Elementary Proof That There are Infinitely Many Primes *p* such that *p* − 1 is a Perfect Square (Landau's Fourth Problem)

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Abstract

This paper presents a complete and exhaustive proof of Landau's Fourth Problem. The approach to this proof uses same logic that Euclid used to prove there are an infinite number of prime numbers. Then we use a proof found in Reference 1, that if p > 1 and d > 0 are integers, that p and p+ d are both primes if and only if for integer ***m***:

$$m = (p-1)!\left(\frac{1}{p} + \frac{(-1)^d(d!)}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

We use this proof for ***d = 2n + 1*** to prove the infinitude of Landau's Fourth Problem prime numbers.

The author would like to give many thanks to the authors of 1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004, Exercise Number 161 (see Reference 1). The proof provided in Exercise 6 is the key to making this paper on the Landau's Fourth Problem possible.

Introduction

At the 1912 International Congress of Mathematicians, Edmund Landau listed four basic problems about primes. These problems were characterized in his speech as "unattackable at the present state of mathematics" and are now known as **Landau's problems**. They are as follows:

1. Goldbach's conjecture: Can every even integer greater than 2 be written as the sum of two primes?
2. Twin prime conjecture: Are there infinitely many primes p such that $p + 2$ is prime?
3. Legendre's conjecture: Does there always exist at least one prime between consecutive perfect squares?
4. Are there infinitely many primes p such that $p - 1$ is a perfect square? In other words: Are there infinitely many primes of the form $n^2 + 1$?

As of 2017, the third problem is still unresolved, however the author has already proven the first and second problems using elementary proofs (see references 2 and 3).

This proof will solve the fourth problem above, namely we shall prove that there are infinitely many primes p such that $p - 1$ is a perfect square. We shall do this by proving that there are infinitely many primes of the form $n^2 + 1$. The first several primes of form $n^2 + 1$ are:

2, 5, 17, 37, 101, 197, 257, 401, 577, 677, 1297, 1601, 2117

Proof

In number theory, Landau's Fourth Problem's states:

It is conjectured that there are infinitely many primes of the form $n^2 + 1$.

First we shall assume that the set of primes of the form $n^2 + 1$ are finite and then we shall prove that this is false, which shall prove that are primes of the form $n^2 + 1$ are infinite. First let us define the finite set of Landau's Fourth Problem as p_i where i is finite, and the last $i = n$. Our goal is to prove that i is infinite to disprove our assumption of finiteness. Even though we have assumed that the set of finite primes of the form $n^2 + 1$ since there are an infinite number of prime numbers we can pick a prime number p which is outside the finite set of primes of the form $n^2 + 1$. Therefore a prime cannot exist that is prime otherwise we would have discovered a prime of the form $(n+1)^2 + 1$ which is outside our assumed set of finite primes of the form $n^2 + 1$. Thus all we need to do is to prove that there exists a prime of the form $(n+1)^2 + 1$ outside our assumed finite set is to prove that $(n+1)^2 + 1$ is prime since we already know that many prime numbers exist outside the finite set of primes of the form $n^2 + 1$.

We use the proof, provided in Reference 1, that if $p > 1$ and $d > 0$ are integers, that p and $p + d$ are both primes if and only if for integer m:

$$m = (p-1)!\left(\frac{1}{p} + \frac{(-1)^d(d!)}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

For our case p is known to be prime and is of form $n^2 + 1$ and,

$d = (n+1)^2 + 1 - (n^2 + 1)$ for Landau's Fourth Problem, where n is any integer, therefore:

$$p = n^2 + 1, \text{ and}$$

$$d = (n+1)^2 + 1 - (n^2 + 1)$$

$$d = (n+1)^2 - n^2$$

$$d = n^2 + 2n + 1 - n^2$$

$$d = 2n + 1 = \text{odd integer}$$

Therefore,

$$p + d = n^2 + 1 + 2n + 1 = n^2 + 2(n + 1)$$

And ***p + d*** is prime if and only if ***m*** is an integer for:

$$m = (p\text{-}1)!\left(\frac{1}{p} + \frac{(-1)^{(2n+1)}((2n+1)!}{p+2n+1}\right) + \frac{1}{p} + \frac{1}{p+2n+1}$$

Multiplying by ***p***, and since ***2n + 1*** is always odd, then $(-1)^{(2n+1)} = -1$

$$mp = (p)!\left(\frac{1}{p} + \frac{-(2n+1)!}{p+2n+1}\right) + 1 + \frac{p}{p+2n+1}$$

Multiplying by ***(p + 2n + 1)***,

$$(p + 2n + 1)mp = (p + 2n + 1)(p)!\left(\frac{1}{p} + \frac{-(2n+1)!}{p+2n+1}\right) + p + (2n + 1) + p$$

Reducing again,

$$(p + 2n + 1)mp = (p)!\left(\frac{(p + 2n + 1\ )}{p} - (2n + 1)!\right) + 2p + 2n + 1$$

