Rigorous Proof for Riemann Hypothesis Using Sigma-Power Laws

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Abstract: The triple countable infinite sets of (i) x-axis intercepts, (ii) y-axis intercepts, and (iii) both x- and y-axes [formally known as the 'Origin'] intercepts in Riemann zeta function are intimately related to each other simply because they all constitute complementary points of intersection arising from the single [exact same] countable infinite set of curves generated by this function. This [complete] relationship amongst all three sets of intercepts will enable us to simultaneously study important intrinsic properties derived from all those intercepts in a mathematically consistent manner which then provides the rigorous proof for Riemann hypothesis as well as fully explain x-axis intercepts (which is the usual traditionally-dubbed 'Gram points') and y-axis intercepts. Riemann hypothesis involves analysis of all nontrivial zeros of Riemann zeta function and refers to the celebrated proposal by famous German mathematician Bernhard Riemann in 1859 whereby all nontrivial zeros are conjectured to be located on the critical line [or equivalently stated as all nontrivial zeros are conjectured to exactly match the Origin intercepts]. Concepts from the Hybrid method of Integer Sequence classification, together with our key formulae coined Sigma-Power Laws, are some of the important mathematical tools employed in this paper to successfully achieve our proof. Not least in [again] using the same 'Virtual container' method in this current research paper, there are other additional deeply inseparable mathematical connections between the content of this paper and our 2017-dated publication on the dual source of prime number infiniteness entitled "Rigorous Proofs for Polignac's and Twin Prime Conjectures using Information-Complexity Conservation" http://viXra.org/abs/1703.0115.

Keywords: Gram points; Polignac's conjecture; Riemann hypothesis; Sigma-Power Laws; Twin prime conjecture

Mathematics Subject Classification: 11A41, 11M26

1. Introduction

This landmark 2017-dated research paper concentrates on solving the intractable open problem Riemann hypothesis. Most of the underlying fundamental mathematical principles for composing this paper to solve Riemann hypothesis were outlined in our other publication on the dual source of prime number infiniteness with resulting rigorous proofs for Polignac's and Twin prime conjectures [1]. Therefore, readers are strongly encouraged to read this other research paper as well for optimal comprehension. We will shortly explore some preliminary preambles after rendering all our claims and statements in this paper meaningful by defining the following terms "Completely Predictable" and "Incompletely Predictable" particularly in the context of numbers.

Completely Predictable numbers, and Incompletely Predictable numbers, are (respectively) defined as numbers whose associated details are able to be fully specified without, and with, needing to know the associated details of preceding numbers in the neighborhood. This property is demonstrated below using the example of odd number '99' (generated by a 'simple' formula and is thus a Completely Predictable number obeying Simple Elementary Fundamental Laws) and prime number '97' (generated by a 'complicated' algorithm and is a Incompletely Predictable number obeying Complex Elementary Fundamental Laws). We must emphasize here that this property is equally applicable to, for instance, all the numerical digits of any [Incompletely Predictable] transcendental number after its decimal point.

Randomly picked odd number '99': Can we completely predict its associated details (i) specifying whether it can be classified as an odd number and (ii) whether its precise position can be specified without needing to know all preceding odd numbers? '99' satisfy the Odd number Label "Always end with a digit of 1, 3, 5, 7, or 9" and hence is truly an odd number. Its precise position can be calculated as follows: i = (99 + 1) / 2 = 50. This implies that 99 is the 50^{th} odd number. Note that '99' is odd and composite but not prime as it consists of factors derived as 99 = 3 X 33 = 3 X 3 X 11.

Randomly picked prime number '97': Can we completely predict its associated details (i) specifying whether it can be classified as a prime number and (ii) whether its precise position can be specified without needing to know all preceding prime numbers? '97' satisfy the Prime number Label "Always evenly divisible only by 1 or itself & must be whole numbers greater than 1" as tested by the trial division method resulting in 97 = 1 X 97. Hence '97' is truly prime. However, its precise position can only be determined by knowing all preceding 24 prime numbers to eventually determine that 97 is the 25^{th} prime number. We would have already realize that '97' is both odd [(97 + 1) / 2 = 49^{th} odd number] and prime as it also satisfy the Odd number Label "Always end with a digit of 1, 3, 5, 7, or 9".

We advocate for the Hybrid method of Integer Sequence classification as a simple mathematical tool to meaningfully enable division of all integer sequences into either Hybrid integer sequences and non-Hybrid integer sequences. In regards to usage of the terms "Hybrid integer sequence" and "non-Hybrid integer sequence" in this paper, we now mention here the curious A228186 integer sequence [2] - the first ever novel (infinite length) Hybrid integer sequence artificially synthesized from Combinatorics Ratio [which could be defined as a Ratio Study with an inequality criteria]. A228186 is equal to non-Hybrid (usual 'garden-variety') integer sequence A100967 [3] except for the 21 'exceptional' terms at positions 0, 11, 13, 19, 21, 28, 30, 37, 39, 45, 50, 51, 52, 55, 57, 62, 66, 70, 73, 77, and 81 with their values given by the relevant A100967 term plus 1; and was previously published by us on The On-line Encyclopedia of Integer Sequences website in 2013.

The famous Riemann hypothesis was proposed in 1859 by German mathematician Bernhard Riemann (September 17, 1826 – July 20, 1866). Riemann conjecture / hypothesis refers to the famous conjecture explicitly equivalent to the mathematical statement that the critical line in the critical strip of Riemann zeta function is the location for all nontrivial zeros. At the most rudimentary level, Riemann conjecture / hypothesis simply refers to the generated curves of Riemann zeta function graphically intersecting both x- and y-axes [formally named the 'Origin'] an infinite number of times and these Origin intercepts are Incompletely Predictable [Pseudorandom] numbers constituting a countable infinite set (CIS) of irrational [transcendental] numbers. It is precisely because these nontrivial zeros are Incompletely Predictable thus obeying Complex Elementary Fundamental Laws that we will need to once again use our previously devised 'Virtual container' method to solve Riemann conjecture / hypothesis. As before in our recent paper on Polignac's and Twin prime conjectures, we will also designate Riemann conjecture / hypothesis as belonging to the Special-Class-of-Mathematical-Problems with Solitary-Proof-Solution because the successful proof for this conjecture / hypothesis seems to compulsorily require the aid of this [solitary] Virtual container method.

Remark 1.1. Computationally checking for nontrivial zeros to be correctly located on the critical line philosophically implies [but does not rigorously proof] Riemann hypothesis to be true.

In regards to Riemann conjecture / hypothesis, it was postulated in 1859 that the nontrivial zeros of Riemann zeta function must all lie on the critical line ($\sigma = \frac{1}{2}$) or [quivalently stated in this paper] must all match the Origin intercepts. This has previously been computationally checked for the first 10,000,000,000,000 identities (solutions). But the exercise of checking these nontrivial zeros identities only represent the 'tip of the iceberg' as they have already been shown to be infinitely many of them lying on this critical line by Hardy and Littlewood [4][5]. However this discovery by Hardy and Littlewood does not constitute the rigorous proof for Riemann conjecture / hypothesis because they have not mathematically exclude the possible existence of nontrivial zeros which are located away from the critical line. As we shall see, the use of our Virtual container [essentially embodied in Theorem I to IV below] in "containing" all nontrivial zeros is paramount in allowing us to convincingly prove Riemann conjecture / hypothesis via subsequent / concurrent "correct analysis" of this container. Resulting crucial primary and secondary beneficiary by-products arising out of this monumental feat (predominantly achieved through using the mathematical tool Sigma-Power Laws and concepts from the Hybrid method of Integer Sequence classification) promise to be aplenty.

For the purpose of this research paper, we are interested in various generated curves [Output] of Riemann zeta function which consist of "continuous" uncountable infinite set (UIS) when given the relevant values [Input] which again consist of "continuous" UIS. The "discrete" CIS of trivial and nontrivial zeros are subsets of this UIS Output. All the negative even integer number values constituting trivial zeros are Completely Predictable [obeying Simple Elementary Fundamental Laws] whereas all the irrational [transcendental] number values for nontrivial zeros, x- & y-axes intercepts are Incompletely Predictable [obeying Complex Elementary Fundamental Laws]. We note that each nontrivial zero / transcendental number once again has a CIS of Incompletely Predictable numerical digits after the decimal point. This Input and Output concepts for Riemann zeta function are succinctly summarized in the next two paragraphs.

Input: Continuous-type real number values denoted by σ and t variables –> Black Box: Unique Equations related to Riemann zeta function (with alternative + and - summation carried over all integers n = 1, 2, 3,...,∞ in the critical strip specifically via its surrogate Dirichlet eta function) –> Output: Continuous-type real values curves / spirals / loops denoted by $Re\{\zeta(\frac{1}{2} + \imath t)\}$ and $Im\{\zeta(\frac{1}{2} + \imath t)\}$ parameters. Then the Discrete-type integer number values of trivial zeros (via Riemann functional equation) and the Discrete-type real number values of nontrivial zeros are, respectively, graphical intercepts at the negative part of horizontal x-axis and the Origin. Note that there are two other possible graphical intercepts occurring at the remaining parts of horizontal x- and vertical y-axes coined respectively as the usual / traditional 'Gram points' (or Gram [y=0] points) and Gram [x=0] points. Then nontrivial zeros can also be dubbed Gram [x=0,y=0] points.

Thus all the Completely Predictable trivial zeros occur at σ = -2, -4, -6,... (all negative even numbers) and all the Incompletely Predictable nontrivial zeros [are conjectured to] occur at $\sigma = \frac{1}{2}$ with the locations given by parameter t [rounded off to six decimal places] = 14.134725, 21.022040, 25.010858, 30.424876, 32.935062, 37.586178,.... We begin our initial mission to prove the phenomenal Riemann conjecture / hypothesis (and explain its closely related Gram points) with the starting geometrical principle ”Any given 2-variable equation able to be computationally depicted by a 2-dimensional graph with its x- and y-axes relevantly defined often have point(s) of intersection on (i) x-axis, and/or (ii) y-axis, and/or (iii) both x- and y-axes (formally known as the Origin)” to provide equivalent geometrical-format-version of this conjecture / hypothesis - itself deemed to logically constitute one of three components of the ’glorified’ Dirichlet-Gram-Riemann conjecture / hypothesis. These, and highlight explanations for the closely-related Gram points, will be systematically expounded when we correctly utilize various selected fundamental mathematical principles in this paper. [Note that the three components of Dirichlet-Gram-Riemann conjecture / hypothesis refer to the study of x-axis, y-axis, and both x- & y-axes intercepts.]

The term ’hypothesis’ is generally taken to connote a ’conjecture’ once it has been rigorously proven to be true. For instance, the traditionally-dubbed ’Riemann hypothesis’ should ideally be previously labeled as ’Riemann conjecture’ because this ’Riemann hypothesis’ entity was chronologically used [incorrectly] in the era prior to rigorous proof being obtained for this conjecture. Therefore utilizing more accurate terms and metaphorically speaking in this paper, we aim to produce easy-to-understand [initially] geometrical-format-version & [subsequently] mathematical-format-version of the wider proposed-in-2017 Dirichlet-Gram-Riemann conjecture which also encompasses the proposed-in-1859 Riemann conjecture; with both conjectures now being able to be denoted by (respectively) the Dirichlet-Gram-Riemann hypothesis and Riemann hypothesis. In this paper, unless stated otherwise, the symbol ’log’ will refer to natural logarithm.

2. Intercepts of 2-Variable Equations

In the Class of n-Variable Equations with n = 2 [which translate to 2-Variable Equations], when computationally depicted by 2-dimensional graphs with their x- and y-axes relevantly defined; they often have one or more points of intersection on (i) x-axis, and/or (ii) y-axis, and/or (iii) both x- and y-axes [formally known as the ’Origin’]. The Origin, usually labeled with capital letter ’O’, is defined as the point where the vertical y-axis and the horizontal x-axis intersect each other. Not all functions, though, will have intercepts; which are where the graph crosses either the x-axis (viz. the x-axis intercept, usually referred to as ”zeros or roots of the equation”), or the y-axis (viz. the y-axis intercept), or both the x- and y-axes (viz. the Origin intercept).

