Hyper-Naturals: A Counter-example to Tennenbaum's Theorem

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Abstract. In the following we define a set of hyper-naturals on N x N with the lexicographic ordering and a novel definition of the arithmetical operation, multiplication. These hyper-naturals are isomorphic to ω^2 yet have recursive arithmetical operations defined on them, demonstrating a counter-example to Tennenbaum's Theorem.

1. Introduction. In what follows we define a set, N_H, of hyper-naturals as ordered pairs of natural numbers; existence, then, follows from the existence of N x N. We then show that N_H is linearly ordered by the standard lexicographic ordering and strengthen (N_H, <) to well-ordering by demonstrating an obvious isomorphism, $f:\omega^2 \rightarrow N_H$. Finally, using recursion we demonstrate the existence of unique arithmetical operations,$+:N_H \times N_H \rightarrow N_H$ and $*: N_H \times N_H \rightarrow N_H$,.

The existence of N_H and unique recursive functions, + and *, defined on it, has profound theoretical and philosophical implications for both model theory and mathematics in general. "On Non-Standard Models of Peano Arithmetic and Tennenbaum's Theorem," [SR] by Samuel Reid, provides a lucid and economical review of the pertinent issues. Specifically, we demonstrate that (N, <) is isomorphic to an initial segment of (N_H, <) excluding the existence of an isomorphism between the standard model of Peano Arithmetic (PA) and the model which assumes N_H as universe. In spite of this, the arithmetical operations, + and *, defined on (N_H, <) are recursive, demonstrating a counter-example to Tennenbaum's Theorem. And of course, the hyper-naturals can be extended to the hyper-integers, the hyper-integers to the hyper-rationals, the hyper-rationals, using Cauchy Sequences, to a novel set of hyper-reals, the hyper^2-reals, and the hyper^2-reals to the hyper-complex, which induces the philosophical question: What makes a model standard? Historical considerations aside, it would seem that, with the introduction of the hyper-naturals, the determination of which model is "standard" becomes a contextual consideration.

Notation. We use the standard notation together with:

I_H	|	a hyper-inductive set
N_H	|	the set of all hyper-naturals

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2. Definitions. We define our mathematical entities using standard terminology:

Definition 2.01. A hyper-natural number is an ordered pair of natural numbers, (a, b), such that (a, b) = a.b.

Definition 2.02. The hyper-successor of a hyper-natural number has two distinct components, S(a).b = (a U {a}).b = (a + 1).b and a.S(b) = a.(b U {b}) = a.(b + 1) (reference [HJ], Chapter 3, page 52). These distinct components are independent.

Definition 2.03. A set, I_H, is hyper-inductive if:

1. 0.0 ϵ I_H;
2. if a.b ϵ I_H, then, S(a).S(b) ϵ I_H.

Definition 2.04. The set of all hyper-natural numbers is the set

N_H = {x | x ϵ I_H for every hyper-inductive set I_H}

Definition 2.05. The relation "<" on N_H is defined by:

For all a.b,c.d ϵ N_H, a.b < c.d iff a.b < c.d ˅ (a = c ^ b < d), where <(a, b) is the natural order (reference [HJ], Chapter 3 , page 42) and <(a.b, c.d) is the hyper-natural or lexicographic order (reference [HJ], Chapter 4, page 81).

Definition 2.06. The operation "+" (addition) on N_H is defined by:

For all a.b, c.d ϵ N_H, a.b + c.d = (a + c).(b + d), where +(a, c) is as defined on the set of natural numbers (reference [HJ], Chapter 3, page 52).

Definition 2.07. The operation "*" (multiplication) on N_H is defined by:

For all a.b, c.d ϵ N_H, a.b * c.d = a.b * c.a.b * d

= a * c.b * c.a * d.b * d

= a * c.(b * c) + (a * d) + (b * d),

where *(a,c) is as defined on the set of natural numbers (reference [HJ], Chapter 3, page 54) and +(b, d) is as defined immediately above.

3.Arguments. We demonstrate our arguments using the standard methods and terminology of mathematical logic and ZFC or generalizations thereof. Specific to the current work, we generalize the Principle of Induction to the Principle of Hyper-Induction and we reproduce certain arguments, verbatim, from reference [HJ].

