Q-Naturals: A Counter-Example to Tennenbaum's Theorem

By Wes Hansen

Abstract. In what follows we develop foundations for a set of non-standard natural numbers we call q-naturals, where q stands for quanta, by the recursive generation of reflexive sets. From the practical perspective, these q-naturals correspond to ordered pairs of natural numbers with the lexicographic ordering, hence, they are isomorphic to ω^2. In addition, we demonstrate a novel definition of the arithmetical operation, multiplication, which turns out to be recursive. This, in turn, enables our demonstration of a counter-example to Tennenbaum's Theorem.

1. Introduction. Motivated primarily by the demands of Computer Science, Peter Aczel extended Zermelo-Fraenkel set theory with Choice (ZFC) to include non-well-founded sets by replacing the Axiom of Foundation with the Anti-Foundation Axiom (AFA).[PA] Shortly thereafter, Aczel's ZFC/AFA was greatly popularized by Jon Barwise and John Etchemendy in their modern classic, "The Liar: An Essay in Truth and Circularity."[BE] Barwise and Etchemendy use ZFC/AFA to analyze the Liar paradox from the perspective of Russellian semantics and Austinian semantics. They dedicate all of Chapter 3 to the introduction of ZFC/AFA and their very first exercise in that chapter, Exercise 7, asks one to draw two graphs representing the von Neumann ordinal four. Just below that exercise is the key passage informing the present work.

"The liberating element is that we allow arbitrary graphs, including graphs which contain proper cycles. Of course graphs with cycles cannot depict sets in the well-founded universe. Thus, for example, in Aczel's universe there is a set $\Omega = \{\Omega\}$, simply because we can picture the membership relation on Ω by means of the graph G_Ω shown in Figure 4. Furthermore, on Aczel's conception this graph ***unambiguously*** *(emphasis theirs) depicts a set; that is, there is only one set with G_Ω as its graph. Consequently, there is only one set in Aczel's universe equal to its own singleton."* (pg. 37)

In our development of foundations, we leverage this uniqueness to formally extend the set of standard natural numbers to a set of q-natural numbers, where q stands for quanta, by introducing a one-place operation we call the hyperloop, which, when applied to any set, generates that set's unique reflexive set; we apply this operator recursively in such a manner that it generates countably many representations for the one same set where these representations exist in a state of perfect symmetry. We then break this symmetry by imposing an order, the lexicographic ordering, generating a countable and meaningful hierarchy of distinct elements. These elements, of course, are existent in between any two standard von Neumann ordinals. This provides our foundation.

For practical reasons, the set of q-naturals, N_Q, is then interpreted as ordered pairs of standard natural numbers and the properties of these numbers follow, in a natural and straight-forward way, from the properties of standard natural numbers. The recursive arithmetical operations, + and *, are defined in a consistent manner and this consistency leads to what is, perhaps, a less than obvious "multiplication;" however, the existence of these operations and their recursive nature is unassailable.

This existence of N_Q and unique recursive functions, + and *, defined on it, has profound theoretical and philosophical implications for both model theory and mathematics in general. "On Non-Standard

Models of Peano Arithmetic and Tennenbaum's Theorem,"[SR] by Samuel Reid, provides a lucid and economical review of the pertinent issues. Specifically, we demonstrate that $(N, <)$ is isomorphic to an initial segment of $(N_Q, <)$ excluding the existence of an isomorphism between the standard model of Peano Arithmetic (PA) and the model which assumes N_Q as universe. In spite of this, the arithmetical operations, + and *, defined on $(N_Q, <)$ are recursive, demonstrating a counter-example to Tennenbaum's Theorem.

And of course, the q-naturals can be extended to the q-integers, the q-integers to the q-rationals, the q-rationals, using Cauchy Sequences, to the q-reals, and the q-reals to the q-complex, which induces the philosophical question: What makes a model standard? Historical considerations aside, it would seem that, with the introduction of the q-naturals, the determination of which model is "standard" becomes a contextual consideration – something, I'm sure, Barwise, Etchemendy, and John Austin would well appreciate.

