The Q-Naturals: A Recursive Arithmetic Which Extends the "Standard" Model

By Wes Hansen

Abstract. In what follows we develop foundations for a set of non-standard natural numbers we call q-naturals, where q stands for quanta, by the recursive generation of reflexive sets. From the practical perspective, these q-naturals correspond to ordered pairs of natural numbers with the lexicographic ordering, hence, they are isomorphic to ω^2. In addition, we demonstrate a novel definition of the arithmetical operation, multiplication, which turns out to be recursive. This operation, together with lexicographic order and coordinate-wise addition, defines an arithmetical structure which extends the "standard" model but yet has a recursive order relation and recursive arithmetical operations defined on the entire domain.

1. Introduction. In his paper, "On Non-Standard Models of Peano Arithmetic and Tennenbaum's Theorem," [SR] Samuel Reid briefly covers the history of Peano Arithmetic (PA) up to and including Tennenbaum's Theorem. Tennenbaum's Theorem demonstrates that the so-called "standard" model of PA is the only recursive model of PA possible. Furthermore, as a careful reading of [SR] and the related, "Tennenbaum's Theorem," [PS] by Peter Smith would seem to indicate, it has been a long and widely held belief within the mathematical community, that the Peano Axioms capture the only recursive model of arithmetic possible, period; any model of arithmetic, Peano or otherwise, is either isomorphic to PA or is not recursive, where "or" here is exclusive. In this paper we construct an arithmetical structure which is not isomorphic to PA but yet has a recursive relation, $<_q$, and recursive arithmetic operations, $+_q$ and $*_q$, defined on it, and we suggest that the "definition" of a recursive L_A-structure (Definition 9 in [SR]) be updated to reflect this development.

In a subsequent work, "The Q-Universe: A Set Theoretic Construction," [WH] we show how to generalize this relation and these operations in countably many straightforward ways leading to a countable subsumption hierarchy of recursive arithmetics. If you represent the domain of each structure in the hierarchy with an ordinal, then the exponents of the domains correspond to the geometric sequence $\{2^n\}$. In particular, the "standard" model is simply the zeroeth-order structure in the hierarchy.

The present work was motivated by discussion on a blog post of John Baez, "Computing the Uncomputable," [JB] which I read while working through, "The Liar: An Essay in Truth and Circularity," [BE] by Jon Barwise and John Etchemendy. Specifically, it was motivated by the blog discussion and the discussion in [BE] following Exercise 25 of Chapter 3. In Exercise 25, the authors define the well-founded ordinals inductively and then instruct the reader to use co-induction to define the largest fixed point and to show that Ω is a member; in the discussion immediately following, they say: "Thus one might consider the set Ω a hyperordinal. However, this is a good example of a case where one would want to use the inductive definition, since the point of defining the ordinals is as representatives of well-orderings. Hyperordinals like Ω are of no use for such purposes."

In our development of foundations, we formally extend the set of standard natural numbers to a set of q-natural numbers, where q stands for quanta, by introducing a one-place operation we call the hyperloop, which, when applied to any set, generates that set's unique reflexive set (hyperset); we apply this operator recursively in such a manner that it generates countably many representations for the one same set, where these representations exist in a state of perfect symmetry. We then break this symmetry by imposing an order, the lexicographic ordering, generating a countable and meaningful hierarchy of distinct elements - hyperordinals. These elements, of course, are existent in between any two standard von Neumann ordinals. This provides our foundation.

For practical reasons, the set of q-naturals, N_Q, is then interpreted as ordered pairs of standard natural numbers and the properties of these numbers follow, in a natural and straightforward way, from the properties of standard natural numbers. Lexicographic order and coordinate-wise addition are certainly other than novel and consistency constraints lead immediately to the only "multiplication" possible, hence, the recursive arithmetical operations, $+_q$ and $*_q$.

And of course, the q-naturals can be extended to the q-integers, the q-integers to the q-rationals, the q-rationals, using Dedekind Cuts, to the q-reals, and the q-reals to the q-complex, a project undertaken in the subsequent paper [WH] already mentioned.

Notation. We use the standard notation together with:

@	|	a one-place non-logical symbol called the hyperloop
I_H	|	a hyper-inductive set
I_Q	|	a q-inductive set
N_Q	|	the set of all q-naturals

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2. Definitions. We define our mathematical entities using standard terminology:

Definition 2.01. A set is reflexive if $X = \{X\}$; a reflexive set is called a hyperset.[BE]

Definition 2.02. A one-place operation, @, when applied to any set X, generates a reflexive set; this operation, called a hyperloop, can be applied recursively.