There are eight possible Categories of Intercepts for 2-Variable Equations, as detailed below:

- Category I Intercept - comprising of nil intercept
- Category II Intercept - comprising of the single x-axis intercept(s) only
- Category III Intercept - comprising of the single y-axis intercept(s) only
- Category IV Intercept - comprising of the single Origin intercept(s) only
- Category V Intercept - comprising of the double x- and y-axes intercept(s)
- Category VI Intercept - comprising of the double x-axis and Origin intercept(s)
- Category VII Intercept - comprising of the double y-axis and Origin intercept(s)
- Category VIII Intercept - comprising of the triple x-, y-axes and Origin intercept(s)

Combinatorics language wise, the eight categories in total is obviously given by the sum total of (a) choosing zero item from three as in Category I; (b) choosing one item from three as in Category II, III, and IV; (c) choosing two items from three as in Category V, VI, and VII; and (d) choosing three items from three as in Category VIII. Theoretically, the various intercepts could numerically consist of whole numbers from $0, 1, 2, ..., \infty$. Permutational wise, this observation would apparently result in (not unexpectedly) limitless, or almost limitless, ’exotic-flavored’ equations; typified (for instance) by the simple case of 2-Variable Equation $y = \sin(x)$ (shown as part of Figure 1), which is easily seen as belonging to Category VI Intercept containing a Completely Predictable solitary Origin intercept combined with a Completely Predictable infinite number of x-axis intercepts.

3. Riemann zeta and Dirichlet eta functions

The iconic Riemann zeta function, $\zeta(s)$, is a function of the complex variable, s ($= \sigma \pm \imath t$), that analytically continues the sum of an infinite series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$. The arguments of this function are traditionally denoted by two letters: sigma (σ) for the real part, and t for the imaginary part where $i = \sqrt{-1}$ is the imaginary number. Alternatively stated, $\zeta(s)$ is the famous complex number infinite series constituting a real and an imaginary part determined by its

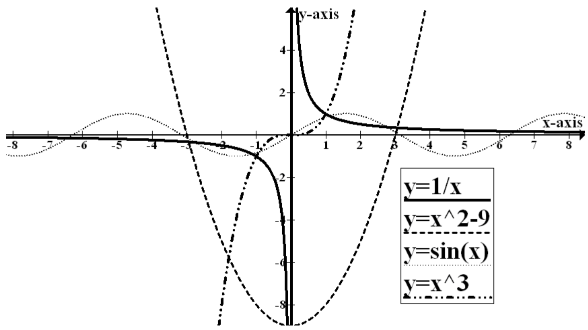


Figure 1: Sample graphs for Category I Intercept using $y = \frac{1}{x}$ with nil intercept; Category V Intercept using $y = (x+3)(x-3) = x^2 - 9$ with two x-axis intercepts and one y-axis intercept; Category VI Intercept using $y = \sin(x)$ with one Origin intercept and infinite number of x-axis intercepts; and Category VIII Intercept using $y = x^3$ with one Origin intercept.

complex variable, s; whereby s itself is further constituted by a real part, σ, and an imaginary part, t. In practice, the positive ($0 < t < +\infty$) and numerically equal to the negative ($-\infty < t < 0$), counterpart of the conjugate pairs for the x-axis, y-axis, and Origin intercepts is usually quoted or employed for calculation purposes.

The Dirichlet eta function, $\eta(s)$, also known as the alternating zeta function, must act as the proxy / surrogate for $\zeta(s)$ in the critical strip ($0 < \sigma < 1$) whereby the critical line ($\sigma = \frac{1}{2}$) lies. This is because $\zeta(s)$ only converges when $\sigma > 1$, implying that it is essentially undefined to the left of this region [viz. $0 < \sigma < 1$] which then requires its proxy $\eta(s)$ representation instead. Their mathematical relationship is defined by $\zeta(s) = \gamma.\eta(s)$, whereby the proportionality factor γ is defined as $\gamma = \dfrac{1}{(1-2^{1-s})}$ and $\eta(s) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^s} = \dfrac{1}{1^s} - \dfrac{1}{2^s} + \dfrac{1}{3^s} - \cdots$.

The above paragraphs are further discussed below in terms of Simplicity and Complexity. The concept of Simplexity [as opposed to Simplicity] could be amplified as the process by which nature strives towards simple ends by complex or complicated means. In other words, Simplexity may be defined as the combination of Simplicity and Complexity within the context of a dynamic relationship between means and ends. We observe the extra presence of alternating + and - signs in $\eta(s)$ conferring an extra layer of Complexity as opposed to just the + sign present in $\zeta(s)$ with relative Simplicity - bearing in mind that there are in general "Subjective to Semi-objective to Objective views on the Simplicity to Complexity continuum-spectrum range". Stated in a slightly different manner, the words 'Simplicity' and 'Complexity' can be seen to roughly progress along the following equivalent but opposing continuum-spectrum range.

Left——————————————>Right

Decreasing Order from Left to Right for

Maximal Simplicity to Minimal Simplicity range

should correspond to:

Increasing Order from Left to Right for

Minimal Complexity to Maximal Complexity range

Left——————————————>Right

$$
\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (1)\\
&= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \\
&= \Pi_{p\ prime} \frac{1}{(1-p^{-s})} \\
&= \frac{1}{(1-2^{-s})} . \frac{1}{(1-3^{-s})} . \frac{1}{(1-5^{-s})} . \frac{1}{(1-7^{-s})} . \frac{1}{(1-11^{-s})} \cdots \frac{1}{(1-p^{-s})} \cdots
\end{aligned}
$$

Eq. (1) can only be defined for the $1 < \sigma < \infty$ region whereby $\zeta(s)$ is absolutely convergent. There are no zeros located in this region. Note the equivalent Euler product formula with product over prime numbers [instead of summation over

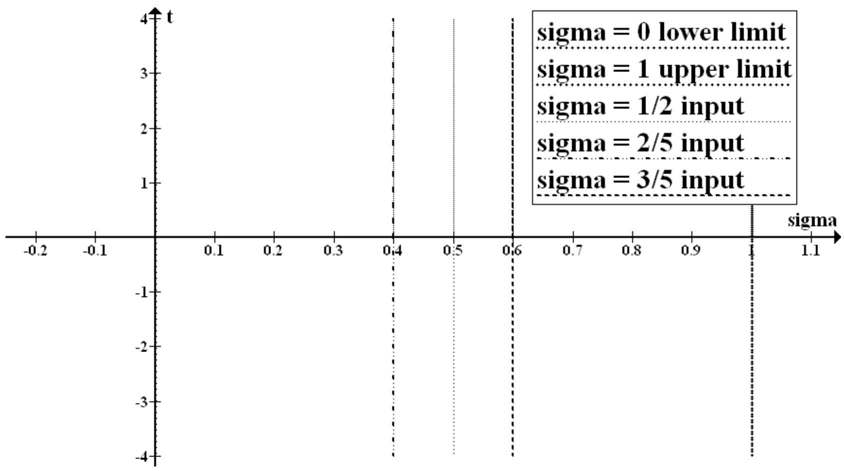


Figure 2: INPUT for $\sigma = \frac{1}{2}$, $\frac{2}{5}$, and $\frac{3}{5}$. For Riemann zeta function, the set of zeros, or roots, in ζ(s) consist of the easily identifiable (Completely Predictable) trivial zeros located at σ = negative even numbers -2, -4, -6,... and the not-so-easily identifiable (Incompletely Predictable) nontrivial zeros located at $\sigma = \frac{1}{2}$ for various computed t values. Both trivial and nontrivial zeros are of infinite magnitude.

natural numbers] can also be used to represent Riemann zeta function.

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right).\Gamma(1-s).\zeta(1-s) \tag{2}$$

With $\sigma = \frac{1}{2}$ as a symmetry line of reflection, Eq. (2) is the Riemann's functional equation fully satisfying $-\infty < \sigma < \infty$ and can be used to find all trivial zeros on the horizontal line at $\imath t = 0$ and σ = -2, -4, -6, . . . , ∞ [all negative even integer] whereby $\zeta(s) = 0$ because the factor $\sin(\frac{\pi s}{2})$ vanishes. Γ is the gamma function, an extension of the factorial function [a product function denoted by the ! notation; n! = n(n-1)(n-2). . . (n-(n-1))] with its argument shifted down by 1, to real and complex numbers. That is, if n is a positive integer, $\Gamma(n) = (n-1)!$

$$\begin{aligned} \zeta(s) &= \frac{1}{(1-2^{1-s})}.\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \\ &= \frac{1}{(1-2^{1-s})}.\left(\frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots\right) \end{aligned} \tag{3}$$

Eq. (3) is defined for all $\sigma > 0$ except for a simple pole at $\sigma = 1$. As just alluded to above, ζ(s) without the $\frac{1}{(1-2^{1-s})}$ proportionality factor, viz. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ is also known as Dirichlet eta (η) or alternating zeta function. This $\eta(s)$ function is a holomorphic function of s as defined by analytic continuation and can mathematically be seen to be defined at $\sigma = 1$ whereby an analogous trivial zeros [with presence only] for η(s) [and not for ζ(s)] on the vertical straight line $\sigma = 1$ are obtained at $s = 1 \pm i.\frac{2\pi k}{\log(2)}$ where k = 1, 2, 3, . . . , ∞. All nontrivial zeros are conjectured to be located on the critical line ($\sigma = \frac{1}{2}$) in the critical strip ($0 < \sigma < 1$) of this region.

Any given function or equation including our $\zeta(s)$ can be supplied with an INPUT and resulting in an OUTPUT. Figure 2 pictorially depict complex variable s (= $\sigma \pm \imath t$) as INPUT with x-axis denoting the real part Re{s}, equating to σ; and y-axis denoting the imaginary part Im{s}, equating to t. Figures 3, 4, and 5 schematically depict (respectively) $\zeta(s)$ as OUTPUT for real values of t running from 0 to 34 at $\sigma = \frac{1}{2}$ (critical line), $\sigma = \frac{2}{5}$, and $\sigma = \frac{3}{5}$ with x-axis denoting the real part Re{$\zeta(s)$} and y-axis denoting the imaginary part Re{$\zeta(s)$}. Riemann conjecture / hypothesis can be computationally visualized as the appearance of infinite number of Origin intercepts by its generated spirals / curves occurring only when $\sigma = \frac{1}{2}$ - this being previously computed for the first 10,000,000,000,000 nontrivial zeros solutions.

There will be infinite types-of-spirals possibilities associated with each and every σ value arising from all possible infinite

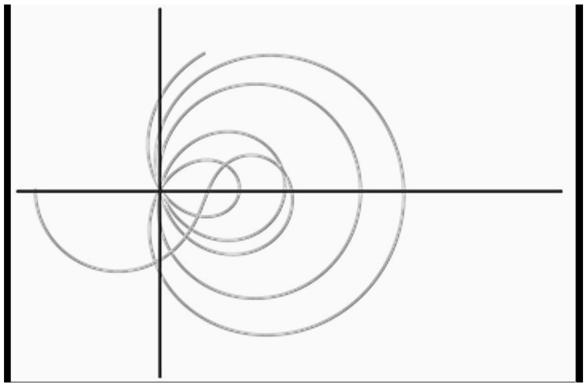


Figure 3: OUTPUT for $\sigma = \frac{1}{2}$. Schematically depicted polar graph of $\zeta(\frac{1}{2} + \imath t)$ with plot of $\zeta(s)$ along the critical line for real values of t running from 0 to 34, horizontal axis: $Re\{\zeta(\frac{1}{2} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{1}{2} + \imath t)\}$.

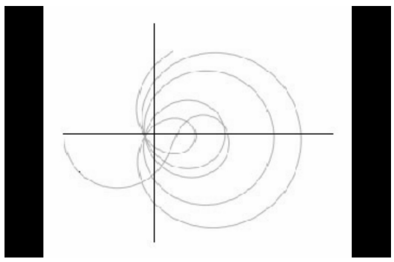


Figure 4: OUTPUT for $\sigma = \frac{2}{5}$ with identical axes definitions as that used in Figure 3.

