Naive Risk Parity Portfolio with Fractal Estimation of Volatility

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Abstract

A fractal approach to long-only portfolio optimization is proposed. The quantitative system is based on naive risk parity approach. The core of the optimization scheme is a fractal distribution of returns, applied to estimation of the volatility law. Out-of-sample performance data has been represented in ten period of observation with half year and one year horizons. Implementation of fractal estimator of volatility improves all performance metrics of portfolio in comparison to the standard estimator of volatility. The efficiency of fractal estimator plays a significant protective role for the periods of market abnormal volatility and drawdowns, which allows beating the market in the long term perspective. The provided results may be useful for a wide range of quantitative investors, including hedge funds, robo-advisors and retails investors.

Introduction

The market crashes of 2000s and 2007-2008 have rose questions about applicability of Modern Portfolio Theory (MPT) [**1**]. For the past two decades investors observed long memory highly correlated behavior of several assets and asset classes which raised a problem of the alternative hedging of systematic risks. Although correlations and standard deviations of returns are still basic concepts of MPT these parameters do not take into account nonlinear effects of abnormal volatility [**1**]. In long term perspective the infrequent large scale declines can sufficiently affect the portfolio performance. In 2006-2016 the drawdown of S&P500 index by 36% has decreased the accumulated ten year return twice and annual Sharpe ratio by three times. Naive risk parity approach has partially reduces the "nonlinear gap" by avoiding calculation of equally weighted expected returns and linear correlations. It turned out that naive risk parity strategy with global allocation has outperformed S&P500 index by 2% for the past forty years with annual Sharpe ratio improvement by 63% [**2**]. However the volatility estimation of global risk parity model is still based on normal distribution of returns which fails to explain effects [**2**] of the panic based persistency. It has been shown that fractal distributions provide more realistic asymptotic for large scale returns [**3**]. The factors of these distributions allow modifying a rule for volatility estimation. In this research we introduce a fractal law of volatility into the naive risk parity buy-and-hold global portfolio and compare the performance of this portfolio to the original model of normally distributed returns.

Estimation of volatility

According to [8] the alpha stable distributions of returns can generalize the normal distribution and simultaneously allow heavy tails and "three-sigma" declines. The stability of this class of distributions is supported by the Central Limit Theorem [**3**]. Probability alpha stable distribution function of returns *r* is characterized by the following relation:

$$F(r \mid \alpha, \beta, t) = \frac{1}{\pi}\int_0^{\infty} \cos(h(r \mid \alpha, \beta, t)) \exp(-t^{\alpha}) dt \qquad (1)$$

Here $\alpha, \beta, \sigma, \mu$ are static parameters of distribution, t is time. The core of the integral is represented by the piecewise function *h*:

$$h(r \mid \alpha, \beta, t) = rt + \beta \tan\left(\frac{\pi\alpha}{2}\right)\left(t - t^{\alpha}\right), \alpha \neq 1 \qquad (2)$$

$$h(r \mid \alpha, \beta, t) = \mu + \frac{2\beta\sigma}{\pi} \ln \sigma, \alpha = 1 \quad (3)$$

The stability of this distribution is defined by α parameter which lies in the interval $0 \prec \alpha \leq 2$. Rescaling of this distribution with rescale factor *k* corresponds to the following relation [9]:

$$F(r \mid \alpha, \beta, kt) = k^H F(r \mid \alpha, \beta, t) \quad (4)$$

Alpha parameter quantifies the tail thickness of the probability distribution which allows its implementation into volatility analysis. According to Mandelbrot [**4**] this key parameter is also directly related to Hurst factor H of time series persistency: $\alpha = 1/H$. We should recall that positive memory (momentum) of returns is characterized by $H \in (0.5, 1]$ while negative memory (mean reversion) corresponds to $H \in [0, 0.5)$. The law for rescaling of volatility may be derived directly from (4):

$$E\left[\left(r(t_2) - r(t_1)\right)^2\right] = \left|t_2 - t_1\right|^{2H} \quad (5)$$

Here the values of two consequent time moments $t_1, t_2 : t_2 \succ t_1$ are involved. In this research the volatility is estimated as the unbiased standard deviation of returns. Accordingly a one period standard deviation STD_0 may be derived [9] for rescaling into the *N*-period horizon standard deviation in the following way: $STD_N = STD_0 N^H$. In this paper we apply an approach of small data basis, suggested in [**5**] for the stable definition of Hurst factor. We recall that MPT and market efficiency imply a random walk of returns and the short memory model of volatility estimation: $H = 0.5$ for any asset class. Typically rescaling is excluded from estimation as it gives a constant factor for the definition of investment weights.