Definition 2.03. Let X be an arbitrary von Neumann ordinal, then $@^nX$ designates the recursive application of @ to X "n" times, where $n \in N$; specifically, $@^0X$, the zeroth-order application, is identical to no application, i.e. $@^0X = X$.

Definition 2.04. Hypersets have two distinct successor functions; let $@^nX$ be an arbitrary hyperset, then $S(@^nX) = @(@^nX) = @^{(n+1)}X$, while $@^nS(X) = @^n(X \cup \{X\}) = @^n(X + 1)$ (reference [HJ], Chapter 3, pages 40 and 41).

Definition2.05. $\phi = 0.0$, $@\phi = 0.1$, $@^2\phi = 0.2$, … , $@^{(\omega-2)}\phi = 0.(\omega - 2)$, $@^{(\omega-1)}\phi = 0.(\omega - 1)$, $@^{\omega}\phi = 0.\omega$, $\{@^{\omega}\phi\} = 1.0$, $@\{@^{\omega}\phi\} = 1.1$, $@^2\{@^{\omega}\phi\} = 1.2$, … , $@^{(\omega-2)}\{@^{\omega}\phi\} = 1.(\omega - 2)$, $@^{(\omega-1)}\{@^{\omega}\phi\} = 1.(\omega - 1)$, $@^{\omega}\{@^{\omega}\phi\} = 1.\omega$, … , $\{@^{\omega}\phi, \{@^{\omega}\phi\}\} = 2$, $@\{@^{\omega}\phi, \{@^{\omega}\phi\}\} = 2.1$, $@^2\{@^{\omega}\phi, \{@^{\omega}\phi\}\} = 2.2$, …

Definition2.06. ϕ and any number $@^{\omega}X$ are examples of base elements; any number $@^nX$, where $n < \omega$, is an example of a hyper-element; for example, ϕ is the only base element of $@^{\omega}\phi$ and for any $m \in N_Q$, $m > @^{\omega}\phi$, $m = @^k\{0.\omega, 1.\omega, \dots, n.\omega\}$, where n is some von Neumann ordinal, and every $x.\omega$ is a base element of m, while every $@^p\{0.\omega, 1.\omega, \dots, n.\omega\}$, $p \in [0, k]$, is a hyper-element of m.

Definition 2.07. A set, I_H, is hyper-inductive if:

1. $\phi \in I_H$;
2. if $X \in I_H$, then $S(X) \in I_H$;
3. if $X \in I_H$, then $@X \in I_H$;
4. if $@X \in I_H$, then $S(@X) \in I_H$.

Definition 2.08. Consistent with Definition 2.05, a q-natural number is an ordered pair of natural numbers, (a, b), such that (a, b) = a_b.

Definition 2.09. Consistent with Definition 2.04, any q-natural number, a_b, has two distinct successor functions which can be applied independently or in conjunction; specifically, S(a)_b = (a U {a})_b = (a + 1)_b and a_S(b) = a_(b U {b}) = a_(b + 1) (reference [HJ], Chapter 3, page 52).

Definition 2.10. A set, I_Q, is q-inductive if:

1. $0\_0 \in I_Q$;
2. if $a\_b \in I_Q$, then, $S(a)\_b \in I_Q$;
3. if $a\_b \in I_Q$, then, $a\_S(b) \in I_Q$.

Definition 2.11. The set of all q-natural numbers is the set

$N_Q = \{ x \mid x \in I_Q \text{ for every q-inductive set } I_Q\}$

Definition 2.12. The relation $=_q$ (equality) on N_Q is defined by:

For all a_b, c_d $\in N_Q$, a_b $=_q$ c_d iff $(a = b) \wedge (c = d)$, where =(a, b) is the natural equality (references [HJ], Chapter 3, pages 39 – 40 and [CD], Chapter 2, page 44).

Definition 2.13. The relation $<_q$ on N_Q is defined by:

For all a_b, c_d $\in N_Q$, a_b $<_q$ c_d iff $(a < c) \vee [(a = c) \wedge (b < d)]$, where <(a, b) is the natural order (reference [HJ], Chapter 3 , page 42), =(a, b) is the natural equality (references [HJ], Chapter 3, pages 39 – 40 and [CD], Chapter 2, page

44), and $<_q$(a_b, c_d) is the lexicographic order (reference [HJ], Chapter 4, page 81).