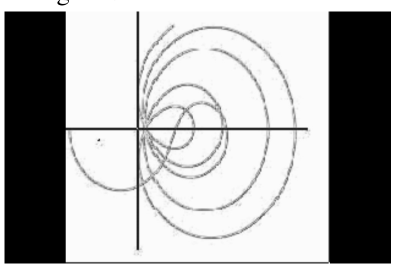


Figure 5: OUTPUT for $\sigma = \frac{3}{5}$ with identical axes definitions as that used in Figure 3.

σ values in the critical strip. From this, we derive the following two "interim" propositions for $\zeta(s)$ with their proofs naturally arising out of the other proofs obtained for subsequent lemmas, propositions and theorems below.

Proposition ["interim"] 3.1. Only at $\sigma = \frac{1}{2}$ value will the [singular] type-of-spirals generated belong to Category VIII Intercept comprising of triple x-axis, y-axis and Origin intercepts with all intercepts characterized by Incompletely Predictable property and of infinite magnitude.

Proposition ["interim"] 3.2. For all other $\sigma \neq \frac{1}{2}$ (viz. $0 < \sigma < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 1$) values, the [infinite] types-of-spirals generated will only belong to Category V Intercept comprising of double x-axis and y-axis intercepts with all intercepts characterized by Incompletely Predictable property and of infinite magnitude.

The foundation mathematics targeted to concisely prove the infinite magnitude existence of all intercepts associated with both propositions above naturally belong to the realm of rigorously proving Dirichlet-Gram-Riemann conjecture / hypothesis which literally consists of Gram [x=0] conjecture / hypothesis, Gram [y=0] conjecture / hypothesis, and Gram [x=0,y=0] conjecture / hypothesis (Riemann conjecture / hypothesis). The expanded explanations on each respective conjecture / hypothesis will be elaborated in the relevant sections below. But, firstly, preliminary $\zeta(s)$ and $\eta(s)$ nomenclature materials and their interpretational meanings are now provided with the following six clear-cut correlation points.

(A) At the one specific $\sigma = \frac{1}{2}$ value whereby the term Gram points is understood to denote the "Critical line-Gram points" official notation;

Point 1. The Origin intercepts are synonymous with Gram [x=0,y=0] points or the traditionally denoted 'nontrivial zeros'. The associated Riemann conjecture / hypothesis is synonymous with Gram [x=0,y=0] conjecture / hypothesis.

Point 2. The x-axis intercepts are synonymous with Gram [y=0] points or the traditionally denoted 'Gram points'. This is associated with Gram [y=0] conjecture / hypothesis.

Point 3. The y-axis intercepts are synonymous with Gram [x=0] points. This is associated with Gram [x=0] conjecture / hypothesis.

(B) For all other infinite $\sigma \neq \frac{1}{2}$ (viz. $0 < \sigma < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 1$) values whereby the term 'near-identical' (virtual) Gram points is understood to denote the "Non-critical lines-Gram points" official notation;

Point 4. The Origin intercepts are non-existent.

Point 5. The x-axis intercepts are synonymous with 'near-identical' (virtual) Gram [y=0] points. These points have totally different numerical values to Gram [y=0] points.

Point 6. The y-axis intercepts are synonymous with 'near-identical' (virtual) Gram [x=0] points. These points have totally different numerical values to Gram [x=0] points.

4. {Geometrical-format-version} conjectures / hypotheses

Our Gram [y=0] {geometrical-format-version} conjecture / hypothesis is explicitly equivalent to the statement that the infinite number of x-axis intercepts / Gram [y=0] points derived from the infinite number of spirals generated by Riemann zeta function in the critical strip occurs only on the critical line denoted by $\sigma = \frac{1}{2}$.

Our Gram [x=0] {geometrical-format-version} conjecture / hypothesis is explicitly equivalent to the statement that the infinite number of y-axis intercepts / Gram [x=0] points derived from the infinite number of spirals generated by Riemann zeta function in the critical strip occurs only on the critical line denoted by $\sigma = \frac{1}{2}$.

From previous reasoning above, we can now justifiably coin the 1859 Riemann {geometrical-format-version} conjecture / hypothesis as Gram [x=0,y=0] {geometrical-format-version} conjecture / hypothesis; which is explicitly equivalent to the statement that the infinite number of Origin intercepts / Gram [x=0,y=0] points / nontrivial zeros derived from the infinite number of spirals generated by Riemann zeta function in the critical strip occurs only on the critical line denoted by $\sigma = \frac{1}{2}$. In descriptive terms, this can succinctly be seen in Figure 3 as Riemann zeta function in the critical strip generating an infinite number of spirals graphically intersecting the Origin an infinite number of times only for the critical line which is denoted by $\sigma = \frac{1}{2}$.

Then, as honor and tribute to the three famous namesake mathematicians, it is pure common sense to create the 'glorified' Dirichlet-Gram-Riemann {geometrical-format-version} conjecture / hypothesis which is compatibly equivalent to the statement that the infinite number of Gram [x=0] points, Gram [y=0] points, and Gram [x=0,y=0] points derived from the infinite number of spirals generated by Riemann zeta function in the critical strip occurs only on the critical line denoted by $\sigma = \frac{1}{2}$.

5. Prerequisite lemma, corollary and propositions for Gram [x=0,y=0] conjecture / hypothesis

We treat and closely analyze both Riemann zeta and Dirichlet eta functions as unique mathematical objects looking for key intrinsic properties and behaviors. As the original true equations containing all possible x-axis, y-axis and Origin intercepts, Riemann zeta and Dirichlet eta functions by themselves (viz. without computationally supplying input data to generate the necessary output intercepts); both equations will intrinsically incorporate the actual presence [but not the actual locations] of the complete set of Gram [x=0] points, Gram [y=0] points, and Gram [x=0,y=0] points. We use the typical case of nontrivial zeros or Gram [x=0,y=0] points to obtain the required lemma, corollary, propositions and their proofs prior to ultimately proving Theorem I to IV. The lemma, corollary, propositions and proofs associated with the other two cases of Gram [x=0] and Gram [y=0] points will simply reflect "slight mathematical variations to the same theme for nontrivial zeros". These are outlined in Appendix 1.

Lemma 5.1. The Riemann-Dirichlet Ratio can be derived from Riemann zeta or Dirichlet eta function with complete ability to incorporate the actual presence [but not the actual locations] of the complete set of nontrivial zeros.

Proof. Euler formula is commonly stated as $e^{\imath x} = \cos x + \imath.\sin x$. The magnificent Euler identity (where $x = \pi$) is $e^{\imath\pi} = \cos\pi + \imath.\sin\pi = -1 + 0$, commonly stated as $e^{\imath\pi} + 1 = 0$. The n^s of Riemann zeta function can be expanded to $n^s = n^{(\sigma+\imath t)} = n^{\sigma}.e^{t.\log(n).\imath}$ since $n^t = e^{t.\log(n)}$. Apply the Euler formula to n^s will result in $n^s = n^{\sigma}.(\cos(t.\log(n)) + \imath.\sin(t.\log(n))$ - designated here with the short-hand notation $n^s(Euler)$ - whereby n^{σ} is the modulus and $t.\log(n)$ is the polar angle.

Apply $n^s(Euler)$ to Eq. (1), we have $\zeta(s) = Re\{\zeta(s)\} + \imath.Im\{\zeta(s)\}$ whereby $Re\{\zeta(s)\} = \sum_{n=1}^{\infty} n^{-\sigma}.\cos(t.\log(n))$ and $Im\{\zeta(s)\}$

$= \text{i}.\sum_{n=1}^{\infty} n^{-\sigma}.\sin(t.\log(n))$. As Eq. (1) is defined only for $\sigma > 1$ where zeros never occur, we will not carry out further treatment related to this subject area.

Apply $n^s(Euler)$ to Eq. (3), we have $\zeta(s) = \gamma.\eta(s) = \gamma.[Re\{\eta(s)\} + \imath.Im\{\eta(s)\}]$ whereby

$Re\{\eta(s)\} = \sum_{n=1}^{\infty}((2n-1)^{-\sigma}.\cos(t.\log(2n-1)) - (2n)^{-\sigma}.\cos(t.\log(2n)))$ and

$Im\{\eta(s)\} = \imath.\sum_{n=1}^{\infty}((2n)^{-\sigma}.\sin(t.\log(2n)) - (2n-1)^{-\sigma}.\sin(t.\log(2n)))$. Here γ is the proportionality factor $\frac{1}{(1-2^{1-s})}$.

Apply the trigonometry identity $\cos(x) - \sin(x) = \sqrt{2}.\sin\left(x + \frac{3}{4}\pi\right)$ to $\sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + Im\{\eta(s)\}$. Then,

$$\begin{aligned}\sum ReIm\{\eta(s)\} &= \sum_{n=1}^{\infty}[(2n-1)^{-\sigma}.\cos(t.\log(2n-1))\mathbf{TAG: + cos2n - 1(Re)} - (2n-1)^{-\sigma}.\sin(t.\log(2n-1)) \\ &\mathbf{TAG: - sin\ 2n - 1(Im)} - (2n)^{-\sigma}.\cos(t.\log(2n))\mathbf{TAG: - cos\ 2n(Re)} + (2n)^{-\sigma}.\sin(t.\log(2n))\mathbf{TAG: +}\boldsymbol{sin}\ \mathbf{2n(Im)}] \\ &= \sqrt{2}\sum_{n=1}^{\infty}[(2n-1)^{-\sigma}.\sin(t.\log(2n-1) + \frac{3}{4}\pi)\mathbf{TAG: + cos\ 2n - 1(Re)\& - sin\ 2n - 1(Im)} + -(2n)^{-\sigma}.\sin(t.\log(2n) \\ &+ \frac{3}{4}\pi)\mathbf{TAG: - cos\ 2n(Re)\& + sin\ 2n(Im)}]\end{aligned} \tag{4}$$

Note our self-explanatory **TAG** legend used to illustrate where each term in the equations above originated from. It can easily be seen that both terms in the final equation consist of a mixture of real and imaginary portions. As Riemann conjecture / hypothesis on nontrivial zeros based on $\zeta(s)$ is identical to that based on its proxy $\eta(s)$, then it is satisfied when

$$\sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + Im\{\eta(s)\} = 0 \tag{5}$$

Ignoring the $\sqrt{2}$ term temporarily and with the application of Eq. (5), Eq. (4) becomes

$$\sum_{n=1}^{\infty}(2n-1)^{-\sigma}.\sin(t.\log(2n-1) + \frac{3}{4}\pi) = \sum_{n=1}^{\infty}(2n)^{-\sigma}.\sin(t.\log(2n) + \frac{3}{4}\pi) \tag{6}$$

We note from the above sequential mathematical derivation of Eq. (6) that this equation will completely and intrinsically fulfills the 'presence of the complete set of nontrivial zeros' criteria.

$$\frac{\sum_{n=1}^{\infty}\sin(t.\log(2n) + \frac{3}{4}\pi)}{\sum_{n=1}^{\infty}\sin(t.\log(2n-1) + \frac{3}{4}\pi)} = \frac{\sum_{n=1}^{\infty}(2n)^{\sigma}}{\sum_{n=1}^{\infty}(2n-1)^{\sigma}} \tag{7}$$

Eq. 7 above will also abide to this specified criteria as it is simply the result of rearranging the terms in Eq. 6 thus giving rise to our desired Riemann-Dirichlet Ratio. The proof is now complete for Lemma 5.1.

Denote the left hand side ratio as Ratio R1 (of a 'cyclical' nature) and the right hand side ratio as Ratio R2 (of a 'non-cyclical' nature). Then the Riemann-Dirichlet Ratio can be deemed to be representing a more complicated 'dynamic' version of non-Hybrid integer sequence in that besides consisting of identical 'Class function' in each of the two functions when expressed in Ratio R1's numerator and denominator, this first Ratio R1 is again given as an equality to another seemingly different Ratio R2 whose numerator and denominator also consist of identical 'Class function'. One may intuitively think of a Hybrid integer sequence to metaphorically arise from a non-Hybrid integer sequence "in the limit" the non-identical 'Class function' in Hybrid integer sequence becomes the identical 'Class function' in the new non-Hybrid integer sequence. Note the absence and presence of σ variable in Ratio R1 and R2 respectively.