Notation. We use the standard notation together with:

@	|	a one-place non-logical symbol called the hyperloop
I_H	|	a hyper-inductive set
I_Q	|	a q-inductive set
N_Q	|	the set of all q-naturals

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2. Definitions. We define our mathematical entities using standard terminology:

Definition 2.01. A set is reflexive if $X = \{X\}$; a reflexive set is called a hyperset.[BE]

Definition 2.02. A one-place operation, @, when applied to any set X, generates a reflexive set; this operation, called a hyperloop, can be applied recursively.

Definition 2.03. Let X be an arbitrary von Neumann ordinal, then $@^n X$ designates the recursive application of @ to X "n" times, where $n \in N$; specifically, $@^0 X$, the zeroth-order application, is identical to no application, i.e. $@^0 X = X$.

Definition 2.04. Hypersets have two distinct successor functions; let $@^{n}X$ be an arbitrary hyperset, then $S(@^{n}X) = @(@^{n}X) = @^{(n+1)}X$, while $@^{n}S(X) = @^{n}(X \cup \{X\}) = @^{n}(X + 1)$ (reference [HJ], Chapter 3, pages 40 and 41).

Definition2.05. $\phi = 0.0$, $@\phi = 0.1$, $@^{2}\phi = 0.2$, … , $@^{(\omega-2)}\phi = 0.(\omega - 2)$, $@^{(\omega-1)}\phi = 0.(\omega - 1)$, $@^{\omega}\phi = 0.\omega$, $\{@^{\omega}\phi\} = 1.0$, $@\{@^{\omega}\phi\} = 1.1$, $@^{2}\{@^{\omega}\phi\} = 1.2$, … , $@^{(\omega-2)}\{@^{\omega}\phi\} = 1.(\omega - 2)$, $@^{(\omega-1)}\{@^{\omega}\phi\} = 1.(\omega - 1)$, $@^{\omega}\{@^{\omega}\phi\} = 1.\omega$, … , $\{@^{\omega}\phi, \{@^{\omega}\phi\}\} = 2$, $@\{@^{\omega}\phi, \{@^{\omega}\phi\}\} = 2.1$, $@^{2}\{@^{\omega}\phi, \{@^{\omega}\phi\}\} = 2.2$, …

Definition2.06. ϕ and any number $@^{\omega}X$ are examples of base elements; any number $@^{n}X$, where $n < \omega$, is an example of a hyper-element; for example, ϕ is the only base element of $@^{\omega}\phi$ and for any $m \in N_Q$, $m > @^{\omega}\phi$, $m = @^{k}\{0.\omega, 1.\omega, … , n.\omega\}$, where n is some von Neumann ordinal, and every $x.\omega$ is a base element of m, while every $@^{p}\{0.\omega, 1.\omega, … , n.\omega\}$, $p \in [0, k]$, is a hyper-element of m.

Definition 2.07. A set, I_H, is hyper-inductive if:

1. $\phi \in I_H$;
2. if $X \in I_H$, then $S(X) \in I_H$;
3. if $X \in I_H$, then $@X \in I_H$;
4. if $@X \in I_H$, then $S(@X) \in I_H$.

Definition 2.08. Consistent with Definition 2.05, a q-natural number is an ordered pair of natural numbers, (a, b), such that (a, b) = a.b.

Definition 2.09. Consistent with Definition 2.04, any q-natural number, a.b, has two distinct successor functions which can be applied independently or in conjunction; specifically, $S(a).b = (a \cup \{a\}).b = (a + 1).b$ and $a.S(b) = a.(b \cup \{b\}) = a.(b + 1)$ (reference [HJ], Chapter 3, page 52).

Definition 2.10. A set, I_Q, is q-inductive if:

1. $0.0 \in I_Q$;
2. if $a.b \in I_Q$, then, $S(a).b \in I_Q$;
3. if $a.b \in I_Q$, then, $a.S(b) \in I_Q$.

Definition 2.11. The set of all q-natural numbers is the set

$N_Q = \{x \mid x \in I_Q \text{ for every q-inductive set } I_Q\}$

Definition 2.12. The relation "<" on N_Q is defined by:

For all $a.b, c.d \in N_Q$, $a.b < c.d$ iff $a < c \vee (a = c \wedge b < d)$, where <(a, b) is the natural order (reference [HJ], Chapter 3 , page 42) and <(a.b, c.d) is the q-natural or lexicographic order (reference [HJ], Chapter 4, page 81).