Definition 2.14. The relation $\leq_q$ on N_Q is defined by:

For all a_b, c_d ϵ N_Q, a_b $\leq_q$ c_d iff (a_b $<_q$ c_d) $^\vee$ (a_b $=_q$ c_d).

Definition 2.15. The operation $+_q$ (addition) on N_Q is defined by:

For all a_b, c_d ϵ N_Q, a_b $+_q$ c_d $=_q$ (a + c)_(b + d), where +(a, b) is as defined on the set of natural numbers (reference [HJ], Chapter 3, page 52).

Definition 2.16. The operation $*_q$ (multiplication) on N_Q is defined by:

For all a_b, c_d ϵ N_Q, a_b $*_q$ c_d $=_q$ (a * c)_[(b * c) + (a * d) + (b * d)],

where *(a,c) and +(b, d) are both as defined on the set of natural numbers (reference [HJ], Chapter 3, page 54).

3.Arguments. We demonstrate our arguments using the standard methods and terminology of mathematical logic and ZFC/AFA or generalizations thereof. Specific to the current work, we generalize the Principle of Induction to the Principle of q-Induction and we reproduce certain arguments, verbatim, from reference [HJ].

Theorem 3.01. *The Axiom of Anti-Foundation implies that there exists a unique reflexive set.*

Proof. This theorem is reproduced verbatim from, “Introduction to Set Theory,”[HJ] by Karel Hrbacek and Thomas Jech (Chapter 14, page 263) and the proof can be found therein, as desired. □

Theorem 3.02. *A hyper-inductive set, I_H, defined by Definition 2.07, exists.*

Proof. Let I be an arbitrary set satisfying properties “1” and “2” of Definition 2.07, then I is an inductive set (reference [HJ], Chapter3, page 40) and, by the Axiom of Infinity, I exists. Let K be a family of intervals, [n, n + 1), such that n ϵ I and [n, n + 1) satisfies properties “3” and “4” of Definition 2.07. Let [n, n + 1) be an arbitrary element of K, then, by the Axiom of Infinity and Theorem 3.01, [n, n + 1) exists. Since [n, n + 1) was arbitrary, every [n, n + 1) ϵ K exists, hence, K exists. Finally, by the Axiom of Union, UK = I_H exists, as desired. □

Theorem 3.03. *For any hyper-inductive set, I_H, and any X ϵ I_H, X can be represented as an ordered pair of natural numbers, (a, b), such that (a, b) = a.b.*

Proof. This follows immediately from Definition 2.03, 2.04, 2.05, and the properties of natural numbers, (reference [HJ], Chapter3), as desired. □

Theorem 3.04. *A q-inductive set, I_Q, defined by Definition 2.10, exists.*

Proof. This is a direct consequence of a number of facts about the set of natural numbers,N:

1. N exists and is inductive (reference [HJ], Chapter 3, page 41);

2. By the Axiom of Power Set, the power set of N exists (reference [HJ], Chapter 1, page 10);
3. By the definition of ordered pair (reference [HJ], Chapter 2, page 17) and the definition of cartesian product (reference [HJ], Chapter 2, page 21), N x N exists;

together with Definition 2.08, 2.09, and 2.10, as desired. □

Theorem 3.05. *The set, N_Q, defined by Definition 2.11 exists and is q-inductive.*

Proof. Let X be the family of all q-inductive sets I_Q, then, by the Axiom of Union, the set UX exists and, by Definition 2.10, UX is q-inductive. By Definition 2.11, UX contains N_Q, hence, N_Q exists and is q-inductive, as desired. □

Theorem 3.06. *(The Principle of Q-Induction) Let P(x_y) be a property and assume that:*

1. *P(0_0) is true;*
2. *for all n_k ϵ N_Q, P(n_k) → P[(n + 1)_k] ^ P[n_(k + 1)].*

Then P holds for all q-natural numbers n_k.