The Riemann-Dirichlet Ratio calculations, valid for all continuous real number values of t, would theoretically result in infinitely many non-Hybrid integer sequences [here arbitrarily] for the $0 < \sigma < 1$ critical strip region of interest with n = 1, 2, 3,..., ∞ being discrete integer number values, or n being continuous real numbers from 1 to ∞ with Riemann integral applied in the interval from 1 to ∞. This infinitely many integer sequences can geometrically be interpreted to representatively cover the entire plane of the critical strip bounded by σ values of 0 and 1, thus (at least) allowing our proposed proof to be of a 'complete' nature.

Proposition 5.2. The Sigma-Power Laws can be rigorously derived from Riemann-Dirichlet Ratio.

Proof. In Calculus, integration is defined as the reverse process of differentiation which is geometrically viewed as the area enclosed by the curve of the function and the axis. Using the definite integral I between the points a and b (i.e. in the interval [a, b] where a <b) and computing the value when $\Delta x \longrightarrow 0$, we get $I = \lim_{\Delta x \longrightarrow 0} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$ - this is the Riemann integral of the function f(x) in the interval [a, b]. We apply Riemann integral to the four continuous functions of Ratio R1 and Ratio R2 in Eq. (7) thus depicting the Riemann-Dirichlet Ratio in the integral forms - see the subsequent Eq. (12) below.

Thereafter, step-by-step we derive the closely related Dirichlet σ-Power Law [expressed in real numbers] and the Riemann σ-Power Law [expressed in real and complex numbers]. Due to the resemblance to various power-law functions in that the σ variable from s (= $\sigma + \imath t$) being the exponent of a power function n^{σ}, the log scale use, and the harmonic ζ(s) series connection in Zipf's law; we explain here why we have elected to endow our newly derived formula with the name Sigma-Power Law. Its Dirichlet and Riemann versions are directly related to each other via Dirichlet η(s) being the equivalence of Riemann ζ(s) but without the $\frac{1}{(1-2^{1-s})}$ proportionality factor. We stress that it is the main underlying mathematically-consistent properties of *symmetry* and *constraints* arising from this power law that also allowed our most direct, basic and elementary proof for the Riemann conjecture / hypothesis to mature. An important characteristic to note of σ-Power Law is that its exact formula expression in the usual mathematical language [$y = f(x_1, x_2)$ format description for a 2-variable function] consists of $y = \{2n\}$ or $\{2n-1\} = f(t, \sigma)$ with n = 1, 2, 3,. . . , ∞ or n = 1 to ∞ with Riemann integral application; $-\infty$<t< + ∞; and σ being of real number values 0 <σ <1 corresponding to the [arbitrarily defined] critical strip of interest in this particular case scenario.

For the, initially, {2n} parameter integration of R1, $\int_1^{\infty} \sin\left(t.\log(2n) + \frac{3}{4}\pi\right).dn$

Use integration by u-substitution technique to obtain $u = t.\log(2n) + \frac{3}{4}\pi$, $n = \frac{1}{2}e^{\frac{1}{t}(u-\frac{3}{4}\pi)}$, $\frac{du}{dn} = \frac{2t}{2n} = \frac{t}{n}, du = t.\frac{dn}{n}, dn = 2n.\frac{du}{2t} = n.\frac{du}{t}$

$\int_1^{\infty} \sin(u).\frac{n}{t}.du = \int_1^{\infty} \sin(u).\frac{1}{t}.\frac{1}{2}.e^{\frac{1}{t}(u-\frac{3}{4}\pi)}.du = \frac{1}{2t}.e^{\frac{3}{4}\pi}\int_1^{\infty} \sin(u).e^{\frac{1}{t}u}.du$

Use the Products of functions proportional to their second derivatives, namely the indefinite integral

$\int \sin(a.u).e^{b.u}du = \frac{e^{bu}}{a^2+b^2}(b.\sin(a.u) - a.\cos(a.u)) + C.$

Then a = 1, $b = \frac{1}{t}$, and temporarily ignore the $\frac{1}{2t}e^{\frac{3}{4}\pi}$ term, we have

$$\int_1^{\infty} \sin(u).e^{\frac{1}{t}u}.du$$
$$= [(e^{\frac{1}{t}u})/(1 + \frac{1}{t^2})).(\frac{1}{t}.\sin(u) - \cos(u)) + C]_1^{\infty}$$
$$= [(t^2.e^{\frac{1}{t}u})/(t^2 + 1).(\frac{1}{t}.\sin(u) - \cos(u)) + C]_1^{\infty}$$

Now apply the non-linear combination of sine and cosine functions identity, namely
$a.\sin(u) + b.\cos(u) = c.\sin(u + \varphi)$ where $c = \sqrt{(a^2 + b^2)}$ and $\varphi = atan2(b, a)$.

Here $a = \frac{1}{t}, b = -1, c = \sqrt{\left(\left(\frac{1}{t}\right)^2 + 1\right)} = \frac{\sqrt{(t^2+1)}}{t}$. Then we have

$$\int_1^{\infty} \sin(u).e^{\frac{1}{t}u}.du$$
$$= [(t^2.e^{\frac{1}{t}u})/(t^2 + 1).\frac{\sqrt{(t^2 + 1)}}{t}.\sin(u + atan2(b, a)) + C]_1^{\infty}$$
$$= [(t.e^{\frac{1}{t}u})/\sqrt{(t^2 + 1)}).\sin(u + \arctan(t)) + C]_1^{\infty}$$

But there was a $\frac{1}{2t}.e^{\frac{3}{4}\pi}$ term in front of this integral as can be seen above. Then after substituting this term and simplifying, the integral

$$\int_1^\infty \sin(u).e^{\frac{1}{t}u}.du$$
$$= [\left(e^{\frac{1}{t}u-\frac{3}{4}\pi}\right)/2\sqrt{(t^2+1)}.\sin(u-\arctan(t))+C]_1^\infty$$

But $u = t.\log(2n) + \frac{3}{4}\pi$. Reverting back to the n variable, and incorporating $\sqrt{2}$ originating from the beginning during Eq. (6) derivation, the equation for the {2n} parameter finally becomes

$$\sqrt{2}\int_1^\infty \sin(t.\log(2n)+\frac{3}{4}\pi).dn$$
$$= [\sqrt{2}.(\{2n\}.e^{\frac{1}{t}\cdot\frac{3}{4}\pi})/(2\sqrt{(t^2+1)}.e^{\frac{3}{4}\pi}.\sin(t.\log(2n)+\frac{3}{4}\pi-\arctan(t))+C)]_1^\infty \tag{8}$$

In a similar manner integration for the {2n-1} parameter, this equation becomes

$$[\sqrt{2}.(\{2n-1\}.e^{\frac{1}{t}\cdot\frac{3}{4}\pi})/(2\sqrt{(t^2+1)}.e^{\frac{3}{4}\pi}.\sin(t.\log(2n-1)+\frac{3}{4}\pi-\arctan(t))+C)]_1^\infty \tag{9}$$

In R2 using {2n} parameter,

$$\int_1^\infty (2n)^\sigma dn$$
$$= [1/(2(\sigma+1)).(2n)^{\sigma+1}+C)]_1^\infty$$
$$= [\frac{1}{3}\{2n\}(2n)^{\frac{1}{2}}+C)]_1^\infty \text{ when } \sigma = \frac{1}{2} \tag{10}$$

For the equivalent R2 based on {2n-1} parameter,

$$\int_1^\infty (2n-1)^\sigma dn$$
$$= [1/(2(\sigma+1)).(2n-1)^{\sigma+1}+C)]_1^\infty$$
$$= [\frac{1}{3}\{2n-1\}(2n-1)^{\frac{1}{2}}+C)]_1^\infty \text{ when } \sigma = \frac{1}{2} \tag{11}$$

The Ratio R1 and Ratio R2 of Riemann-Dirichlet Ratio (for $\sigma = \frac{1}{2}$) is defined by the integral

$$\frac{[(\{2n\}.(e^{\frac{1}{t}\cdot\frac{3}{4}\pi}/2\sqrt{(t^2+1)}.e^{\frac{3}{4}\pi}).\sin(t.\log(2n)+\frac{3}{4}\pi-\arctan(t))]_1^\infty}{[(\{2n-1\}.e^{\frac{1}{t}\cdot\frac{3}{4}\pi}/2\sqrt{(t^2+1)}.e^{\frac{3}{4}\pi}).\sin(t.\log(2n-1)+\frac{3}{4}\pi-\arctan(t))]_1^\infty} = \frac{[\frac{1}{3}\{2n\}(2n)^{\frac{1}{2}}]_1^\infty}{[\frac{1}{3}\{2n-1\}(2n-1)^{\frac{1}{2}}]_1^\infty}$$

Canceling out the common parameter {2n} and {2n-1} terms,

$$\frac{[(e^{\frac{1}{t}\cdot\frac{3}{4}\pi}/2\sqrt{(t^2+1)}.e^{\frac{3}{4}\pi}).\sin(t.\log(2n)+\frac{3}{4}\pi-\arctan(t))]_1^\infty}{[(e^{\frac{1}{t}\cdot\frac{3}{4}\pi}/2\sqrt{(t^2+1)}.e^{\frac{3}{4}\pi}).\sin(t.\log(2n-1)+\frac{3}{4}\pi-\arctan(t))]_1^\infty} \leftarrow \textbf{this is R1}$$
$$= \frac{[\frac{1}{3}(2n)^{\frac{1}{2}}]_1^\infty}{[\frac{1}{3}(2n-1)^{\frac{1}{2}}]_1^\infty} \leftarrow \textbf{this is R2} \tag{12}$$

The γ proportionality factor term in Riemann ζ function, viz. $\frac{1}{(1-2^{1-s})}$, can also be expressed with the aid of Euler formula as follows (with the formula for $\sigma = \frac{1}{2}$ substitution depicted last).

$$\begin{aligned}
&\frac{1}{(1-2^{1-s})}\\
&=\frac{(2^{\sigma}.2^{it})}{(2^{\sigma}.2^{it}-2)}\\
&=\frac{(2^{\sigma}.e^{t.\log(2\imath)})}{(2^{\sigma}.e^{t.\log(2\imath)}-2)}\\
&=\frac{(2^{\sigma}.(\cos(t.\log(2)+\imath.\sin(t.\log(2))}{(2^{\sigma}.(\cos(t.\log(2)+\imath.\sin(t.\log(2)-2)}\\
&=\frac{(2^{\frac{1}{2}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2))}{(2^{\frac{1}{2}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2)-2)}
\end{aligned} \tag{13}$$

The Dirichlet and Riemann σ-Power Laws are given by the exact formulae in Eqs. (14) to (17) below with ψ being the same proportionality constant valid for both power laws. We can now dispense with the constant of integration C. **Using Dimensional analysis approach we can easily conclude that the 'fundamental dimension' [Variable / Parameter / Number X to the power of Number Y] has to be represented by the particular 'unit of measure' [Variable / Parameter / Number X to the power of Number Y whereby Number Y needs to be of the specific value $\frac{1}{2}$] for Dimensional analysis homogeneity to occur. This *de novo* Dimensional analysis homogeneity equates to the location of the complete set of nontrivial zeros and is crucially a fundamental property present in all laws of Physics. The 'unknown' σ variable, now endowed with the value of $\frac{1}{2}$, is treated as Number Y.**

Dirichlet σ-Power Law using the {2n} parameter:

$$\left[2^{\frac{1}{2}}.\{2n\}.\frac{e^{\frac{1}{t}\cdot\frac{3}{4}\pi}}{2(t^2+1)^{\frac{1}{2}}.e^{\frac{3}{4}\pi}}.\sin(t.\log(2n)+\frac{3}{4}\pi-\arctan(t))\right]_1^{\infty}=\left[\psi.\frac{1}{3}\{2n\}(2n)^{\frac{1}{2}})\right]_1^{\infty}$$

With the common parameter {2n} canceling out on both sides, the equation reduces to

$$\left[2^{\frac{1}{2}}.\frac{e^{\frac{1}{t}\cdot\frac{3}{4}\pi}}{2(t^2+1)^{\frac{1}{2}}.e^{\frac{3}{4}\pi}}.\sin(t.\log(2n)+\frac{3}{4}\pi-\arctan(t))-\psi.\frac{1}{3}(2n)^{\frac{1}{2}})\right]_1^{\infty}=0 \tag{14}$$