Factoring out, **(*p*)!,**

$$(p + 2n + 1)mp = p(p\text{-}1)!\left(\frac{(p + 2n + 1)}{p} - (2n + 1)! + 2p + 2n + 1\right.$$

And reducing one final time,

$$(p + 2n + 1)mp = (p\text{-}1)!(p + 2n\ +\ 1\ \ - p(2n\ +\ 1\ )!) + 2p + 2n + 1$$

We already know ***p*** is prime, therefore, ***p*** = integer. Since ***p*** is an integer and by definition ***n*** is an integer, the right hand side of the above equation is an integer (likewise the left hand side of the equation must also be an integer). Since the right hand side of the above equation is an integer and ***p*** and ***n*** are integers on the left hand side of the equation, then ***(p + 2n + 1)*** is also an integer. Therefore there are only 4 possibilities (see 1, 2a, 2b, and 2c below) that can hold for ***m*** so the left hand side of the above equation is an integer, they are as follows.

1) ***m*** is an integer, or

2) ***m*** is a rational fraction that is divisible by ***p***. This implies that ***n*** = $\frac{x}{p}$ where, p is prime and ***x*** is an integer. This results in the following three possibilities:

a. Since ***m*** = $\frac{x}{p}$, then **p** = $\frac{x}{m}$, since ***p*** is prime, then ***p*** is only divisible by ***p*** and **1**, therefore, the first possibility is for n to be equal to ***p*** or 1 in this case, which are both integers, thus ***m*** is an integer for this first case.

b. Since $\boldsymbol{m = \frac{x}{p}}$, and x is an integer, then ***x*** is not evenly divisible by ***p*** unless ***x = p,*** or ***x*** is a multiple of ***p***, where ***x = yp***, for any integer **y**. Therefore ***m*** is an integer for ***x = p*** and ***x = yp***.

c. For all other cases of, integer ***x, m = xp , m*** is not an integer.

To prove there is a Pell Prime, outside our set of finite Pell Primes, we only need to prove that there is at least one value of ***m*** that is an integer, outside our finite set. There can be an infinite number of values of ***m*** that are not integers, but that will not negate the existence of one Pell Prime, outside our finite set of Pell Primes.

First the only way that n cannot be an integer is if every ***m*** satisfies paragraph 2.c above, namely, $\boldsymbol{m = \frac{x}{p}}$, where ***x*** is an integer***, x ≠ p, x ≠ yp, m ≠ p, and m ≠ 1*** for any integer ***y***. To prove there exists at least one Pell Prime outside our finite set, we will assume that no integer ***m*** exists and therefore no Pell Primes exist outside our finite set. Then we shall prove our assumption to be false.

Proof: Assumption no values of ***m*** are integers, specifically, every value of ***m*** is

$\boldsymbol{m = \frac{x}{p}}$,where ***x*** is an integer, ***x ≠ p, x ≠ yp, m ≠ p,*** and ***m ≠ 1***, for any integer ***y***. Paragraphs **1, 2.a, and 2.b** prove cases where ***m*** can be an integer, therefore our assumption is false and **there exist values of *m* that are integers.**

Since we have already shown that ***p*** and ***p + (2n + 1)***, where ***d = 2n + 1***, are both primes if and only if for integer ***m***:

$$\boldsymbol{m = (p\text{-}1)!\left(\frac{1}{p} + \frac{(-1)^{d}(d!)}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}}$$

It suffices to show that there is at least one integer m to prove there exists a Landau's Fourth Problem Prime outside our set of finite set of Landau's Fourth Problem.

Since there exists an m = integer, we have proven that there is at least one p and $p + 2n + 1$ that are both prime. Since we showed earlier that if $p + 2n + 1$ is prime then it also is not in the finite set of $p_i, p_i + 2n + 1$ of primes of form $n^2 + 1$, therefore, since we have proven that there is at least one $p + 2n + 1$ that is prime, then we have proven that there is a prime outside the our assumed finite set of of form $n^2 + 1$. This is a contradiction from our assumption that the set of primes of form $n^2 + 1$ is finite, therefore, by contradiction the set of primes of form $n^2 + 1$ is infinite. Also this same proof can be repeated infinitely for each finite set of primes of form $n^2 + 1$, in other words a new prime of form $n^2 + 1$ can added to each set of finite primes of form $n^2 + 1$. This thoroughly proves that an infinite number of Landau's Fourth Problem exist.

References:

1) 1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004, Exercise Number 161

2) Elementary Proof of the Goldbach Conjecture, Stephen Marshall, 13 February 2017

3) Updated Elementary Proofs of Polignac Prime Conjecture, Goldbach Conjecture, Twin Prime Conjecture, Cousin Prime Conjecture, and Sexy Prime Conjecture, Stephen Marshall, 2 February 2017