Proof. By Definition 2.10, "1" and "2" above define a q-inductive set I_Q. By Definition 2.11, that set, I_Q, contains N_Q, as desired. □

Lemma 3.07. *For all a_b ϵ N_Q, a, b ϵ N.*

Proof. This follows immediately from Definition 2.10, Theorem 3.05, and the fact that N is inductive (reference [HJ], Chapter 3, page 41), as desired. □

Theorem 3.08. *(N, <) is a linearly ordered set.*

Proof. This theorem is reproduced verbatim from reference [HJ] (Chapter 3, page 43) and the proof can be found therein, as desired. □

Lemma 3.09. *For all a_b, c_d ϵ N_Q:*

1. *0_0 $\leq_q$ c_d;*
2. *a_b $<_q$ c_(d + 1) iff a_b $\leq_q$ c_d.*

Proof. The proof is in two parts:

1) We proceed by q-induction. Let P(x_y) be the property, "0_0 $\leq_q$ x_y," then:

P(0_0). By Definition 2.12, 0_0 $=_q$ 0_0, hence, by Definition 2.14, 0_0 $\leq_q$ 0_0.

Suppose P(n_k) is true, then (0_0 $<_q$ n_k) ˅ (0_0 $=_q$ n_k) and:

P[(n + 1)_k] ^ P[n_ (k + 1)]. In both cases, by Lemma 3.07 and Theorem 3.08, [0_0 $<_q$ (n + 1)_k] ^ [0_0 $<_q$ n_(k + 1)].

Therefore, P(n_k) → P[(n + 1)_k] ^ P[n_(k + 1)] and, by the Principle of Q-Induction, for all n_k ϵ N_Q, 0_0 $\leq_q$ n_k, as desired. □

2) Suppose $a\_b <_q c\_(d + 1)$, then, by Definition 2.13, $(a < c) \vee [(a = c) \wedge (b < (d + 1))]$. If $a < c$, then, by Definition 2.13, $a\_b <_q c\_d$; otherwise, if $(a = c) \wedge [b < (d + 1)]$, then, by Lemma 3.07 and Theorem 3.08, $a\_b \leq_q c\_d$.

   In both cases $a\_b \leq_q c\_d$, hence, $a\_b <_q c\_(d + 1) \rightarrow a\_b \leq_q c\_d$.

   Suppose $a\_b \leq_q c\_d$, then, by Definition 2.13, $\{(a < c) \vee [(a = c) \wedge (b < d)]\} \vee [(a = c) \wedge (b = d)]$ and three cases arise:

   Case 1. Suppose $a < c$, then, by Definition 2.13, $a\_b <_q c\_(d + 1)$.

   Case 2. Suppose $(a = c) \wedge (b < d)$, then, by Lemma 3.07 and Theorem 3.08, $a\_b <_q c\_(d + 1)$.

   Case 3. Suppose $(a = c) \wedge (b = d)$, then , by Lemma 3.07 and Theorem 3.08, $a\_b <_q c\_(d + 1)$.

   In all three cases $a\_b <_q c\_(d + 1)$, hence, $a\_b \leq_q c\_d \rightarrow a\_b <_q c\_(d + 1)$.

   Therefore, $a\_b <_q c\_(d + 1)$ iff $a\_b \leq_q c\_d$, as desired.□

Theorem 3.10. *$(N_Q, <_q)$ is a linearly ordered set.*

Proof. The proof is in three parts:

1) *Transitivity.* Let $k\_p, m\_q, n\_r \in N_Q$ be arbitrary but such that $(k\_p <_q m\_q) \wedge (m\_q <_q n\_r)$. Then, by Definition 2.13, $(k < m) \vee [(k = m) \wedge (p < q)]$ and $(m < n) \vee [(m = n) \wedge (q < r)]$ and four cases arise:

   Case 1. Suppose $(k < m) \wedge (m < n)$, then, by Lemma 3.07 and Theorem 3.08, $k < n$, and, by Definition 2.13, $k\_p <_q n\_r$.

   Case 2. Suppose $(k < m) \wedge (m = n) \wedge (q < r)$, then, by Lemma 3.07 and Theorem 3.08, $k < n$, and, by Definition 2.13, $k\_p <_q n\_r$.

   Case 3. Suppose $(k = m) \wedge (p < q) \wedge (m < n)$, then, by Lemma 3.07 and Theorem 3.08, $k < n$, and, by Definition 2.13, $k\_p <_q n\_r$.

   Case 4. Suppose $(k = m) \wedge (p < q) \wedge (m = n) \wedge (q < r)$, then, by Lemma 3.07 and Theorem 3.08, $(k = n) \wedge (p < r)$, and, by Definition 2.13, $k\_p <_q n\_r$.