Similarly for the {2n-1} parameter, this equivalent equation is

$$\left[2^{\frac{1}{2}}.\frac{e^{\frac{1}{t}\cdot\frac{3}{4}\pi}}{2(t^2+1)^{\frac{1}{2}}.e^{\frac{3}{4}\pi}}.\sin(t.\log(2n-1)+\frac{3}{4}\pi-\arctan(t))-\psi.\frac{1}{3}(2n-1)^{\frac{1}{2}})\right]_1^{\infty}=0 \tag{15}$$

Finally, the Riemann σ-Power Law is given by the exact formulae using {2n} and {2n-1} parameters with the $\gamma = (2^{\frac{1}{2}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2))/(2^{\frac{1}{2}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2)-2)$ substitution.

$$\begin{aligned}
&\left[2^{\frac{1}{2}}.\frac{e^{\frac{1}{t}\cdot\frac{3}{4}\pi}}{2(t^2+1)^{\frac{1}{2}}.e^{\frac{3}{4}\pi}}.\sin(t.\log(2n)+\frac{3}{4}\pi-\arctan(t))-\psi.\gamma.\frac{1}{3}(2n)^{\frac{1}{2}})\right]_1^{\infty}=0\\
&\left[2^{\frac{1}{2}}.\frac{e^{\frac{1}{t}\cdot\frac{3}{4}\pi}}{2(t^2+1)^{\frac{1}{2}}.e^{\frac{3}{4}\pi}}.\sin(t.\log(2n)+\frac{3}{4}\pi-\arctan(t))-\psi.\frac{2^{\frac{1}{2}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2))))}{2^{\frac{1}{2}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2)-2)))}.\frac{1}{3}(2n)^{\frac{1}{2}})\right]_1^{\infty}=0
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\left[2^{\frac{1}{2}}.\frac{e^{\frac{1}{t}\cdot\frac{3}{4}\pi}}{2(t^2+1)^{\frac{1}{2}}.e^{\frac{3}{4}\pi}}.\sin(t.\log(2n-1)+\frac{3}{4}\pi-\arctan(t))-\psi.\gamma.\frac{1}{3}(2n-1)^{\frac{1}{2}})\right]_1^{\infty}=0\\
&\left[2^{\frac{1}{2}}.\frac{e^{\frac{1}{t}\cdot\frac{3}{4}\pi}}{2(t^2+1)^{\frac{1}{2}}.e^{\frac{3}{4}\pi}}.\sin(t.\log(2n-1)+\frac{3}{4}\pi-\arctan(t))-\psi.\frac{2^{\frac{1}{2}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2))))}{2^{\frac{1}{2}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2)-2)))}.\frac{1}{3}(2n-1)^{\frac{1}{2}})\right]_1^{\infty}=0
\end{aligned} \tag{17}$$

The proof is now complete for Proposition 5.2.

Proposition 5.3. Application of Dimensional analysis homogeneity to Sigma-Power Laws will enable the rigorous proof for Riemann conjecture / hypothesis to mature.

Proof. We notice the γ proportionality factor given by Eq. (13) above when depicted with the $2^{\frac{1}{2}}$ constant numerical value (derived using $\sigma = \frac{1}{2}$ as conjectured in the original Riemann conjecture / hypothesis) further allowing, and enabling, *de novo* Dimensional analysis homogeneity compliance in Riemann σ-Power Law in Eqs. (16) and (17) above. This mathematical statement essentially complete the proof for Proposition 5.3 with complimentary demonstration below for the Dimensional analysis non-homogeneity case scenario.

Corollary 5.4. Dimensional analysis non-homogeneity in Sigma-Power Laws will never be associated with the one specific $\sigma = \frac{1}{2}$ value for Gram [x=0,y=0] points.

Proof. We illustrate the Dimensional analysis non-homogeneity property for a $\sigma = \frac{1}{4}$ arbitrarily chosen value [clear-cut case with {2n}-parameter] of Riemann σ-Power Law lying on a non-critical line (with total absence of nontrivial zeros) in the following formula derived using Eqs. (13) and (16). **As Ratio R1 component of Riemann-Dirichlet Ratio is independent of σ variable, unlike the Ratio R2 component of Riemann-Dirichlet Ratio and the γ proportionality factor which are dependent on σ variable, we now note the mixture of $\frac{1}{4}$ and $\frac{1}{2}$ exponents subtly, but nonetheless, present in this formula indicating Dimensional analysis non-homogeneity.** Also the replacement of $\frac{1}{3}$ fraction with $\frac{2}{5}$ fraction [derived from substituting $\sigma = \frac{1}{4}$ into $\frac{1}{2(\sigma+1)}$] has occurred. Mathematically, this Dimensional analysis non-homogeneity property for any real number value of σ, when $\sigma \neq \frac{1}{2}$ and $0 < \sigma < 1$, will always be present.

$$\left[2^{\frac{1}{2}}.\frac{e^{\frac{1}{t}\cdot\frac{3}{4}\pi}}{2(t^2+1)^{\frac{1}{2}}.e^{\frac{3}{4}\pi}}.\sin(t.\log(2n)+\frac{3}{4}\pi-\arctan(t))-\psi.\frac{2^{\frac{1}{4}}.(\cos(t.\log(2)+\iota.\sin(t.\log(2))))}{2^{\frac{1}{4}}.(\cos(t.\log(2)+\iota.\sin(t.\log(2)-2)))}.\frac{2}{5}(2n)^{\frac{1}{4}}\right]_1^{\infty}=0 \tag{18}$$

The proof is now complete for Corollary 5.4.

6. {Mathematical-format-version} conjectures / hypotheses

We now explore the corresponding Riemann, Gram [y=0], Gram [x=0], and Dirichlet-Gram-Riemann {mathematical-format-version} conjectures / hypotheses. The beautiful conjectures / hypotheses given in their geometrical-format-versions provide convincing but still insufficient evidence for their rigorous proofs. In this regard, mathematicians still demand that in reference to the 'grand' Dirichlet-Gram-Riemann {mathematical-format-version} conjecture / hypothesis fully constituted by the three subsets (i) Riemann or Gram [x=0,y=0], (ii) Gram [y=0], and (iii) Gram [x=0] {mathematical-format-version} conjectures / hypotheses; only when all are perfectly correct can they fulfill the absolute requirements for these rigorous proofs to be completely valid with [figuratively-speaking] 100% certainty. Fortunately, these mathematical-format-versions have conveniently been solved and proven not just beyond reasonable doubts, but beyond all doubts. The succinct summary on this point is appropriately expressed by the "Common Master Proof" outlined below for these three subsets using "Generic Gram conjecture / hypothesis" with their individualized "Generic Gram points". This Common Master Proof is centered on the four theorems (Virtual container) below [with their proofs automatically obtained from the preceding section above and Appendix 1].

Theorem I. The exact same {Generic Gram points}-Riemann-Dirichlet Ratio, directly derived from either the Riemann zeta or Dirichlet eta function, is an irrefutably accurate mathematical expression on the *de novo* criteria for the actual presence [but not the actual locations] of the complete set of (identical) infinite Generic Gram points in both functions.

Proof. This 'overall' proof for Theorem I is now complete as it will successfully incorporate the required proofs from Lemmas 5.1 and [Appendix] 1.1 which are associated with the complete set of the three types of Gram points [thus constituting the Generic Gram points].

Theorem II. Both the near-identical (by proportionality factor-related) {Generic Gram points}-Riemann Sigma-Power Law and {Generic Gram points}-Dirichlet Sigma-Power Law with their derivations based on either the numerator or denominator of {Generic Gram points}-Riemann-Dirichlet Ratio have Dimensional analysis homogeneity only when their common and unknown σ variable has a value of $\frac{1}{2}$ as its solution.

Proof. This 'overall' proof for Theorem II is now complete as it has successfully incorporated the required proofs from Propositions 5.2 & [Appendix] 1.2 on Sigma-Power Laws and Propositions 5.3 & [Appendix] 1.3 on Dimensional analysis homogeneity. These are applicable to the complete set of the three types of Gram points [which constitute the Generic Gram points].

Theorem III. The σ variable with value of $\frac{1}{2}$ derived using the Generic Gram points-Sigma-Power Law [from Theorem II above] is the exact same σ variable in Generic Gram conjecture which proposed σ to also have the value of $\frac{1}{2}$ (representing the critical line with $\sigma = \frac{1}{2}$ in the critical strip with $0 < \sigma < 1$) for the location of all Generic Gram points of Riemann zeta function [and Dirichlet eta function by default], thus providing irrefutable evidence for this Generic Gram conjecture to be correct with further clarification from Theorem IV below.

Proof. This 'overall' proof for Theorem III is now complete as Theorem III simply reflect Theorem II with its Generic Gram points-Sigma-Power Laws having the exact same σ variable as that referred to by the Generic Gram conjecture with each individual case independently referring to (i) the Generic Gram points being endowed with the same value of $\frac{1}{2}$ for their σ variable and (ii) the critical line also simultaneously being endowed with the same value of $\frac{1}{2}$ for its σ variable. In relation to Dimensional analysis homogeneity in Sigma-Power Laws, the condition "If σ parameter is endowed with the $\frac{1}{2}$ value, then Dimensional analysis homogeneity will be satisfied" acts as the one (and only one) possible mathematical solution.

Theorem IV. Condition 1. Any other values of σ apart from the $\frac{1}{2}$ value arising from $0 < \sigma < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 1$ in the critical strip does not contain any Generic Gram points ["the DA-wise mathematical impossibility argument" with resulting *de novo* DA non-homogeneity], together with Condition 2. The one and only one value of $\frac{1}{2}$ for σ in the critical strip contains all the Generic Gram points ["the DA-wise one and only one mathematical possibility argument" with resulting *de novo* DA homogeneity] from Theorem III, fully support the rather mute, but nevertheless the whole, point of study in this paper that Generic Gram conjecture is proven to be true when these two (mutually inclusive) conditions are met.

Proof. This 'overall' proof for Theorem IV is now complete as Theorem IV simply reflect the proofs from Theorem III on Generic Gram points with the additional proofs from Corollaries 5.4 & [Appemdix] 1.4 on non-Generic Gram points being tightly incorporated into this mathematical framework.

Dimensions are properties which can be measured. With global consensus, Systeme International d'Unites (SI Units) is the standard elements we use to scientifically quantify dimensions. In Dimensional analysis (DA), we are only concerned with the nature of the dimension i.e. its quality (and not its quantity). The following abbreviations are commonly used for examples of various dimensions (expressed in their SI base or SI derived units):

angle = θ in radian

length = L in meter

mass = M in kilogram

time = T in second

force = F in newton

temperature = Q in kelvin

For instance, the common and traditional unit of measurement of angles is degree. Radian is considered the SI derived unit of measurement of angles equivalent to the angle subtended at the centre of a circle by an arc equal in length to the radius. One radian is equal to about 57.3 deg and π radian is exactly 360 deg. Thus the term 'dimension' is traditionally used to refer to the units of measurement associated with various terms of an equation. However, we arbitrarily utilized 'dimension' to also refer to other mathematical properties such as the power or exponent associated with various terms of an equation. In other words, it is nothing more than a convenient language tool when using the term 'dimension' to directly refer to 'power' or 'exponent' in the sense that we could legitimately coin parallel terms such as Power analysis (PA) or Exponent analysis (EA) homogeneity [and non-homogeneity]. So what is this mysterious DA homogeneity (and non-homogeneity)? Any equation describing a physical situation will only be true (false) if both sides of the equation have the same (different) dimensions; that is, it must possess DA homogeneity (non-homogeneity). Examples: 2 kg + 3 kg = 5 kg is a valid equation because it possess DA homogeneity. 2 kg + 3 meter = 5 'something undefined nonsense unit' is definitely not a valid equation because it possess DA non-homogeneity; and it is a straight-forward exercise to arrive at this verdict.