Definition 2.13. The operation "+" (addition) on N_Q is defined by:

For all $a.b, c.d \in N_Q$, $a.b + c.d = (a + c).(b + d)$, where +(a, c) is as defined on the set of natural numbers (reference [HJ], Chapter 3, page 52).

Definition 2.14. The operation "*" (multiplication) on N_Q is defined by:

For all a.b, c.d ϵ N_Q, a.b * c.d = (a.b * c).(a.b * d)

= (a * c).(b * c).(a * d).(b * d)

= (a * c).(b * c) + (a * d) + (b * d),

where *(a,c) and +(b, d) are both as defined on the set of natural numbers (reference [HJ], Chapter 3, page 54) and +(b, d).

3.Arguments. We demonstrate our arguments using the standard methods and terminology of mathematical logic and ZFC/AFA or generalizations thereof. Specific to the current work, we generalize the Principle of Induction to the Principle of q-Induction and we reproduce certain arguments, verbatim, from reference [HJ].

Theorem 3.01. *The Axiom of Anti-Foundation implies that there exists a unique reflexive set.*

Proof. This theorem is reproduced verbatim from, "Introduction to Set Theory,"[HJ] by Karel Hrbacek and Thomas Jech (Chapter 14, page 263) and the proof can be found therein, as desired. □

Theorem 3.02. *A hyper-inductive set, I_H, defined by Definition 2.07, exists.*

Proof. Let I be an arbitrary set satisfying properties "1" and "2" of Definition 2.07, then I is an inductive set (reference [HJ], Chapter3, page 40) and, by the Axiom of Infinity, I exists. Let K be a family of intervals, [n, n + 1), such that n ϵ I and [n, n + 1) satisfies properties "3" and "4" of Definition 2.07. Let [n, n + 1) be an arbitrary element of K, then, by the Axiom of Infinity and Theorem 3.01, [n, n + 1) exists. Since [n, n + 1) was arbitrary, every [n, n + 1) ϵ K exists, hence, K exists. Finally, by the Axiom of Union, UK = I_H exists, as desired. □

Theorem 3.03. *For any hyper-inductive set, I_H, and any X ϵ I_H, X can be represented as an ordered pair of natural numbers, (a, b), such that (a, b) = a.b.*

Proof. This follows immediately for Definition 2.03, 2.04, 2.05, and the properties of natural numbers, (reference [HJ], Chapter3), as desired. □

Theorem 3.04. *A q-inductive set, I_Q, defined by Definition 2.10, exists.*

Proof. This is a direct consequence of a number of facts about the set of natural numbers,N:

1. N exists and is inductive (reference [HJ], Chapter 3, page 41);
2. By the Axiom of Power Set, the power set of N exists (reference [HJ], Chapter 1, page 10);
3. By the definition of ordered pair (reference [HJ], Chapter 2, page 17) and the definition of cartesian product (reference [HJ], Chapter 2, page 21), N x N exists;

together with Definition 2.08, 2.09, and 2.10, as desired. □

Theorem 3.05. *The set, N_Q, defined by Definition 2.11 exists and is q-inductive.*

Proof. Let X be the family of all q-inductive sets I_Q, then, by the Axiom of Union, the set UX exists and, by Definition 2.10, UX is q-inductive. By Definition 2.11, UX contains N_Q, hence, N_Q exists and is q-inductive, as desired. □

Theorem 3.06. *(The Principle of Q-Induction) Let P(x) be a property, and assume that:*

1. *P(0.0) is true;*
2. *for all n.k ϵ N_Q, P(n.k) → P[(n + 1).(k + 1)].*

Then P holds for all q-natural numbers n.k.