   In all four cases, $k\_p <_q n\_r$, hence, $(k\_p <_q m\_q) \wedge (m\_q <_q n\_r) \rightarrow k\_p <_q n\_r$.
2) *Asymmetry.* Let $k\_p, m\_q \in N_Q$ be arbitrary and suppose, for contradiction, that $(k\_p <_q m\_q) \wedge (m\_q <_q k\_p)$, then, by transitivity, $k\_p <_q k\_p$, contradicting Definition 2.13.
3) *Linearity.* We proceed by q-induction. Let $P(x\_y)$ be the property, "for all $m\_p \in N_Q$, $(m\_p <_q x\_y) \vee (m\_p =_q x\_y) \vee (x\_y <_q m\_p)$," then:

   P(0_0). This is an immediate consequence of Lemma 3.09.

Suppose $P(n\_k)$ is true, then for all $m\_p \in N_Q$, $(m\_p <_q x\_y) \vee (m\_p =_q x\_y) \vee (x\_y <_q m\_p)$ and: *$P[(n + 1)\_k] \wedge P[n\_(k + 1)]$.* There are three cases to consider:

Case 1. Suppose $m\_p <_q n\_k$, then, by Lemma 3.07, Theorem 3.08, and Definition 2.13, $[n\_k <_q (n + 1)\_k] \wedge [n\_k <_q n\_(k + 1)]$, hence, by transitivity, $[m\_p <_q (n + 1)\_k] \wedge [m\_p <_q n\_(k + 1)]$.

Case 2. Suppose $m\_p =_q n\_k$, then, by Lemma 3.07, Theorem 3.08, and Definition 2.13, $[m\_p <_q (n + 1)\_k] \wedge [m\_p <_q n\_(k + 1)]$.

Case 3. Suppose $n\_k <_q m\_p$, then, by Definition 2.13, $(n < m) \vee [(n = m) \wedge (k < p)]$ and two cases arise:

Case 3a. Suppose $n < m$, then, by Lemma 3.07 and Theorem 3.08, $[(n + 1) < m] \vee \{[(n + 1) = m] \wedge [((k + 1) < p) \vee ((k + 1) = p) \vee (p < (k + 1))]\}$ and four cases arise:

Case 3a.1. Suppose $(n + 1) < m$, then, by Lemma 3.07, Theorem 3.08, and Definition 2.13, $[(n + 1)\_k <_q m\_p] \wedge [n\_(k + 1) <_q m\_p]$.

Case 3a.2. Suppose $[(n + 1) = m] \wedge [(k + 1) < p]$, then, by Lemma 3.07, Theorem 3.08, and Definition 2.13, $[(n + 1)\_k <_q m\_p] \wedge [n\_(k + 1) <_q m\_p]$.

Case 3a.3. Suppose $[(n + 1) = m] \wedge [(k + 1) = p]$, then, by Lemma 3.07, Theorem 3.08, and Definition 2.13, $[(n + 1)\_k <_q m\_p] \wedge [n\_(k + 1) <_q m\_p]$.

Case 3a.4. Suppose $[(n + 1) = m] \wedge [p < (k + 1)]$, then, by Lemma 3.07, Theorem 3.08, and Definition 2.13, $[(n + 1)\_k <_q m\_p] \wedge [n\_(k + 1) <_q m\_p]$.

In all four cases, $[((n + 1)\_k <_q m\_p) \wedge (n\_(k + 1) <_q m\_p)] \vee [(m\_p \leq_q (n + 1)\_k) \wedge (n\_(k + 1) <_q m\_p)]$, hence, $(n < m) \rightarrow \{[((n + 1)\_k <_q m\_p) \vee ((n + 1)\_k =_q m\_p) \vee (m\_p <_q (n + 1)\_k)] \wedge [(n\_(k + 1) <_q m\_p) \vee (n\_(k + 1) =_q m\_p) \vee (m\_p <_q n\_(k + 1))]\}$.

Case 3b. Suppose $(n = m) \wedge (k < p)$, then, by Lemma 3.07 and Theorem 3.08, $m < (n + 1)$ and $(k + 1) \leq p$, and, by Definition 2.13, $[m\_p <_q (n + 1)\_k] \wedge [(n\_(k + 1) <_q m\_p) \vee (n\_(k + 1) =_q m\_p)]$.