The original equation $y^{\frac{1}{2}}=x^{\frac{1}{2}}+3$ which is equivalent to $y=x+6x^{\frac{1}{2}}+9$ possess DA or PA or EA homogeneity, having the same power or exponent $\frac{1}{2}$ [at least] in the original equation. But the original equation $y^{\frac{1}{2}}=x^{\frac{1}{3}}+3$ which is equivalent to $y=x^{\frac{2}{3}}+6x^{\frac{1}{3}}+9$ possess DA or PA or EA non-homogeneity, having different powers or exponents $\frac{1}{2}$, $\frac{1}{3}$ or $\frac{2}{3}$ in both the original and equivalent equation. We make a brief comment here that determing the validity of the last two equations endowed with DA or PA or EA non-homogeneity is intuitively not such a straight-forward exercise in this setting. Alternatively stated, those last two equations looks like being physically invalid but may still be mathematically valid.

In this paper, it is to be explicitly elaborated here that an invented phrase such as "Dimensional analysis homogeneity can prove Riemann hypothesis" will contextually never be used by us as a connotation that "Laws of Physics along with Scientific Principles even when all of them put together fully satisfying DA homogeneity *per se* can purportedly prove mathematical theorems". Rather, of utmost significance is this DA homogeneity (and non-homogeneity) being the secondary consequence [in a mathematical consistent albeit seemingly indirect manner] arising naturally out of our research methods used to fully prove Theorems I - IV above. As shown in the above section and in Appendix 1, the process to ultimately prove Theorems I - IV involves important mathematical tools such as Euler formula application, Ratio Study, Riemann integral, Calculus (Integration and Differentiation), Dimensional analysis, and concepts from the Hybrid method of Integer Sequence Classification.

Overall, the 6 steps ('mathematical foot-prints') in specific sequence required to prove Theorem I - IV are:

Step 1: Riemann zeta or Dirichlet eta function [for the critical strip $0 < \sigma < 1$] → *Step 2:* Riemann zeta or Dirichlet eta function [with Euler formula application] → *Step 3:* Riemann zeta or Dirichlet eta function [simplified and identical version specifically indicating the criteria for the presence of the complete set of Generic Gram points] → *Step 4:* Riemann-Dirichlet Ratio [in discrete summation format] → *Step 5:* Riemann-Dirichlet Ratio [in continuous integral format] → *Step 6:* Riemann Sigma-power law and Dirichlet Sigma-power law [both with Dimensional analysis homogeneity].

In the process of deriving our rigorous proof, the seemingly small but utterly essential mathematical step in recognizing and representing a 2-variable function with parameters {2n} or {2n-1} allows crucial moments where cancellation of the relevant "common" parameters in Riemann-Dirichlet Ratio and various Sigma-Power Laws can occur, further allowing the proper DA process to happen in the absolute correct way. These "common" parameters must be mathematically viewed as $(2n)^1$ or $(2n-1)^1$, viz. raised to a power (exponent) of 1 which will hamper proper DA if not serendipitously deleted – contrast this scenario with the presence of parameters $(2n)^{\frac{1}{2}}$ or $(2n-1)^{\frac{1}{2}}$, viz. raised to a power (exponent) of $\frac{1}{2}$ which will then enable proper DA (homogeneity) to proceed.

The key end-product equations arising out of, and fully conforming with, Theorem I - IV above then act as ultimate and final evidences for the complete proof of Riemann conjecture / hypothesis, Gram [y=0] conjecture / hypothesis, and Gram [x=0] conjecture / hypothesis. These closely related equations, for the {2n}-parameter case with ψ being the proportionality constant, will **subtly manifest the necessary DA homogeneity for the single $\sigma = \frac{1}{2}$ value involving the $\frac{1}{2}$ exponents [and DA non-homogeneity for all other infinite σ values with the given example of $\sigma = \frac{1}{4}$ involving the mixture of $\frac{1}{4}$ and $\frac{1}{2}$ exponents]**. They are regurgitated from Section 5 and Appendix 1 for convenience-sake and are respectively listed below:

Gram[x=0,y=0] or Riemann conjecture / hypothesis

$$[2^{\frac{1}{2}}.\frac{e^{\frac{1}{t}\cdot\frac{3}{4}\pi}}{2(t^2+1)^{\frac{1}{2}}.e^{\frac{3}{4}\pi}}.\sin(t.\log(2n)+\frac{3}{4}\pi-\arctan(t))-\psi.\frac{2^{\frac{1}{2}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2))))}{2^{\frac{1}{2}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2)-2)))}.\frac{1}{3}(2n)^{\frac{1}{2}})]_1^{\infty}=0 \tag{19}$$

$$[2^{\frac{1}{2}}.\frac{e^{\frac{1}{t}\cdot\frac{3}{4}\pi}}{2(t^2+1)^{\frac{1}{2}}.e^{\frac{3}{4}\pi}}.\sin(t.\log(2n)+\frac{3}{4}\pi-\arctan(t))-\psi.\frac{2^{\frac{1}{4}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2))))}{2^{\frac{1}{4}}.(\cos(t.\log(2)+\imath.\sin(t.\log(2)-2)))}.\frac{2}{5}(2n)^{\frac{1}{4}})]_1^{\infty}=0 \tag{20}$$

Gram [y=0] conjecture / hypothesis

$$[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n)-\arctan(t))-\psi.\frac{2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2)))}{2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2))-2)}.\frac{1}{3}.(2n)^{\frac{1}{2}}]_1^{\infty}=0 \tag{21}$$

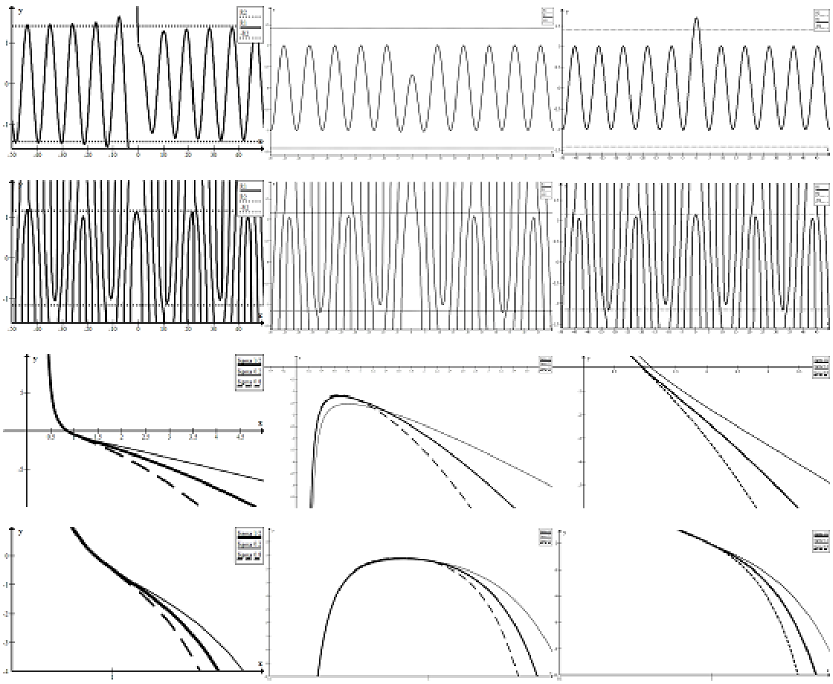
$$[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n)-\arctan(t))-\psi.\frac{2^{\frac{1}{4}}.(\cos(t.\log(2)+i.\sin(t.\log(2)))}{2^{\frac{1}{4}}.(\cos(t.\log(2)+i.\sin(t.\log(2))-2)}.\frac{2}{5}.(2n)^{\frac{1}{4}}]_1^{\infty}=0 \tag{22}$$

Gram [x=0] conjecture / hypothesis

$$[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n)+\arctan(t))-\psi.\frac{2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2)))}{2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2))-2)}.\frac{1}{3}.(2n)^{\frac{1}{2}}]_1^{\infty}=0 \tag{23}$$

$$[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n)+\arctan(t))-\psi.\frac{2^{\frac{1}{4}}.(\cos(t.\log(2)+i.\sin(t.\log(2)))}{2^{\frac{1}{4}}.(\cos(t.\log(2)+i.\sin(t.\log(2))-2)}.\frac{2}{5}.(2n)^{\frac{1}{4}}]_1^{\infty}=0 \tag{24}$$

The near-identical roughly speaking "physical manifestations" [6, Section 5, Figure 2a & Figure 2b, p. 17, Section 6, Figure 3a & Figure 3b, p. 18][7, Section 4, Figure 2a & 2b, p. 7, Section 5, Figure 3a & Figure 3b, pp. 8-9, Section 6, Figure 4a & 4b, p. 12, Figure 5a & 5b, p. 13] of various Ratio and Law formulae derived out of our three conjectures / hypotheses demonstrating properties such as Mathematical Symmetry, Chaos (Chaos theory with Chaotic non-linear deterministic dynamics), and Fractals (Fractal geometry with Self-similarity) are graphically portrayed in Figure 6. These miniaturized version graphs are grouped together from Left to Right as corresponding to Gram [x=0,y=0] (Riemann), Gram [y=0], and Gram [x=0] conjectures / hypotheses; and depicted using $\sigma = \frac{1}{5}, \frac{1}{2}, \& \frac{4}{5}$ values.



Graphs.png

Figure 6: Combined graphs giving us a snapshot on the "physical manifestations" of various Ratio and Law formulae derived out of our three conjectures / hypotheses.

7. Conclusions

We outline the overlapping pure and applied mathematics on our rigorous proof for Riemann conjecture / hypothesis in this paper only after more than 150 years from its initial proposal in 1859. Conforming to all logical arguments above, one can firmly believe that this lengthy delay is simply because Riemann zeta function contains an infinite number of Incompletely Predictable intercepts demonstrating Supraminimal Simplicity, or alternatively stated, contains none of the Completely Predictable infinite intercepts demonstrating Supramaximal Simplicity [whereby Supramaximal Simplicity does allow multiple type solutions to prove a particular conjecture / hypothesis]. This will then require a proviso that there is only one [solitary] way using the Virtual container method to solve this problem which belongs to the 'Special-Class-of-Mathematical-Problems with Solitary-Proof-Solution'.

Named after Danish mathematician Jørgen Pedersen Gram (June 27, 1850 – April 29, 1916), Gram points are the other conjugate pairs values on the critical line defined by $Im\{\zeta(\frac{1}{2} \pm \imath t)\} = 0$ whereby they obey Gram's Rule and Rosser's Rule with many other interesting characteristics. We have additionally provided a detailed explanation on the closely related Gram points in this paper.

The Z function is a function used for studying the Riemann zeta function along the critical line. It is also called the Riemann-Siegel Z function, the Riemann-Siegel zeta function, the Hardy function, the Hardy Z function, and the Hardy zeta function. It can be defined in terms of the Riemann-Siegel theta function and the Riemann zeta function by $Z(t) = e^{\imath\theta(t)}\zeta(\frac{1}{2} + \imath t)$ whereby $\theta(t) = \arg(\Gamma(\frac{(2\imath t+1)}{4})) - \frac{\log \pi}{2}t$. In the next paragraph, we will only outline a brief exposition of some of the useful properties of Gram points.

The algorithm used to compute Z(t) is called the Riemann-Siegel formula. The zeta function on the critical line, $\zeta(\frac{1}{2} + \imath t)$, will be real when $\sin(\theta(t)) = 0$. Positive real values of t where this occurs are called Gram points and can also be described as the points where $\frac{\theta(t)}{\pi}$ is an integer. The real part of zeta function on the critical line tends to be positive, while the imaginary part alternates more regularly between positive and negative values. That means that the sign of Z(t) must be opposite to that of the sine function most of the time, so one would expect the nontrivial zeros of Z(t) to alternate with zeros of the sine term, i.e. when θ takes on integer multiples of π. This turns out to hold most of the time and is known as Gram's Rule (Law) - a law which is violated infinitely often though. Thus Gram's Law is the statement that nontrivial zeros of Z(t) alternate with Gram points. Gram points which satisfy Gram's Law are called 'good', while those that do not are called 'bad'. A Gram block is an interval such that its very first and last points are good Gram points and all Gram points inside this interval are bad. The exercise of counting nontrivial zeros then reduces to that of counting all Gram points where Gram's Law is satisfied, and adding to that the count of nontrivial zeros inside each Gram block. With this process we do not have to locate nontrivial zeros exactly, and we just have to compute Z(t) accurately enough to show that it changes sign.