Theorem 3.01. *A hyper-inductive set, I_H, defined by Definition 2.07, exists.*

Proof. This is a direct consequence of a number of facts about the set of natural numbers N:

1. N exists and is inductive (reference [HJ], Chapter 3, page 41);
2. By the Axiom of Power set, the power set of N exists (reference [HJ], Chapter 1, page 10);

3. By the definition of ordered pair (reference [HJ], Chapter 2, page 17) and the definition of Cartesian product (reference [HJ], Chapter 2, page 21), N x N exists;

Therefore, by Definition 2.01, 2.02, and 2.03, every hyper-inductive set, I_H, is a subset of N x N and exists, as desired. □

Theorem 3.03. *N_H exists and N_H is hyper-inductive.*

Proof. Let X be the family of all hyper-inductive sets I_H, then, by the Axiom of Union, the set UX exists and, by Definition 2.03, UX is hyper-inductive. By Definition 2.04, UX contains N_H, hence, N_H exists and is hyper-inductive, as desired. □

Theorem 3.04. *(The Principle of Hyper-Induction) Let P(x) be a property, and assume that:*

1. *P(0.0) is true;*
2. *for all n.k ϵ N_H, P(n.k) → P[(n + 1).(k + 1).*

Then P holds for all hyper-natural numbers n.k.

Proof. By Definition 2.03, "1" and "2" above define a hyper-inductive set I_H. By Definition 2.04, that set, I_H, contains N_H, as desired. □

Lemma 3.05. *For all a.b ϵ N_H, a, b ϵ N.*

Proof. This follows immediately from Definition 2.03, Theorem 3.03, and the fact that N is inductive (reference [HJ], Chapter 3, page 41), as desired. □

Theorem 3.06. *(N, <) is a linearly ordered set.*

Proof. This theorem is reproduced verbatim from reference [HJ] (Chapter 3, page 43) and the proof can be found therein, as desired. □

Lemma 3.07. *For all a.b, c.d ϵ N_H:*

1. *0.0 ≤ c.d;*
2. *a.b < c.(d + 1) iff a.b ≤ c.d.*

Proof. The proof is in two parts:

1) We proceed by hyper-induction. Let P(x.y) be the property, "0.0 ≤ x.y," then:

P(0.0). 0.0 = 0.0, hence, 0.0 ≤ 0.0.

Suppose P(n.k) is true, then 0.0 < n.k or 0.0 = n.k and:

P[(n + 1).(k + 1)]. There are two cases to consider:

Case 1. Suppose 0.0 < n.k, then, by Definition 2.05, 0 < n ˅ (0 = n ^ 0 < k) and, by Definition 2.02, n.k < (n + 1).(k + 1), hence, by Lemma 3.05 and Theorem 3.06, 0.0 < (n + 1).(k + 1).

Case 2. Suppose 0.0 = n.k, then, by Definitions 2.02, 0. 0 < (n + 1).(k + 1).

Therefore, P(n.k) → P[(n + 1).(k + 1)] and, by the Principle of Hyper-Induction, for all n.k ϵ N_H, 0.0 ≤ n.k, as desired. □

2) Suppose a.b < c.(d + 1), then, by Definition 2.05, a < c ˅ (a = c ^ b < d +1) and two cases arise:

Case 1. Suppose a < c, then, by Definition 2.05, a.b < c.d.

Case 2. Suppose a = c ^ b < (d + 1), then, by Lemma 3.05 and Theorem 3.06, a = c ^ b ≤ d, hence, a.b ≤ c.d.

In both cases a.b ≤ c.d, hence, a.b < c.(d + 1) → a.b ≤ c.d.

Suppose a.b ≤ c.d, then, by Definition 2.05, [a < c ˅ (a = c ^ b < d)] ˅(a = c ^ b = d) and three cases arise:

Case 1. Suppose a < c, then, by Definition 2.05, a.b < c.(d + 1).

Case 2. Suppose (a = c ^ b < d), then, by Lemma 3.05 and Theorem 3.06, a.b < c.(d + 1).

Case 3. Suppose (a = c ^ b = d), then , by Lemma 3.05 and Theorem 3.06, a.b < c.(d + 1).

In all three cases a.b < c.(d + 1), hence, a.b ≤ c.d → a.b < c.(d + 1).