In both cases, $[((n + 1)\_k <_q m\_p) \vee ((n + 1)\_k =_q m\_p) \vee (m\_p <_q (n + 1)\_k)] \wedge [(n\_(k + 1) <_q m\_p) \vee (n\_(k + 1) =_q m\_p) \vee (m\_p <_q n\_(k + 1))]$, hence, $(n\_k <_q m\_p) \rightarrow \{[((n + 1)\_k <_q m\_p) \vee ((n + 1)\_k =_q m\_p) \vee (m\_p <_q (n + 1)\_k)] \wedge [(n\_(k + 1) <_q m\_p) \vee (n\_(k + 1) =_q m\_p) \vee (m\_p <_q n\_(k + 1))]\}$.

Therefore, $P(n\_k) \rightarrow P[(n + 1)\_k] \wedge P[n\_(k + 1)]$ and, by the Principle of Q-Induction, linearity.

Therefore, $(N_Q, <_q)$ is a linearly ordered set, as desired. □

Theorem 3.11. *$(N_Q, <_q)$ is a well-ordered set.*

Proof. This is an immediate consequence of Lemma 3.07, Theorem 3.08, and Lemma 3.09, as desired. □

Theorem 3.12. *(N_Q, $<_q$) is isomorphic to ω^2.*

Proof. Let Y = {S_i | i ϵ N} = ran S for some index function S, where each S_i is the set of natural numbers. Then ω^2 = { a_i | [a_i ϵ S_i ϵ Y] ^ [for all i,j,a,b ϵ N, $a_i < b_j$ iff (i < j) ˅ [(i = j) ^ (a < b)]]} and there is an obvious isomorphism, f:ω^2 → (N_Q, $<_q$), defined by f(a_i) = i_a, as desired. □

Theorem 3.13. *There is a unique function, $+_q$:N_Q x N_Q → N_Q, such that:*

1. *$+_q$(m_p, 0_0) $=_q$ m_p, for all m_p ϵ N_Q;*
2. *$+_q$(m_p, n_q + 1_0) $=_q$ $+_q$(m_p, n_q) +1_0, for all m_p, n_q ϵ N_Q.*

Proof. In the parametric version of the Recursion Theorem (reference [HJ], Chapter 3, page 51), let a:N_Q → N_Q be the identity function, and let g:N_Q x N_Q x N_Q → N_Q be defined by g(k_p, m_q, n_r) $=_q$ m_q $+_q$ 1_0, for all k_p, m_q, n_r ϵ N_Q. Then, by the Recursion Theorem, there exists a unique function, f:N_Q x N_Q → N_Q, such that:

1. f(k_p, 0_0) $=_q$ a(k_p) $=_q$ k_p, for all k_p ϵ N_Q;
2. f(k_p, m_q $+_q$ 1_0) $=_q$ g(k_p, f(k_p, m_q), m_q) $=_q$ f(k_p, m_q) $+_q$ 1_0, for all k_p, m_q ϵ N_Q.

Let $+_q$ = f, as desired. □

Theorem 3.14. *There is a unique function, $*_q$:N_Q x N_Q → N_Q, such that:*

1. *$*_q$(m_p, 0_0) $=_q$ 0_0, for all m_p ϵ N_Q;*
2. *$*_q$(m_p, n_q + 1_0) $=_q$ $*_q$(m_p, n_q) $+_q$ m_p, for all m_p, n_q ϵ N_Q.*

Proof. In the parametric version of the Recursion Theorem (reference [HJ], Chapter 3, page 51), let a:N_Q → N_Q be the constant function defined by a(m_p) $=_q$ 0_0, for all m_p ϵ N_Q, and let g:N_Q x N_Q x N_Q → N_Q be defined by g(k_p, m_q, n_r) $=_q$ m_q $+_q$ n_r, for all k_p, m_q, n_r ϵ N_Q. Then, by the Recursion Theorem, there exists a unique function, f: N_Q x N_Q → N_Q, such that:

1. f(k_p, 0_0) $=_q$ a(k_p) $=_q$ 0_0, for all k_p ϵ N_Q;
2. f(k_p, m_q $+_q$ 1_0) $=_q$ g(k_p, f(k_p, m_q), m_q) $=_q$ f(k_p, m_q) $+_q$ m_q, for all k_p, m_q ϵ N_Q.