Up to this point, a crucial obervation to note from the above is that Riemann zeta or Dirichlet eta function will always generate an infinite number of relevant curves / spirals / loops on which will be located all nontrivial zeros and 'usual / traditional' Gram points (and our Gram [x=0] points) in a fixed relationship manner. Thus the afore-mentioned Gram's Law and its violation, Gram block, etc will predictably occur an infinite number of times.

Hadamard product:

$$\begin{aligned}&\zeta(s)\\&= \frac{e^{(\log(2\pi)-1-\frac{\gamma}{2}).s)}}{2(s-1).\Gamma(1+\frac{s}{2})}.\Pi_\rho\left(1-\frac{s}{\rho}\right).e^{\frac{s}{\rho}}\\&= \pi^{\frac{s}{2}}.\frac{\Pi_\rho\left(1-\frac{s}{\rho}\right)}{2(s-1).\Gamma\left(1+\frac{s}{2}\right)}\end{aligned}$$

Euler product formula:

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{n^s} \qquad \{\text{which is } \zeta(s)\}\\&= \Pi_{p\ prime}\frac{1}{(1-p^{-s})}\\&= \frac{1}{(1-2^{-s})}.\frac{1}{(1-3^{-s})}.\frac{1}{(1-5^{-s})}.\frac{1}{(1-7^{-s})}.\frac{1}{(1-11^{-s})}\cdots\frac{1}{(1-p^{-s})}\cdots\end{aligned}$$

The beautiful Hadamard product above is the infinite product expansion of Riemann zeta function, $\zeta(s)$, based on Weierstrass's factorization theorem - this product simultaneously contains both trivial and nontrivial zeros. The beautiful Euler product formula above connects Riemann zeta function and prime numbers and was discovered by Euler - this identity has, by definition, the left hand side being $\zeta(s)$ and the infinite product on the right hand side extends over all prime numbers p. The form of the Hadamard product clearly displays the simple pole at s = 1, the trivial zeros at all even negative integers due to the gamma function term in the denominator, and the nontrivial zeros at $s = \rho$; with the letter γ in the expansion here specifically denoting the Euler-Mascheroni constant. Note that with the second simpler infinite product expansion formula of Hadamard, to ensure convergence, the product should be taken over "matching pairs" of zeroes, i.e. the factors for a pair of zeroes of the form ρ and $1-\rho$ should be combined.

The usual primary by-products arising out of the rigorous proof for Riemann conjecture / hypothesis are often stated as "With this one solution, we have proven five hundred theorems or more at once". This apply to the many important theorems in number theory (mostly about prime numbers) that rely on properties of Riemann zeta or Dirichlet eta functions such as where trivial and nontrivial zeros are, and are not, located. A classical example of this primary by-product is the resulting absolute and full delineation of the prime number theorem, which relates to prime counting function. This function is usually denoted by $\pi(x)$ and is defined as the number of prime numbers less than or equal to x. In mathematics, the logarithmic integral function or integral logarithm li(x) is a special function. It is relevant in problems of physics and has number theoretic significance, occurring in prime number theorem as an estimate of the number of prime numbers less than a given value. In prime number theorem, the form of this function is defined so that li(2)=0; viz. li(x) $\equiv \int_2^x \frac{du}{\ln u}$ = li(x)-li(2). The symbol 'ln' here denotes natural logarithm. Thus the rigorous proof for Riemann hypothesis on nontrivial zeros location at $\sigma=\frac{1}{2}$, together with the negative even number locations for trivial zeros, is instrumental in proving the efficacy of techniques that estimate $\pi(x)$ efficiently and reasonably well. In particular, our rigorous proof of Riemann conjecture / hypothesis will now confirm the "best possible" bound for the error / the "smallest possible" error of the prime number theorem.

We mention here that there are other less accurate ways of estimating $\pi(x)$ such as that conjectured by Gauss and Legendre at the end of the 18th century. This is approximately x/ln x in the sense $\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$. For prime counting function, other functions more convenient to work with can also be utilized and they open up a whole new world of marvelous mathematical relationships. An example is Riemann prime counting function (aka prime power counting function), commonly denoted by J(x). This non-infinite series function has jumps of 1/n for prime powers p^n, and with it taking a value halfway between the two sides at discontinuities. Amazingly, the prime counting function $\pi(x)$ is related to J(x) by the Mobius transform. More amazingly still, J(x) is related to Riemann zeta function through the Mellin transform (which is an integral transform).

In number theory, Skewes' number is any of several extremely large numbers used by South African mathematician Stanley Skewes as upper bounds for the smallest natural number x for which li(x)<$\pi(x)$. These bounds have since been improved by others: there is a crossing near $e^{727.95133}$. It is not known whether it is the smallest. John Edensor Littlewood, who was Skewes' research supervisor, had proved in 1914 [8] that there is such a number (and so, a first such number); and indeed found that the sign of the difference $\pi(x)$ - li(x) changes infinitely often. This then refute all prior numerical evidence available that seem to suggest li(x) was always more than $\pi(x)$. The massive key point here is that the 100% accurate $\pi(x)$ mathematical tool being "wrapped around" by the approximate mathematical tool li(x) infinitely often via this 'sign of difference' changes meant that li(x) must be the most efficient approximate mathematical tool. Contrast this with the x/ln x approximate mathematical tool where values obtained diverge away from $\pi(x)$ at increasingly greater rate when larger range of prime numbers are being studied.

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Appendix 1. Prerequisite lemma, corollary and propositions for Gram [x=0] and Gram [y=0] conjectures / hypotheses

For the mathematical treatment of our two cases on Gram [x=0] and Gram [y=0] points here, we will follow similar procedure carried out for the above case on nontrivial zeros (Gram [x=0,y=0] points).

Lemma [Appendix] 1.1. The {Modified-for-Gram [x=0] & [y=0] points}-Riemann-Dirichlet Ratio can be derived from Riemann zeta or Dirichlet eta function with the complete ability to incorporate the actual presence [but not the actual locations] of the complete set of Gram [x=0] and Gram [y=0] points.

Proof. We hereby depict the case on Gram [y=0] points (which is the usual 'Gram points') to obtain the relevant [simply named] {Modified-for-Gram points}-Riemann-Dirichlet Ratio. Apply n^2(Euler) to Eq. (3), we have $\zeta(s) = \gamma.\eta(s) = \gamma.[Re\{\eta(s)\} + i.Im\{\eta(s)\}]$ whereby

$$Re\{\eta(s)\} = \sum_{n=1}^{\infty}((2n-1)^{-\sigma}.\cos(t.\log(2n-1)) - (2n)^{-\sigma}.\cos(t.\log(2n))) \text{ and}$$
$$Im\{\eta(s)\} = i.\sum_{n=1}^{\infty}((2n)^{-\sigma}.\sin(t.\log(2n)) - (2n-1)^{-\sigma}.\sin(t.\log(2n))) \tag{25}$$

Here γ is the proportionality factor $\frac{1}{(1-2^{1-s})}$.

As Gram [y=0] points based on $\zeta(s)$ is identical to that based on its proxy $\eta(s)$, then Gram [y=0] conjecture is satisfied when

$$\sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + 0, \text{ or simply } Im\{\eta(s)\} = 0 \tag{26}$$

Applying Eq. (26) to Eq. (25), this equation can be simplified and be reduced to

$$\sum_{n=1}^{\infty}((2n)^{-\sigma}.\sin(t.\log(2n)) - (2n-1)^{-\sigma}.\sin(t.\log(2n))) = 0$$
$$\sum_{n=1}^{\infty}(2n)^{\sigma}.\sin(t.\log(2n)) = \sum_{n=1}^{\infty}(2n-1)^{-\sigma}.\sin(t.\log(2n)) \tag{27}$$

We note from the above sequential mathematical derivation of Eq. (27) that this equation will completely and intrinsically fulfills the 'presence of the complete set of nontrivial zeros' criteria.

$$\frac{\sum_{n=1}^{\infty}\sin(t.\log(2n))}{\sum_{n=1}^{\infty}\sin(t.\log(2n-1))} = \frac{\sum_{n=1}^{\infty}(2n)^{\sigma}}{\sum_{n=1}^{\infty}(2n-1)^{\sigma}} \tag{28}$$

Eq. 28 above will also abide to this specified criteria as it is simply the result of rearranging the terms in Eq. 27 thus giving rise to our desired {Modified-for-Gram points}-Riemann-Dirichlet Ratio. This proof is now complete for Lemma [Appendix] 1.1.

Denote the left hand side ratio as Ratio R1 (of a 'cyclical' nature) and the right hand side ratio as Ratio R2 (of a 'non-cyclical' nature). The {Modified-for-Gram points}-Riemann-Dirichlet Ratio calculations, valid for all continuous real number values of t, would theoretically result in infinitely many non-Hybrid integer sequences [here arbitrarily] for the $0 < \sigma < 1$ critical strip region of interest with $n = 1, 2, 3, \ldots, \infty$ being discrete integer number values, or n being continuous real numbers from 1 to ∞ with Riemann integral applied in the interval from 1 to ∞. This infinitely many integer sequences can geometrically be interpreted to representatively cover the entire plane of the critical strip bounded by σ values of 0 and 1, thus (at least) allowing our proposed proof on Gram conjecture to be of a 'complete' nature.

Proposition [Appendix] 1.2. The equivalent Sigma-Power Laws can be rigorously derived from {Modified-for-Gram [x=0] & [y=0] points}-Riemann-Dirichlet Ratio.

Proof. We hereby depict the case on Gram [y=0] points (which is the usual 'Gram points') to obtain its equivalent Sigma-Power Laws. We apply Riemann integral to the four continuous functions of Ratio R1 and Ratio R2 in Eq. (28) thus depicting the {Modified-for-Gram points}-Riemann-Dirichlet Ratio in the integral forms - see the subsequent Eq. (33) below.

Thereafter, step-by-step we derive the closely related {Modified-for-Gram points}-Dirichlet σ-Power Law [expressed in real numbers] and the {Modified-for-Gram points}-Riemann σ-Power Law [expressed in real and complex numbers] - these two laws are further elaborated below. The {Modified-for-Gram points}-Sigma-Power Law has its Dirichlet and Riemann versions directly related to each other via Dirichlet $\eta(s)$ being the equivalence of Riemann $\zeta(s)$ but without the $\frac{1}{(1-2^{1-s})}$ proportionality factor. We stress that it is the main underlying mathematically-consistent properties of *symmetry* and *constraints* arising from this power law that also allowed our most direct, basic and elementary proof for the Gram conjecture to mature. An important characteristic to note of {Modified-for-Gram points}- σ-Power Law is that its exact formula expression in the usual mathematical language [$y = f(x_1, x_2)$ format description for a 2-variable function] consists of $y = \{2n\}$ or $\{2n-1\} = f(t,\sigma)$ with $n = 1, 2, 3, \ldots, \infty$ or n = 1 to ∞ with Riemann integral application; $-\infty < t < +\infty$; and σ being of real number values $0 < \sigma < 1$ corresponding to the [arbitrarily defined] critical strip of interest in this particular case scenario.

