

An Expression For The Argument of ζ at Zeros on the Critical Line

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Abstract

It is conjectured that $\arg\zeta\left(\frac{1}{2} + it_n\right) = S_n(t_n)$ where $S_n(t_n) = \pi\left(\frac{3}{2} - \text{frac}\left(\frac{\vartheta(t_n)}{\pi}\right) - \lfloor \tilde{g}^{-1}(n) \rfloor - n\right)$ and $\tilde{g}^{-1}(t) = \frac{t \ln\left(\frac{t}{2\pi e}\right)}{2\pi} + \frac{7}{8}$ is the inverse of $\tilde{g}(n) = \frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}$ which accurately approximates the Gram points $g(n)$ and that all of the non-trivial zeros of ζ , enumerated by n , are on the critical line. Therefore, if $\arg\zeta\left(\frac{1}{2} + it_n\right) = S_n(t_n)$ then the exact transcendental equation for the Riemann zeros has a solution for each positive integer n which proves that Riemann's hypothesis is true since the counting function for zeros on the critical line is equal to the counting function for zeros on the critical strip if the transcendental equation has a solution for each n .

1 Introduction

1.1 The Riemann-Siegel $\vartheta(t)$ Function

The Riemann-Siegel ϑ function is defined by

$$\begin{aligned}\vartheta(t) &= -\frac{i}{2} \left(\ln \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \ln \Gamma\left(\frac{1}{4} - \frac{it}{2}\right) \right) - \frac{\ln(\pi)t}{2} \\ &= \arg\left(\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right) - \frac{\ln(\pi)t}{2}\end{aligned}$$

Let

$$\tilde{\vartheta}(t) = \frac{t \ln\left(\frac{t}{2\pi e}\right)}{2} - \frac{\pi}{8}$$

be the approximate ϑ function where the Γ function has been replaced with its Stirling approximation.

$$\Gamma(s) \simeq \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s} \tag{1}$$

The $\vartheta(t)$ function is not invertible but the inverse of its approximation $\tilde{\vartheta}(t)$ is defined by a linear function of the Lambert W function given by

$$\tilde{\vartheta}^{-1}(t) = \frac{\pi + 8t}{4W\left(\frac{\pi + 8t}{8\pi e}\right)} \tag{2}$$

Let $\text{frac}(x) = \begin{cases} x - \lfloor x \rfloor & x \geq 0 \\ x - \lceil x \rceil & x < 0 \end{cases} \forall x \in \mathbb{R}$ be the function which gives the fractional part of a real number by subtracting either the floor $\lfloor x \rfloor$ or the ceiling $\lceil x \rceil$ of x from x , depending upon its sign. Furthermore, let

$$S(t) = \arg\left(\zeta\left(\frac{1}{2} + it\right)\right) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} ((S(\rho + i\varepsilon) - S(\rho - i\varepsilon)))$$

be the argument of ζ on the critical line.

Conjecture 1. *The exact equation for the imaginary part of the n -th zero of $\zeta(\frac{1}{2} + it)$ [1, Equation 20]*

$$\vartheta(t_n) + S(t_n) = \left(n - \frac{3}{2}\right)\pi \quad (3)$$

has a solution for each integer $n \geq 1$ where t_n enumerate the zeros of Z on the real line and the zeros of ζ on the critical line

$$\zeta\left(\frac{1}{2} + it_n\right) = 0 \forall n \in \mathbb{Z}^+ \quad (4)$$

where \mathbb{Z}^+ denotes the positive integers. [1, Equation 14]

1.2 The Gram Points

The n -th Gram point is defined to be the solution of the equation

$$\vartheta(t) = (n-1)\pi \quad (5)$$

A very accurate approximation $\tilde{g}(n)$ to the Gram points is $g(n)$ is found by inverting $\tilde{\vartheta}(t)$ to get the exact solution

$$\begin{aligned} \tilde{g}(n) &= \{t: \tilde{\vartheta}(t) - (n-1)\pi = 0\} \\ &= \left\{ t: \left(\frac{t \ln\left(\frac{t}{2\pi e}\right)}{2} - \frac{\pi}{8} \right) - (n-1)\pi = 0 \right\} \\ &= \frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)} \\ &= g(n) + O(\delta_n) \end{aligned} \quad (6)$$

where W is the Lambert W function, and the approximation bounds δ_n when $n = 1$ is $\delta_1 = 0.00223698\dots$, followed by $\delta_2 = 0.00137812\dots$ and decreases monotonically with increasing n , that is, $\delta_{n+1} < \delta_n$. The inverse of $\tilde{g}(n)$ is given by

$$\begin{aligned} \tilde{g}^{-1}(n) &= \{t: \tilde{g}(n) = 0\} \\ &= \frac{t \ln\left(\frac{t}{2\pi e}\right)}{2\pi} + \frac{7}{8} \end{aligned} \quad (7)$$