Investment method

Global diversification between four asset classes is suggested in this research. Fundamentally uncorrelated asset classes were suggested for justification of algorithm. However we admit the possibility of wider selections. We use adjusted closing prices of following low cost liquid exchange traded funds (ETFs):

ETF	Index	Expense ratio,%	Asset class
SPY	S&P 500	0.09	US equity
TLT	U.S. 20+ Year Treasury Bond	0.15	US Treasuries
IYR	U.S. Real Estate Index	0.43	US real estate
GLD	Gold Bullion	0.4	Gold

Table 1. Investment blocks.

The applied history is limited by the inception date of ETFs: 2005-2016 time period is the core of algorithm justification. The optimization sample period *N* and hold period correspond to desired investment horizon and coincide. In this paper we consider half year and annual horizons. Although risk parity approach implies that an equal amount of volatility is designated to each class, in this research a long-only filter is introduced: an investment into the asset class is switched on if the historical expected return is positive. Risk free rate parameter is excluded in this research. Below we compare fractal and standard algorithms in frame of portfolio long term performance. The fractal algorithm may be represented as the consequence of steps:

<u>Fractal risk parity</u>:

- Historical returns of asset i are calculated through the differences of adjusted price logarithms $r_i = 100\% \cdot {}_\Delta \log(p_i)$

- Expected Hurst factor H_i for each asset is defined on the basis of calculated time series of returns according to the small date method [10]
- Expected daily return $\mu_i = mean(r_i)$ is defined as simple average of returns in the sample period *N*, the asset class weight is zero if $\mu_i \leq 0$
- Daily standard deviations of returns STD_0^i are calculated
- Expected asset volatility is rescaled to the horizon *N* of portfolio: $STD_N^i = STD_0^i N^{H_i}$
- Non zero investment weights are calculated on the basis of rescaled volatilities $w_i = 1 / STD_N^i$, the sum of weights is normalized: $w_i \rightarrow w_i / \sum w_i$

The standard MPT estimation of volatility implies market efficiency and $H = 0.5$. Therefore the standard filtered risk parity algorithm may be represented in the following way:

Standard risk parity:

- Historical returns of asset *i* are calculated through the differences of adjusted price logarithms with percentage scale $r_i = 100\% \cdot {}_{\Delta} \log(p_i)$
- Expected daily return $\mu_i = mean(r_i)$ is defined as simple unweighted mean of returns in the sample period *N*, the asset class weight is zero $\mu_i \leq 0$
- Daily standard deviations of returns STD_0^i are calculated
- Expected asset volatility is rescaled to the horizon *N* of portfolio: $STD_N^i = STD_0^i \sqrt{N}$
- Non zero investment weights are calculated on the basis of rescaled volatilities $w_i = 1 / STD_N^i$, the sum of weights is normalized: $w_i \rightarrow w_i / \sum w_i$

Simulation results

The portfolio simulation is based on the out-of-sample analysis. The algorithmic system analyzes past *N* prices of given assets, construct portfolio and observe cumulated performance for the next *N* days. The performance is recorded into the datebase. The efficiency of both algorithms is compared by use of several investment metrics: Sharpe ratio, Treynor ratio, average return, capital protection, standard deviation (STD) and beta (SPY benchmark has been used). Capital protection is defined as the percentage difference between initial capital and maximum drawdown. It is a useful measure of hedging properties of current system, particularly during 2007-2008 debt crises. The simulation does not take into account slippage and spreads, but process commissions, expense ratios and corporate events. We used trading conditions of Interactive Brokers for simulation of both algorithms in MatLab 2015a package. All simulations are provided for non leveraged long only passive investments. Firstly we provide simulation results for the investment capital of $500,000 which corresponds to institutional format. Relative percentage improvements are provided in the last raw.

	Sharpe	Treynor x 0.01	Return,%	Protection,%	STD,%	beta
Fractal parity A	1.29	0.37	9.09	93	7.06	0.25
Standard parity B	1.19	0.32	8.91	92	7.49	0.28
Benchmark (SPY)	0.52	0.08	8.18	62%	15.68	1
Improvement A-B	8%	16%	2%	1%	6%	11%

Table 2. Annual horizon, $500,000

The interpretation of results shows that most significant advantage of the fractal risk parity model is hedging against systematic declines with equal opportunity of return.

	Sharpe	Treynor x 0.01	Return,%	Protection,%	STD,%	beta
Fractal parity A	1.31	0.61	9.45	91	5.10	0.15
Standard parity B	1.16	0.40	9.00	90	5.48	0.23
Benchmark (SPY)	0.5	0.08	8.00	64	11.32	1
Improvement A-B	13%	53%	5%	1%	7%	35%

Table 3. Half year horizon, $500,000

The decrease of horizon strengthens the non efficiency influence. As a result almost all metrics are improved in spite of more frequent commission charging. In average the system beats the market by 1.5% for the twenty out-of-sample periods (Table 3). In the Table 3 we annualized all represented metrics. Commissions are included in the benchmark investment cycle as well which explains lower annual return of SPY investment for the second simulation. We also experimented with lower capital of $10,000 which corresponds to robo-advisor and retail format:

	Sharpe	Treynor x 0.01	Return,%	Protection,%	STD,%	beta
Fractal parity A	1.15	0.35	7.73	92	6.74	0.22
Standard parity B	1.02	0.27	7.36	90	7.23	0.27
Benchmark (SPY)	0.46	0.07	7.18	62	15.65	1
Improvement A-B	13%	30%	5%	1%	7%	19%

Table 4. Annual horizon, $10,000

	Sharpe	Treynor x 0.01	Return,%	Protection,%	STD,%	beta
Fractal parity A	0.93	0.43	6.64	90	5.04	0.15
Standard parity B	0.77	0.26	6.00	89	5.48	0.23
Benchmark (SPY)	0.44	0.07	7.00	63	11.27	1
Improvement A-B	21%	65%	11%	1%	8%	35%

Table 5. Half year horizon, $10,000

As in previous simulation we observe the significant hedging effect. However this time retail commissions liquidate the advantage of the expected return for half-year investment. Still the annual Sharpe ratio is twice higher in relation to the pure SPY investment. This observation allows applying the system for short term conservative investors with horizon of 1-2 years. We should remark that the partial eliminating of systematic risk without decrease of return lies in the range of 6-8 percentage points for all simulations which justify the efficiency of fractal approach to the estimation of volatility. Below we represent the figure of reinvested return for the best simulated strategy – half year optimization with $0.5 mln of investment capital. The bars correspond to the difference between the cumulated return of fractal scheme (Sh=1.31) and standard scheme (Sh=1.16) of investment.

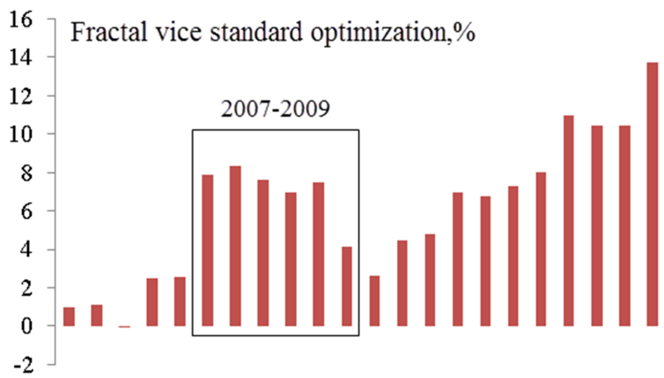

Figure 1. Difference of cumulated returns – two schemes

The period of debt crisis is remarkable due to the significant growth of cumulated return and is marked by black rectangular. This effect confirms the hypothesis of realistic fractal description of heavy tails events. However we observe the gradual growth of difference in “normal” market environment during ten years of observation – slight inefficiency still contributes to the fractal advantage. This advantage reaches 14% at the end of 2016 which remarks a conversion of portfolio hedging and “don’t loose” empirical rule into the long term reward.

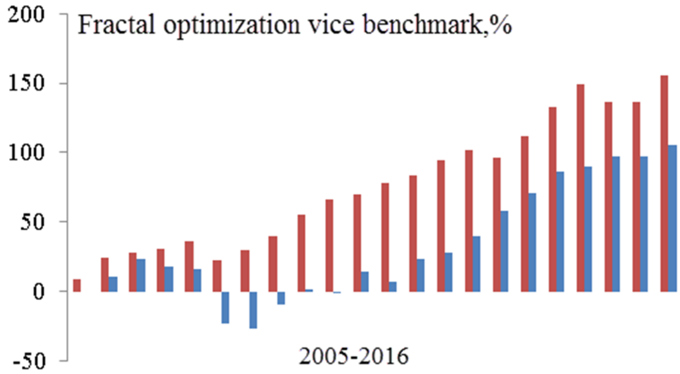

Figure 2. Difference of cumulated returns – two schemes

In Figure 2 the absolute cumulative return of fractal risk parity is compared to the SPY reinvestment strategy. It is remarkable that a 2007-2008 advantage plays the sufficient role in total performance of the strategy and is contradictive to the MPT supporters that consider heavy tail events as statistically non significant.

Conclusions

In current research we investigated the applicability of fractal estimators of volatility to the risk parity long only non leveraged portfolios. The simulation results show that the current approach allows improving all portfolio metrics for large capital investments. The influence on relation between reward and systematic risk is most remarkable which makes the proposed estimator an efficient tool for portfolio hedging. The decrease of horizon strengthens the advantage of fractal estimator in non efficient market environment. We have found that the most remarkable application of the method is possible for institutional investment with half-year optimization of portfolio. This scheme of investment allows increasing the Sharpe ratio by 2.6 times in relation to benchmark investment and increasing average annual return by 1.5%. The application of this scheme in reinvestment cycle will allow increasing an advantage of capital accumulation by 14% percent in ten years. However we still should note that further research of liquidity influence is necessary for strict justification of results especially for institutional investments.

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