Therefore, a.b < c.(d + 1) iff a.b ≤ c.d, as desired.□

Theorem 3.08. *(N_H, <) is a linearly ordered set.*

Proof. The proof is in three parts:

1) Transitivity. Let k.p, m.q, n.r ϵ N_H be arbitrary but such that k.p < m.q ^ m.q < n.r. Then, by Definition 2.05, k < m ˅ (k = m ^ p < q) and m < n ˅ (m = n ^ q < r) and four cases arise:

Case 1. Suppose (k < m) ^ (m < n), then, by Lemma 3.05, Theorem 3.06, and Definition 2.05, k.p < n.r.

Case 2. Suppose (k < m) ^ (m = n ^ q < r), then, by Lemma 3.05, Theorem 3.06, and Definition 2.05, k.p < n.r.

Case 3. Suppose (k = m ^ p < q) ^ (m < n), then, by Lemma 3.05, Theorem 3.06, and Definition 2.05, k.p < n.r.

Case 4. Suppose (k = m ^ p < q) ^ (m = n ^ q < r), then, by Lemma 3.05, Theorem 3.06, and Definition 2.05, k.p < n.r.

In all four cases, k.p < n.r, hence, (k.p < m.q ^ m.q < n.r) → k.p < n.r.

2) *Asymmetry.* Let k.p, m.q ϵ N_H be arbitrary and suppose, for contradiction, that k.p < m.q ^ m.q < k.p, then, by transitivity, k.p < k.p, contradicting Definition 2.05.

3) *Linearity.* We proceed by hyper-induction. Let P(x.y) be the property, "for all m.p ϵ N_H, m.p < x.y ˅ m.p = x.y ˅ x.y < m.p," then:

P(0.0). This is an immediate consequence of Lemma 3.07.

Suppose P(n.k) is true, then for all m.p ϵ N_H, m.p <n.k ˅ m.p = n.k ˅ n.k < m.p and:

P[(n + 1).(k + 1)]. There are three cases to consider:

Case 1. Suppose m.p < n.k, then, by Definition 2.02, Definition 2.06, Theorem 3.03, and Definition 2.05, n.k < (n + 1).(k + 1), hence, by transitivity, m.p < (n + 1).(k + 1).

Case 2. Suppose m.p = n.k, then, by Definition 2.02, Theorem 3.03, and Definition 2.05, m.p < (n + 1).(k + 1).

Case 3. Suppose n.k < m.p, then, by Definition 2.05, n < m ˅ (n = m ^ k < p) and two cases arise:

Case 3a. Suppose n < m, then, by Lemma 3.05 and Theorem 3.06, [(n + 1) < m ˅(n + 1) = m] ^ [(k + 1) <p ˅ (K + 1) = p ˅ p < (k + 1)] and four cases arise:

Case 3a.1. Suppose (n + 1) < m, then, by Definition 2.05, (n + 1).(k + 1) < m.p.

Case 3a.2. Suppose (n + 1) = m ^ (k + 1) <p, then, by Definition 2.05, (n + 1).(k + 1) < m.p.

Case 3a.3. Suppose (n + 1) = m ^ (K + 1) = p, then, (n + 1).(k + 1) = m.p.

Case 3a.4. Suppose (n + 1) = m ^ p < (k + 1), then, by Definition 2.05, m.p < (n + 1).(k + 1).

In all four cases, (n + 1).(k + 1) < m.p ˅ (n + 1).(k + 1) = m.p ˅ m.p < (n + 1).(k + 1), hence, (n.k ϵ m.p ^ n < m) → [(n + 1).(k + 1) < m.p ˅ (n + 1).(k + 1) = m.p ˅ m.p < (n + 1).(k + 1)].

Case 3b. Suppose (n = m ^ k < p), then, by Definition 2.02 and Theorem 3.03, m.p ≥ (n + 1).(k + 1).

In both cases, [(n + 1).(k + 1) < m.p ˅ (n + 1).(k + 1) = m.p ˅ m.p < (n + 1).(k + 1)], hence, (n.k < m.p) → [(n + 1).(k + 1) < m.p ˅ (n + 1).(k + 1) = m.p ˅ m.p < (n + 1).(k + 1)].

Therefore, P(n.k) → P[(n + 1).(k + 1)] and, by the Principle of Hyper-Induction, linearity.

