}\right)^3 + 3 \left(\frac{1}{3} - \frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^3 = \frac{5}{144}.
\]
Therefore, \(\langle \text{area}^3(\text{RST}) \rangle\) is equal to \(\frac{5}{144}\text{area}^3(\text{ABC})\).

3 A Monte Carlo Simulation

A Monte Carlo simulation can be done to approximate the ratio of the average area of \(\text{ABC}\) and randomly generated inscribed triangles. The following program was written in Java to approximate this ratio.

```java
public static void averageArea(int iterations, int power) {
    Random rand = new Random();
    double area = 0, totalArea = 0;
    double r, s, t;
    double Ax, Ay, Bx, By, Cx, Cy;
    Triangle baseTri = new Triangle(
        new Point(rand.nextInt(10), rand.nextInt(10)),
        new Point(rand.nextInt(10), rand.nextInt(10)),
        new Point(rand.nextInt(10), rand.nextInt(10)));
    Point A, B, C;
    for (int i = 0; i < iterations; i++) {
        r = rand.nextDouble();
        s = rand.nextDouble();
        t = rand.nextDouble();
        Ax = baseTri.getPoint(1).getX() + (s *
            (baseTri.getPoint(2).getX() - baseTri.getPoint(1).getX()));
        Ay = baseTri.getPoint(1).getY() + (s *
            (baseTri.getPoint(2).getY() - baseTri.getPoint(1).getY()));
        A = new Point(Ax, Ay);
        Bx = baseTri.getPoint(2).getX() + (t *
            (baseTri.getPoint(0).getX() - baseTri.getPoint(2).getX()));
        By = baseTri.getPoint(2).getY() + (t *
            (baseTri.getPoint(0).getY() - baseTri.getPoint(2).getY()));
        B = new Point(Bx, By);
        Cx = baseTri.getPoint(0).getX() + (r *
            (baseTri.getPoint(1).getX() - baseTri.getPoint(0).getX()));
        Cy = baseTri.getPoint(0).getY() + (r *
            (baseTri.getPoint(1).getY() - baseTri.getPoint(0).getY()));
        C = new Point(Cx, Cy);
        area = new Triangle(C, A, B).area();
        totalArea += Math.pow(area, power);
    }
}
```
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```java
double averageArea = totalArea / iterations;
System.out.println(averageArea);
double ratio = averageArea / Math.pow(baseTri.area(), power); //The ratio of the random area^p to the base triangle's area^p.
System.out.println(ratio);
```

The supporting classes are included below (Triangle.java, Point.java).

### 3.1 Monte Carlo Convergence

<table>
<thead>
<tr>
<th>Iterations</th>
<th>$10^2$</th>
<th>$10^3$</th>
<th>$10^4$</th>
<th>$10^5$</th>
<th>$10^6$</th>
<th>$10^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio</td>
<td>$\frac{1}{4}$</td>
<td>0.0033</td>
<td>-0.0005</td>
<td>-0.0006</td>
<td>-0.0009</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

![Ratio $\frac{1}{4}$ vs. log(Iterations)](image)

### 4 Exact Results from Maxima

Using the following Maxima program, the exact results were computed for $\langle f_1(r,s,t) \rangle, \langle f_2(r,s,t) \rangle, \ldots, \langle f_{10}(r,s,t) \rangle$

```maxima
for n from 1 thru 10 do(
    f(r,s,t) := ((r*s*t) + (1-r)*(1-s)*(1-t))^n,
a : integrate(integrate(integrate(f(r,s,t), t, 0, 1), s, 0, 1), r, 0, 1),
    print(a));
```
5 General Formula for $f^n(r, s, t)$

To be continued...

6 Asymptotic Behavior of $f^n(r, s, t)$

To be continued...

References