Proof. By Definition 2.10, "1" and "2" above define a q-inductive set I_Q. By Definition 2.11, that set, I_Q, contains N_Q, as desired. □

Lemma 3.07. *For all a.b ϵ N_Q, a, b ϵ N.*

Proof. This follows immediately from Definition 2.10, Theorem 3.05, and the fact that N is inductive (reference [HJ], Chapter 3, page 41), as desired. □

Theorem 3.08. *(N, <) is a linearly ordered set.*

Proof. This theorem is reproduced verbatim from reference [HJ] (Chapter 3, page 43) and the proof can be found therein, as desired. □

Lemma 3.09. *For all a.b, c.d ϵ N_Q:*

1. *0.0 ≤ c.d;*
2. *a.b < c.(d + 1) iff a.b ≤ c.d.*

Proof. The proof is in two parts:

1) We proceed by q-induction. Let P(x.y) be the property, "0.0 ≤ x.y," then:

   P(0.0). 0.0 = 0.0, hence, 0.0 ≤ 0.0.

   Suppose P(n.k) is true, then 0.0 < n.k or 0.0 = n.k and:

   P[(n + 1).(k + 1)]. In both cases, by Lemma 3.07 and Theorem 3.08, 0.0 < (n + 1).(k + 1).

   Therefore, P(n.k) → P[(n + 1).(k + 1)] and, by the Principle of Q-Induction, for all n.k ϵ N_Q, 0.0 ≤ n.k, as desired. □

2) Suppose a.b < c.(d + 1), then, by Definition 2.12, a < c ˅ [a = c ^ b < (d +1)]. If a < c, then, by Definition 2.12, a.b < c.d; otherwise, if a = c ^ b < (d +1), then, by Lemma 3.07 and Theorem 3.08, a.b ≤ c.d.

   In both cases a.b ≤ c.d, hence, a.b < c.(d + 1) → a.b ≤ c.d.

   Suppose a.b ≤ c.d, then, by Definition 2.12, [a < c ˅ (a = c ^ b < d)] ˅(a = c ^ b = d) and three cases arise:

Case 1. Suppose a < c, then, by Definition 2.12, a.b < c.(d + 1).

Case 2. Suppose (a = c ^ b < d), then, by Lemma 3.07 and Theorem 3.08, a.b < c.(d + 1).

Case 3. Suppose (a = c ^ b = d), then , by Lemma 3.07 and Theorem 3.08, a.b < c.(d + 1).

In all three cases a.b < c.(d + 1), hence, a.b ≤ c.d → a.b < c.(d + 1).

Therefore, a.b < c.(d + 1) iff a.b ≤ c.d, as desired.□

Theorem 3.10. *(N_Q, <) is a linearly ordered set.*

Proof. The proof is in three parts:

1) *Transitivity.* Let k.p, m.q, n.r ϵ N_Q be arbitrary but such that k.p < m.q ^ m.q < n.r. Then, by Definition 2.12, k < m ˅ (k = m ^ p < q) and m < n ˅ (m = n ^ q < r) and four cases arise:

   Case 1. Suppose (k < m) ^ (m < n), then, by Lemma 3.07 and Theorem 3.08, k < n, and, by Definition 2.12, k.p < n.r.

   Case 2. Suppose (k < m) ^ (m = n ^ q < r), then, by Lemma 3.07 and Theorem 3.08, k < n, and, by Definition 2.12, k.p < n.r.

   Case 3. Suppose (k = m ^ p < q) ^ (m < n), then, by Lemma 3.07 and Theorem 3.08, k < n, and, by Definition 2.12, k.p < n.r.

   Case 4. Suppose (k = m ^ p < q) ^ (m = n ^ q < r), then, by Lemma 3.07 and Theorem 3.08, k = n ^ p < r, and, by Definition 2.12, k.p < n.r.

   In all four cases, k.p < n.r, hence, (k.p < m.q ^ m.q < n.r) → k.p < n.r.
2) *Asymmetry.* Let k.p, m.q ϵ N_Q be arbitrary and suppose, for contradiction, that k.p < m.q ^ m.q < k.p, then, by transitivity, k.p < k.p, contradicting Definition 2.12.
3) *Linearity.* We proceed by q-induction. Let P(x.y) be the property, "for all m.p ϵ N_Q, m.p < x.y ˅ m.p = x.y ˅ x.y < m.p," then:

P(0.0). This is an immediate consequence of Lemma 3.09.