Therefore, (N_H, <) is a linearly ordered set, as desired. □

Theorem 3.08. *(N_H, <) is a well-ordered set.*

Proof. This is an immediate consequence of Lemma 3.06 and Theorem 3.07, as desired. □

Theorem 3.09. *(N_H, <) is isomorphic to ω^2.*

Proof. Let Y = {S_i | i ϵ N} = ran S for some index function S, where each S_i is the set of natural numbers. Then ω^2 = { a_i | $a_i \in S_i$ ^ for all i,j,a,b ϵ N, $a_i < b_j$ iff i < j ˅ (i = j ^ a < b)} and there is an obvious isomorphism, f: $\omega^2 \rightarrow$ (N_H, <), defined by f(a_i) = i.a, as desired. □

Theorem 3.10. *There is a unique function, +:N_H x $N_H \rightarrow N_H$, such that:*

1. *+(m.p, 0.0) = m.p, for all m.p ϵ N_H;*
2. *+(m.p, n.q + 1.0) = +(m.p, n.q) +1.0, for all m.p,n.q ϵ N_H.*

Proof. In the parametric version of the Recursion Theorem (reference [HJ], Chapter 3, page 51), let a:$N_H \rightarrow N_H$ be the identity function, and let g:N_H x N_H x $N_H \rightarrow N_H$ be defined by g(k.p,m.q,n.r) = m.q + 1, for all k.p,m.q,n.r ϵ N_H. Then, by the Recursion Theorem, there exists a unique function, f: N_H x $N_H \rightarrow N_H$, such that:

1. f(k.p, 0.0) = a(k.p) = k.p, for all k.p ϵ N_H;
2. f(k.p, m.q + 1.0) = g(k.p, f(k.p, m.q), m.q) = f(k.p, m.q) + 1, for all k.p,m.q ϵ N_H.

Let + = f, as desired. □

Theorem 3.11. *There is a unique function, *:N_H x $N_H \rightarrow N_H$, such that:*

1. **(m.p, 0.0) = 0.0, for all m.p ϵ N_H;*
2. **(m.p, n.q + 1.0) = *(m.p, n.q) + m.p, for all m.p,n.q ϵ N_H.*

Proof. In the parametric version of the Recursion Theorem (reference [HJ], Chapter 3, page 51), let a:$N_H \rightarrow N_H$ be the constant function defined by a(m.p) = 0.0, for all m.p ϵ N_H, and let g:N_H x N_H x $N_H \rightarrow N_H$ be defined by g(k.p, m.q, n.r) = m.q + n.r, for all k.p,m.q,n.r ϵ N_H. Then, by the Recursion Theorem, there exists a unique function, f: N_H x $N_H \rightarrow N_H$, such that:

1. f(k.p, 0.0) = a(k.p) = 0.0, , for all k.p ϵ N_H;
2. f(k.p, m.q + 1.0) = g(k.p, f(k.p, m.q), m.q) = f(k.p, m.q) + m.q, for all k.p,m.q ϵ N_H.

Let * = f, as desired. □

Theorem 3.12. *If (W_1, $<_1$) and (W_2, $<_2$) are well-ordered sets, then exactly one of the following holds:*

1. *either W_1 and W_2 are isomorphic; or*
2. *W_1 is isomorphic to an initial segment of W_2; or*
3. *W_2 is isomorphic to an initial segment of W_1.*

In each case, the isomorphism is unique.

Proof. This theorem is reproduced verbatim from reference [HJ] (Chapter 6, pages 105 and 106) and the proof can be found therein, as desired. □

4. Demonstration of counter-example. Let Y be the closed/open interval of N_H, [0.0, 1.0), then Y is an initial segment of (N_H, <) (reference [HJ], Chapter 6, page 104), and there is an obvious isomorphism, f:Y $\rightarrow$ (N, <), defined by, f(m.p) = p, for all m.p ϵ N_H. Then, by Theorem 3.12, (N,<)

and (N_H, <) are not isomorphic yet, by Theorem 3.10 and Theorem 3.11, (N_H, <) has recursive arithmetical functions, + and *, defined on it. Therefore, a model of Peano Arithmetic with N_H as universe, represents a counter-example to Tennenbaum's Theorem (reference [SR], page 11), as desired. □

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