Failure of the Diagonal Argument

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Abstract: It is shown that Cantor's diagonal argument fails because either there is no actual infinity and hence no defined diagonal number or there is actual infinity but the diagonal number cannot be distinguished from all real numbers of the Cantor list. Further it is shown by another argument that there are not uncountably many paths in the complete infinite Binary Tree.

I

Real numbers of the unit interval [0, 1] can be visualized as paths in the infinite Binary Tree:

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           0.
         /    \
      0(1)     1(2)
      / \      / \
  0(3)  1(4) 0(5)  1(6)
  ...
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Some numbers possess two representations, namely as a single path and as the limit of a sequence of other paths. The real number *zero* for instance is represented by the path z = 0.000... with a countably infinite (henceforth abbreviated by $\aleph_0$) number of 0-nodes and as the limit of the sequence 0.t, 0.0t, 0.00t, 0.000t, ... → 0.000..., where t is an arbitrary tail. When we assume that every path can be distinguished from all others, then z differs from all paths of a sequence like

$$P = 0.111..., 0.0111..., 0.00111..., 0.000111..., ...$$

The limit of a strictly increasing infinite sequence, like the numbers of 0-nodes in the paths of P, is never a term of the sequence. That means for instance, when each path $p \in P$ is completely coloured, then the path z is not yet completely coloured; precisely $\aleph_0$ 0-nodes are missing. Let [z]

be the number of 0-nodes of z and $[p]$ the number of 0-nodes of p then, with $[z] = \aleph_0$ and $[p] \in \mathbb{N}$,

$$\exists z \, \forall p \in P: [p] < [z].$$

On the one hand, this is clear, because each $p \in P$ has a tail of $\aleph_0$ 1-nodes, while z does not. Let's call this position A.

On the other hand, we cannot find any 0-node of z which is *not* covered by a path of P. That means, we *cannot* distinguish z from *all* paths p of P, let alone from *all* paths of the Binary Tree. Let's call this position B.

According to position B we can completely colour the Binary Tree by a set U of paths with $z \notin U$ as well as by a set V of paths with $z \in V$. According to position A this is *not* possible by *different* sets of infinite paths. Each and every path is required.

These are two opposing statements, only one of which can be and also must be correct.

If position A is correct, then there must be $\aleph_0$ 0-nodes in z that cannot be found and defined, because we can only find such 0-nodes which exclude that z is identical with a *given* path of P, but never evidence for z differing from *all* paths of P. Nodes however that cannot be found and defined cannot (yet) exist. They only can "come into being" in potential infinity. That implies that actual infinity (the complete infinite Binary Tree, and its infinite paths with $\aleph_0$ nodes) does not exist. We have only the potentially infinite path z with "always more" nodes

$$\forall p \in P \, \exists z: [p] < [z].$$

(Note that it is impossible to define *all* nodes of a path by comparing it with *some given* paths.)

If position B is correct, then the actually infinite node-sequence z cannot be distinguished from all paths of P per se. That implies that in a special Cantor list like the following one

0.1
0.01
0.001
0.0001
...

when replacing the diagonal digit 1 by 0, the resulting diagonal number 0.000..., although differing from every entry, cannot be distinguished from all entries either. (Note the significant difference: *Every entry* is followed by infinitely many others while *all entries* are not followed by any further entry.) Therefore Cantor's diagonal argument fails in this special case and hence always, because it is based on a proof by contradiction which never must fail.

As a résumé we can state that there is no uncountable set of paths in the Binary Tree: In case A there is no actual or "completed" infinity and therefore it cannot be surpassed. In case B the possibly existing actual infinity does not support the diagonal argument and does not supply a proof of uncountable sets including the representation of real numbers by paths.

II

But three is an even stronger argument excluding uncountably many *distinct* paths of the Binary Tree. It is easy to see that the set of nodes is countable (cp. the indexes in the above figure). In order to find an upper estimate for the number of *distinct* bundles of paths, note that from every node two distinct bundles of paths emerge. Remove the nodes of a path and put all its nodes side by side on some common level, say the first one. Let from each node emerge two new paths. Then $2 \cdot \aleph_0 - 1 = \aleph_0$ paths have been added. Repeat this with the (remaining) nodes of every path. And even if $\aleph_0$ paths were appended to each node: From all nodes at that common first level not more than $\aleph_0 \cdot \aleph_0 = \aleph_0$ paths could emerge – and that is an upper estimate.

Conclusion: There are not uncountably many real numbers in the unit interval and not uncountably many infinite paths in the Binary Tree.