For the, initially, {2n} parameter integration of R1, $\int_1^\infty \sin(t.\log(2n)).dn$

Use integration by u-substitution technique to obtain $u = t.\log(2n)$, $n = \frac{1}{2}e^{\frac{1}{t}(u)}$, $\frac{du}{dn} = \frac{2t}{2n} = \frac{t}{n}$, $du = t.\frac{dn}{n}$, $dn = 2n.\frac{du}{2t} = n.\frac{du}{t}$ $\int_1^\infty \sin(u).\frac{n}{t}.du = \int_1^\infty \sin(u).\frac{1}{t}.\frac{1}{2}.e^{\frac{1}{t}(u)}.du = \frac{1}{2t}\int_1^\infty \sin(u).e^{\frac{1}{t}u}.du$

Use the Products of functions proportional to their second derivatives, namely the indefinite integral $\int \sin(a.u).e^{b.u}du = \frac{e^{bu}}{a^2+b^2}(b.\sin(a.u) - a.\cos(a.u)) + C$ (Comparatively, we observe that $\int \cos(a.u).e^{b.u}du = \frac{e^{bu}}{a^2+b^2}(b.\cos(a.u) + a.\sin(a.u)) + C$). Then a = 1, b = $\frac{1}{t}$, and temporarily ignore the $\frac{1}{2t}$ term, we have

$$\int_1^\infty \sin(u).e^{\frac{1}{t}u}.du$$
$$= [(e^{\frac{1}{t}u})/(1 + \frac{1}{t^2}).(\frac{1}{t}.\sin(u) - \cos(u)) + C]_1^\infty$$
$$= [(t^2.e^{\frac{1}{t}u})/(t^2 + 1).(\frac{1}{t}.\sin(u) - \cos(u)) + C]_1^\infty$$

Now apply the non-linear combination of sine and cosine functions identity, namely $a.\sin(u) + b.\cos(u) = c.\sin(u + \varphi)$ where $c = \surd(a^2 + b^2)$ and $\varphi = a\tan 2(b, a)$. Here $a = \frac{1}{t}$, b = -1, $c = \surd(\frac{1}{t})2 + 1) = \frac{\surd(t^2+1)}{t}$. Then we have

$$\int_1^\infty \sin(u).e^{\frac{1}{t}u}.du$$
$$= [(t^2.e^{\frac{1}{t}u})/(t^2 + 1).\frac{\surd(t^2 + 1)}{t}.\sin(u + atan2(b, a)) + C]_1^\infty$$
$$= [(t.e^{\frac{1}{t}u})/(\surd(t^2 + 1)).\sin(u + \arctan(-t)) + C]_1^\infty$$

But there was a $\frac{1}{2t}$ term in front of this integral as can be seen above. Then after substituting this term and simplifying, the integral

$$\int_1^\infty \sin(u).e^{\frac{1}{t}u}.du$$
$$= [(e^{\frac{1}{t}u})/2\surd(t^2 + 1).\sin(u - \arctan(t)) + C]_1^\infty$$

But $u = t.\log(2n)$. Reverting back to the n variable, the equation for the {2n} parameter finally becomes

$$\int_1^\infty \sin(t.\log(2n)).dn$$
$$= [(\{2n\}.e^{\frac{1}{t}})/(2\surd(t^2 + 1).\sin(t.\log(2n) - \arctan(t)) + C)]_1^\infty \quad (29)$$

In a similar manner integration for the {2n-1} parameter, this equation becomes

$$[(\{2n-1\}.e^{\frac{1}{t}})/(2\surd(t^2 + 1).\sin(t.\log(2n-1) - \arctan(t)) + C)]_1^\infty \quad (30)$$

In R2 using {2n} parameter,

$$\int_1^\infty (2n)^\sigma dn$$
$$= [1/(2(\sigma + 1)).(2n)^{\sigma+1} + C)]_1^\infty$$
$$= [\frac{1}{3}\{2n\}(2n)^{\frac{1}{2}} + C)]_1^\infty \text{ when } \sigma = \frac{1}{2} \quad (31)$$

For the equivalent R2 based on 2n-1 parameter,

$$\int_1^\infty (2n-1)^\sigma dn$$
$$= [1/(2(\sigma+1)).(2n-1)^{\sigma+1} + C)]_1^\infty$$
$$= [\frac{1}{3}\{2n-1\}(2n-1)^{\frac{1}{2}} + C)]_1^\infty \text{ when } \sigma = \frac{1}{2} \quad (32)$$

The Ratio R1 and Ratio R2 of {Modified-for-Gram points}-Riemann-Dirichlet Ratio (for $\sigma = \frac{1}{2}$) is defined by the integral

$$\frac{[(\{2n\}.(e^{\frac{1}{t}}\cdot/2\sqrt{(t^2+1)}).\sin(t.\log(2n)-\arctan(t))]_1^\infty}{[(\{2n-1\}.e^{\frac{1}{t}}/2\sqrt{(t^2+1)}).\sin(t.\log(2n-1)-\arctan(t))]_1^\infty} = \frac{[\frac{1}{3}\{2n\}(2n)^{\frac{1}{2}}]_1^\infty}{[\frac{1}{3}\{2n-1\}(2n-1)^{\frac{1}{2}}]_1^\infty}$$

Canceling out the common parameter {2n} and {2n-1} terms,

$$\frac{[(e^{\frac{1}{t}}/2\sqrt{(t^2+1)}).\sin(t.\log(2n)-\arctan(t))]_1^\infty}{[(e^{\frac{1}{t}}/2\sqrt{(t^2+1)}).\sin(t.\log(2n-1)-\arctan(t))]_1^\infty} \leftarrow \textbf{this is R1}$$
$$= \frac{[\frac{1}{3}(2n)^{\frac{1}{2}}]_1^\infty}{[\frac{1}{3}(2n-1)^{\frac{1}{2}}]_1^\infty} \leftarrow \textbf{this is R2} \quad (33)$$

The {Modified-for-Gram points}-Dirichlet and {Modified-for-Gram points}-Riemann σ-Power Laws are given by the exact formulae in Eqs. (34) to (37) below with ψ being the same proportionality constant valid for both power laws. We can now dispense with the constant of integration C. **Using Dimensional analysis approach we can easily conclude that the 'fundamental dimension' [Variable / Parameter / Number X to the power of Number Y] has to be represented by the particular 'unit of measure' [Variable / Parameter / Number X to the power of Number Y whereby Number Y needs to be of the specific value $\frac{1}{2}$] for Dimensional analysis homogeneity to occur. This *de novo* Dimensional analysis homogeneity equates to the location of the complete set of Gram [y=0] points and is crucially a fundamental property present in all laws of Physics. The 'unknown' σ variable, now endowed with the value of $\frac{1}{2}$, is treated as Number Y.**

{Modified-for-Gram points}-Dirichlet σ-Power Law using the {2n} parameter:

$$[\{2n\}.\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n)-\arctan(t))]_1^\infty = \psi.\frac{1}{3}\{2n\}(2n)^{\frac{1}{2}}]_1^\infty$$

With the common parameter {2n} canceling out on both sides, the equation reduces to

$$[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n)-\arctan(t)) - \psi.\frac{1}{3}(2n)^{\frac{1}{2}}]_1^\infty = 0 \quad (34)$$

Similarly for the {2n-1} parameter, this equivalent equation is

$$[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n-1)-\arctan(t)) - \psi.\frac{1}{3}(2n-1)^{\frac{1}{2}}]_1^\infty = 0 \quad (35)$$

Finally, the {Modified-for-Gram points}-Riemann σ-Power Law is given by the exact formulae using {2n} and {2n-1} parameters with the $\gamma = \frac{2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2))}{(2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2)-2)}$ substitution.

$$\left[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n)-\arctan(t)) - \psi.\gamma.\frac{1}{3}.(2n)^{\frac{1}{2}}\right]_1^\infty = 0$$
$$\left[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n)-\arctan(t)) - \psi.\frac{2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2)))}{2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2))-2)}.\frac{1}{3}.(2n)^{\frac{1}{2}}\right]_1^\infty = 0 \quad (36)$$

$$\left[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n-1)-\arctan(t))-\psi.\gamma.\frac{1}{3}.(2n-1)^{\frac{1}{2}}\right]_1^{\infty}=0$$

$$\left[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n-1)-\arctan(t))-\psi.\frac{2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2)))}{2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2))-2)}.\frac{1}{3}.(2n-1)^{\frac{1}{2}}\right]_1^{\infty}=0 \quad (37)$$

The proof is now complete for Proposition [Appendix] 1.2.

Proposition [Appendix] 1.3. Application of Dimensional analysis to the equivalent Sigma-Power Laws will enable the rigorous proofs for Gram [x=0] & Gram [y=0] conjectures / hypotheses to mature.

Proof. We again depict the case on Gram [y=0] points here. We note the γ proportionality factor given by Eq. (13) above when depicted with the $2^{\frac{1}{2}}$ constant numerical value (derived using $\sigma = \frac{1}{2}$ as proposed in the original Gram [y=0] conjecture / hypothesis) further allowing, and enabling, *de novo* Dimensional analysis homogeneity compliance in the {Modified-for-Gram points}-Riemann σ-Power Law in Eqs. (36) and (37) above. This mathematical statement essentially complete the proof for Proposition [Appendix] 1.3 with complimentary demonstration below for the Dimensional analysis non-homogeneity case scenario.

Corollary [Appendix] 1.4. Dimensional analysis non-homogeneity in Sigma-Power Laws will never be associated with the one specific $\sigma = \frac{1}{2}$ value for Gram [x=0] & Gram [y=0] points.

Proof. We again depict the case on Gram [y=0] points here. We illustrate the Dimensional analysis non-homogeneity property for a $\sigma = \frac{1}{4}$ arbitrarily chosen value [clear-cut case with {2n}-parameter] of {Modified-for-Gram points}-Riemann σ-Power Law lying on a non-critical line (with total absence of Gram [y=0] points) in the following formula derived using Eqs. (13) and (36). **As Ratio R1 component of {Modified-for-Gram points}-Riemann-Dirichlet Ratio is independent of σ variable, unlike the Ratio R2 component of {Modified-for-Gram points}-Riemann-Dirichlet Ratio and the γ proportionality factor which are dependent on σ variable, we now note the mixture of $\frac{1}{4}$ and $\frac{1}{2}$ exponents subtly, but nevertheless, present in this formula confirming Dimensional analysis non-homogeneity.** Also the replacement of $\frac{1}{3}$ fraction with $\frac{2}{5}$ fraction [derived from substituting $\sigma = \frac{1}{4}$ into $\frac{1}{2(\sigma+1)}$] has occurred. Mathematically, this Dimensional analysis non-homogeneity property for any real number value of σ, when $\sigma \neq \frac{1}{2}$ and $0 < \sigma < 1$, will always be present indicative of the full presence of {Non-critical lines}-Gram [y=0] points, or by the same token, indicative of total absence of Gram [y=0] points.

$$\left[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n)-\arctan(t))-\psi.\frac{2^{\frac{1}{4}}.(\cos(t.\log(2)+i.\sin(t.\log(2)))}{2^{\frac{1}{4}}.(\cos(t.\log(2)+i.\sin(t.\log(2))-2)}.\frac{2}{5}.(2n)^{\frac{1}{4}}\right]_1^{\infty}=0 \quad (38)$$

The proof is now complete for Corollary [Appendix] 1.4.

The $\frac{1}{2}$ exponent in Eq. (38) only occur once in the denominator of the first term. The subtlety of Dimensional analysis non-homogeneity for {Non-critical lines}-Gram [y=0] points is even more pronounced when compared to its closely related cousin Eq. (18) above for Riemann σ-Power Law [with easy clarification and confirmation of the $\frac{1}{2}$ exponent occurring twice in the first term].

For Gram [x=0] points, Gram [x=0] conjecture / hypothesis is satisfied by Eqs. (39) to (41) below, whereby Eq. (39) is the equivalent of Eq. (26) above.

$$\sum ReIm\{\eta(s)\} = 0 + Im\{\eta(s)\}, \text{ or simply } Re\{\eta(s)\} = 0 \quad (39)$$

Not unexpectedly with only minor subtraction (-) operator to addition (+) operator sign change required, the equivalent to Eq. (36) and Eq. (38) above using {2n} parameter for Gram [x=0] points can easily be derived to (respectively) be:

$$\left[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n)+\arctan(t))-\psi.\frac{2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2)))}{2^{\frac{1}{2}}.(\cos(t.\log(2)+i.\sin(t.\log(2))-2)}.\frac{1}{3}.(2n)^{\frac{1}{2}}\right]_1^{\infty}=0 \quad (40)$$

$$\left[\frac{e^{\frac{1}{t}}}{2(t^2+1)^{\frac{1}{2}}}.\sin(t.\log(2n)+\arctan(t))-\psi.\frac{2^{\frac{1}{4}}.(\cos(t.\log(2)+i.\sin(t.\log(2)))}{2^{\frac{1}{4}}.(\cos(t.\log(2)+i.\sin(t.\log(2))-2)}.\frac{2}{5}.(2n)^{\frac{1}{4}}\right]_1^{\infty}=0 \quad (41)$$

Dimensional analysis homogeneity and non-homogeneity are demonstrated once again by Eq. 40 and Eq. 41 respectively.