Now define the infinite sequence of functions indexed by $n \in \mathbb{Z}^+$

$$\begin{aligned} T_n(t) &= 1 + \lfloor \tilde{g}^{-1}(n) \rfloor - n \\ &= 1 + \left\lfloor \frac{t \ln\left(\frac{t}{2\pi e}\right)}{2\pi} + \frac{7}{8} \right\rfloor - n \end{aligned} \quad (8)$$

Near each ‘‘bad’’ Gram point there will be a corresponding zero on the critical line which has an argument not on the principal branch. The function $T_n(t)$ determines how many multiples of π to add or subtract to $-\frac{1}{2} - \lfloor \frac{\vartheta(s)}{\pi} \rfloor$ so that it agrees with the argument of ζ at a zero on the critical line where it is discontinuous, having the value $\lim_{\varepsilon \rightarrow 0} \frac{1}{2}((S(\rho + i\varepsilon) - S(\rho - i\varepsilon)))$ when $\zeta(\rho) = 0$.

Definition 2. *Let*

$$\begin{aligned}
S_n(t) &= \pi \left(\frac{1}{2} - \text{frac} \left(\frac{\vartheta(t)}{\pi} \right) - T_n(t) \right) \\
&= \pi \left(\frac{1}{2} - \text{frac} \left(\frac{\vartheta(t)}{\pi} \right) - (\lfloor \tilde{g}^{-1}(n) \rfloor - n + 1) \right) \\
&= \pi \left(\frac{3}{2} - \text{frac} \left(\frac{\vartheta(t)}{\pi} \right) - (\lfloor \tilde{g}^{-1}(n) \rfloor - n) \right) \\
&= \pi \left(\frac{3}{2} - \text{frac} \left(\frac{\vartheta(t)}{\pi} \right) + \left(\lfloor \frac{t \ln(\frac{t}{2\pi e})}{2\pi} + \frac{7}{8} \rfloor - n \right) \right)
\end{aligned} \tag{9}$$

Let $s_\vartheta(t) = \frac{\vartheta(t)}{|\vartheta(t)|}$ be the sign of $\vartheta(t)$.

Conjecture 3. *The argument $S(t)$ of $\zeta(\frac{1}{2} + it)$ at the n -th non-trivial zero $\zeta(\frac{1}{2} + it_n) = 0 \forall n \geq 1$ on the critical strip is equal to $s_\vartheta(t)S_n(t)$, that is*

$$\begin{aligned}
S(t_n) &= s_\vartheta(t)S_n(t_n) = \frac{1}{2} \left(\lim_{t \rightarrow t_n^-} S(t_n) + \lim_{t \rightarrow t_n^+} S(t_n) \right) \\
&= \frac{\vartheta(t_n)}{|\vartheta(t_n)|} \pi \left(\frac{3}{2} - \text{frac} \left(\frac{\vartheta(t_n)}{\pi} \right) + \left(\lfloor \frac{t_n \ln(\frac{t_n}{2\pi e})}{2\pi} + \frac{7}{8} \rfloor - n \right) \right)
\end{aligned} \tag{10}$$

Theorem 4. *If Conjecture 3 is true then Conjecture 1 is true and, due to Lemma 15, so is Conjecture 13, the Riemann hypothesis.*

Proof. If $s_\vartheta(t)S_n(t_n) = S(t_n)$ then $S(t_n)$ is well-defined $\forall n \geq 1$ since $s_\vartheta(t)S_n(t_n)$ is well-defined $\forall n \geq 1$. \square