Let $*_q$ = f, as desired. □

Theorem 3.15. *If (W_1, $<_1$) and (W_2, $<_2$) are well-ordered sets, then exactly one of the following holds:*

1. *either W_1 and W_2 are isomorphic; or*
2. *W_1 is isomorphic to an initial segment of W_2; or*
3. *W_2 is isomorphic to an initial segment of W_1.*

In each case, the isomorphism is unique.

Proof. This theorem is reproduced verbatim from reference [HJ] (Chapter 6, pages 105 and 106) and the proof can be found therein, as desired. □

Theorem 3.16. *The relation $=_q$ of Definition 2.12 is recursive.*

Proof. This follows immediately from the fact that the natural equality relation is recursive (reference [ER]) and the fact that the logical "and" connective is recursive (reference [SB], Chapter 13, page 89), as desired. □

Theorem 3.17. *The relation $<_q$ of Definition 2.13 is recursive.*

Proof. This follows immediately from the fact that the natural order relation is recursive (reference [OR]), from the fact that the natural equality relation is recursive (reference [ER]), from the fact that the logical "or" connective is recursive (reference [SB], Chapter 13, page 89), and from the fact that the logical "and" connective is recursive (reference [SB], Chapter 13, page 89), as desired. □

Theorem 3.18. *The operation $+_q$ of Definition 2.15 is recursive.*

Proof. This follows immediately from Theorem 3.13, as desired. □

Theorem 3.19. *The operation $*_q$ of Definition 2.16 is recursive.*

Proof. This follows immediately from Theorem 3.14, as desired. □

Theorem 3.20. *The structure K = {N_Q, $<_q$, $+_q$, $*_q$, 1_0, 0_0} is an extension of the "standard" model N' = {N, <, +, *, 1, 0}.*

Proof. Let the inclusion map, i:N' → K, be defined by i(x) = x_0, then, by Theorem 3.12, i is an injective embedding of N' into K (reference [SR], page 4), hence, K is an extension of N' (reference [SR], page 7), as desired. □

Theorem 3.21. *The structure K = {N_Q, $<_q$, $+_q$, $*_q$, 1_0, 0_0} is not isomorphic to the "standard" model N' = {N, <, +, *, 1, 0}.*

Proof. Let Y be the closed/open interval of N_Q, [0_0, 1_0), then Y is an initial segment of (N_Q, $<_q$) (reference [HJ], Chapter 6, page 104), and there is an obvious isomorphism, f:Y → (N, <), defined by, f(m_p) = p, for all m_p ϵ Y. Then, by Theorem 3.15, (N,<) and (N_Q, $<_q$) are not isomorphic, hence, K and N' are not isomorphic (reference [SR], page 4), as desired. □

4. Demonstration of objective. By Theorems 3.16, 3.17, 3.18, and 3.19, {N_Q, $<_q$, $+_q$, $*_q$, 1_0, 0_0} defines a recursive arithmetical structure and, by Theorems 3.20 and 3.21, it extends the "standard" model, as desired. □

5. Closing remarks. What we find most intriguing about this whole development, and something we're certain Kevin Knuth[KK] will appreciate, is the fact that we created a theoretically meaningful mathematical structure from, what amounts to, pure symmetry, simply by imposing an order. Isomorphisms, in general, preserve structure and it is structure which informs – structure is information. And this leads us to view the current work as a rather poignant example of what Knuth has been about with his foundational work in Information Physics: conscious entities impose an order and that imposition of order induces the emergence of structure – information; this information is not necessarily inherent in the systems we study, rather, it is inherent in the way we interact with those systems and it is this interaction which leads to the constraint equations we refer to as laws. It all begins with an object language and every object language begins with a few simple axioms.

In one of his books, we highly recommend them all, dynamical chaos theorist and AGI researcher, Ben Goertzel, expresses the idea that all of the mathematical structures we work with exist, in some undefined sense, within the foundational axioms prior to any human endeavor. He would say that these q-naturals have been there all along, hiding in plain sight within the axioms of ZFC/AFA. If you were privy to the experience which motivated the current work, and knowledgeable of Goertzel's Complex Systems model of mind, you could hardly disagree. One can't help but wonder what else there is, hiding in plain sight (reference [WV]).

Wes Hansen, 1501 E. 1st Street, Los Angeles, California, 90033
mail: PonderSeekDiscover@gmail.com

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