Suppose P(n.k) is true, then for all m.p ϵ N_Q, m.p <n.k ˅ m.p = n.k ˅ n.k < m.p and:

P[(n + 1).(k + 1)]. There are three cases to consider:

Case 1. Suppose m.p < n.k, then, by Lemma 3.07, Theorem 3.08, and Definition 2.12, n.k < (n + 1).(k + 1), hence, by transitivity, m.p < (n + 1).(k + 1).

Case 2. Suppose m.p = n.k, then, by Lemma 3.07, Theorem 3.08, and Definition 2.12, m.p < (n + 1).(k + 1).

Case 3. Suppose n.k < m.p, then, by Definition 2.12, n < m ˅ (n = m ^ k < p) and two cases arise:

Case 3a. Suppose n < m, then, by Lemma 3.07 and Theorem 3.08, [(n + 1) < m ˅ (n + 1) = m] ^ [(k + 1) < p ˅ (K + 1) = p ˅ p < (k + 1)] and four cases arise:

Case 3a.1. Suppose (n + 1) < m, then, by Definition 2.12, (n + 1).(k + 1) < m.p.

Case 3a.2. Suppose (n + 1) = m ^ (k + 1) < p, then, by Definition 2.12, (n + 1).(k + 1) < m.p.

Case 3a.3. Suppose (n + 1) = m ^ (K + 1) = p, then, (n + 1).(k + 1) = m.p.

Case 3a.4. Suppose (n + 1) = m ^ p < (k + 1), then, by Definition 2.12, m.p < (n + 1).(k + 1).

In all four cases, (n + 1).(k + 1) < m.p ˅ (n + 1).(k + 1) = m.p ˅ m.p < (n + 1).(k + 1), hence, (n < m) → [(n + 1).(k + 1) < m.p ˅ (n + 1).(k + 1) = m.p ˅ m.p < (n + 1).(k + 1)].

Case 3b. Suppose (n = m ^ k < p), then, by Lemma 3.07 and Theorem 3.08, m < (n + 1) and, by Definition 2.12, m.p < (n + 1).(k + 1).

In both cases, [(n + 1).(k + 1) < m.p ˅ (n + 1).(k + 1) = m.p ˅ m.p < (n + 1).(k + 1)], hence, (n.k < m.p) → [(n + 1).(k + 1) < m.p ˅ (n + 1).(k + 1) = m.p ˅ m.p < (n + 1).(k + 1)].

Therefore, P(n.k) → P[(n + 1).(k + 1)] and, by the Principle of Q-Induction, linearity.

Therefore, (N_Q, <) is a linearly ordered set, as desired. □

Theorem 3.11. *(N_Q, <) is a well-ordered set.*

Proof. This is an immediate consequence of Lemma 3.07, Theorem 3.08, and Lemma 3.09, as desired. □

Theorem 3.12. *(N_Q, <) is isomorphic to ω^2.*

Proof. Let Y = {S_i | i ϵ N} = ran S for some index function S, where each S_i is the set of natural numbers. Then ω^2 = { a_i | a_i ϵ S_i ϵ Y ^ for all i,j,a,b ϵ N, $a_i < b_j$ iff i < j ˅ (i = j ^ a < b)} and there is an obvious isomorphism, f:ω^2 → (N_Q, <), defined by f(a_i) = i.a, as desired. □

Theorem 3.13. *There is a unique function, +:N_Q x N_Q → N_Q, such that:*

1. *+(m.p, 0.0) = m.p, for all m.p ϵ N_Q;*
2. *+(m.p, n.q + 1.0) = +(m.p, n.q) +1.0, for all m.p, n.q ϵ N_Q.*

Proof. In the parametric version of the Recursion Theorem (reference [HJ], Chapter 3, page 51), let a:N_Q → N_Q be the identity function, and let g:N_Q x N_Q x N_Q → N_Q be defined by g(k.p, m.q, n.r) = m.q + 1, for all k.p, m.q, n.r ϵ N_Q. Then, by the Recursion Theorem, there exists a unique function, f:N_Q x N_Q → N_Q, such that:

1. f(k.p, 0.0) = a(k.p) = k.p, for all k.p ϵ N_Q;
2. f(k.p, m.q + 1.0) = g(k.p, f(k.p, m.q), m.q) = f(k.p, m.q) + 1, for all k.p, m.q ϵ N_Q.

Let + = f, as desired. □

Theorem 3.14. *There is a unique function, $*:N_Q \times N_Q \rightarrow N_Q$, such that:*

1. *$*(m.p, 0.0) = 0.0$, for all $m.p \in N_q$;*
2. *$*(m.p, n.q + 1.0) = *(m.p, n.q) + m.p$, for all $m.p, n.q \in N_Q$.*

Proof. In the parametric version of the Recursion Theorem (reference [HJ], Chapter 3, page 51), let $a:N_Q \rightarrow N_Q$ be the constant function defined by a(m.p) = 0.0, for all $m.p \in N_Q$, and let $g:N_Q \times N_Q \times N_Q \rightarrow N_Q$ be defined by g(k.p, m.q, n.r) = m.q + n.r, for all $k.p, m.q, n.r \in N_Q$. Then, by the Recursion Theorem, there exists a unique function, $f: N_Q \times N_Q \rightarrow N_Q$, such that:

1. f(k.p, 0.0) = a(k.p) = 0.0, , for all $k.p \in N_Q$;
2. f(k.p, m.q + 1.0) = g(k.p, f(k.p, m.q), m.q) = f(k.p, m.q) + m.q, for all $k.p, m.q \in N_Q$.

Let * = f, as desired. □

Theorem 3.15. *If $(W_1, <_1)$ and $(W_2, <_2)$ are well-ordered sets, then exactly one of the following holds:*

1. *either W_1 and W_2 are isomorphic; or*
2. *W_1 is isomorphic to an initial segment of W_2; or*
3. *W_2 is isomorphic to an initial segment of W_1.*

In each case, the isomorphism is unique.

Proof. This theorem is reproduced verbatim from reference [HJ] (Chapter 6, pages 105 and 106) and the proof can be found therein, as desired. □

4. Demonstration of counter-example. Let Y be the closed/open interval of N_Q, [0.0, 1.0), then Y is an initial segment of (N_Q, <) (reference [HJ], Chapter 6, page 104), and there is an obvious isomorphism, $f:Y \rightarrow (N, <)$, defined by, f(m.p) = p, for all $m.p \in Y$. Then, by Theorem 3.15, (N,<) and (N_Q, <) are not isomorphic yet, by Theorem 3.13 and Theorem 3.14, (N_Q, <) has recursive arithmetical functions, + and *, defined on it. Therefore, a model of PA with N_Q as universe, represents a counter-example to Tennenbaum's Theorem (reference [SR], page 11), as desired. □

5. Closing remarks. What we find most intriguing about this whole development, and something we're certain Kevin Knuth[KK] will appreciate, is the fact that we created a theoretically meaningful mathematical structure from, what amounts to, pure symmetry, simply by imposing an order. Isomorphisms are generally defined between ordered sets; the fact that they preserve order is their defining characteristic and the order they preserve is often referred to as structure. And this leads us to view the current work as a rather poignant example of what Knuth has been about with his foundational work in Information Physics: conscious entities impose an order and that imposition of order induces the emergence of structure – information; this information is not necessarily inherent in the systems we study, rather, it is inherent in the way we interact with those systems and it is this interaction which leads to the constraint equations we refer to as laws. It all begins with an object language and every object language begins with a few simple axioms.

In one of his books, we highly recommend them all, dynamical chaos theorist and AGI researcher, Ben Goertzel, expresses the idea that all of the mathematical structures we work with exist, in some undefined sense, within the foundational axioms prior to any human endeavor. He would say that these q-naturals have been there all along, hiding in plain sight within the axioms of ZFC/AFA. If you were privy to the experience which motivated the current work, and knowledgeable of Goertzel's Complex Systems model of mind, you could hardly disagree. According to [SR], there exists a continuum of non-standard models of PA – a continuum! One can't help but wonder what else there is, hiding in plain sight.[WV]

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