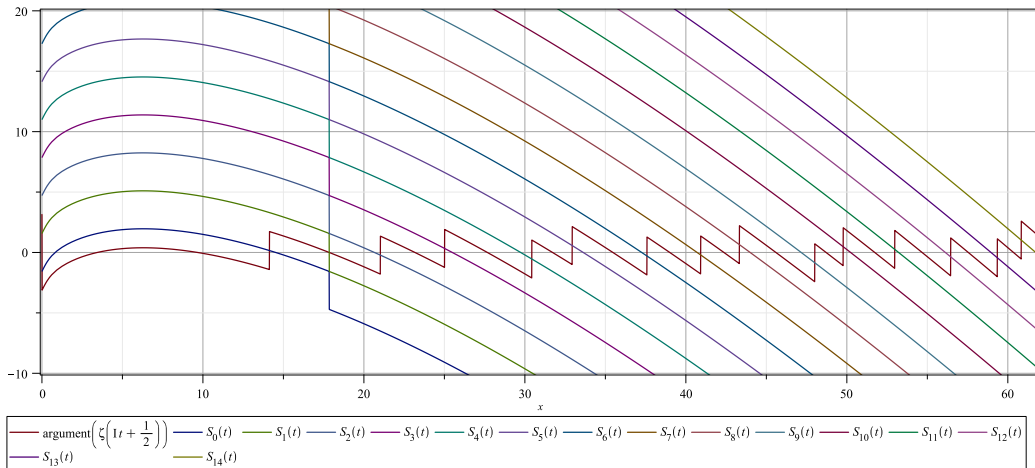


Figure 1. $S_n(t)$ and the imaginary parts of the roots of $\zeta(\frac{1}{2} + it_n)$ marked with verticle lines just touching its touching its corresponding pair of neighboring curves $S_n(t)$ for $n = 0 \dots 14$

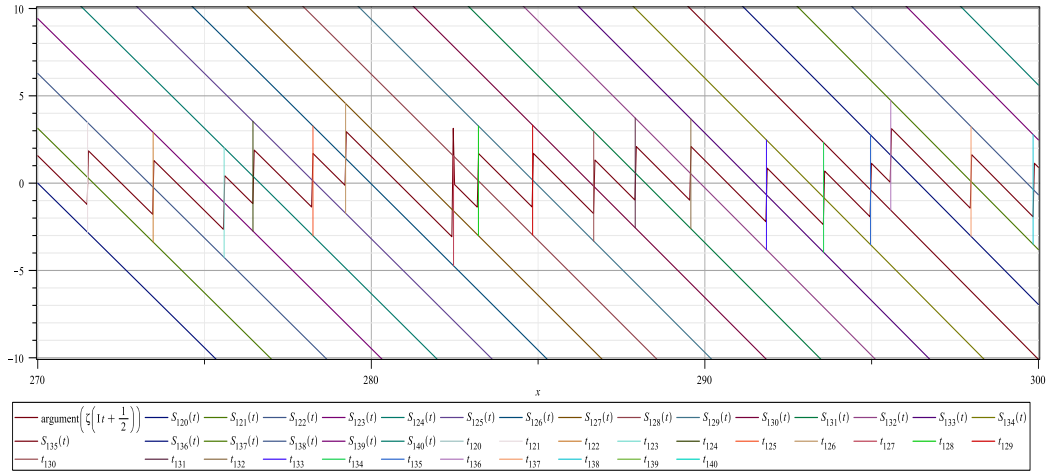


Figure 2. $S_n(t)$ and the imaginary parts of the roots of $\zeta\left(\frac{1}{2} + it_n\right)$ marked with verticle lines just touching its touching its corresponding pair of neighboring curves $S_n(t)$ for $n = 120\dots140$ which includes two “bad” Gram points at $n = 126$ and $n = 134$

Theorem 5. $\arg\left(\zeta\left(\frac{1}{2} + ig(n)\right)\right) = 0 \forall n \in \mathbb{Z}^+$

Proof. The argument of any positive number x with $\text{Im}(x) = 0$ is equal to 0 and $\text{Im}\left(\zeta\left(\frac{1}{2} + ig(n)\right)\right) = 0$ \square

Conjecture 6. $S(t) - f_0(t) \in \{-1, 0, 1\} \forall t \in \mathbb{R}$. That is

$$\text{frac}\left(\frac{\vartheta(t)}{\pi}\right) + \frac{1}{\pi} \arg\left(\zeta\left(\frac{1}{2} + it\right)\right) \in \{-1, 0, 1\} \forall 0 < t \in \mathbb{R}$$

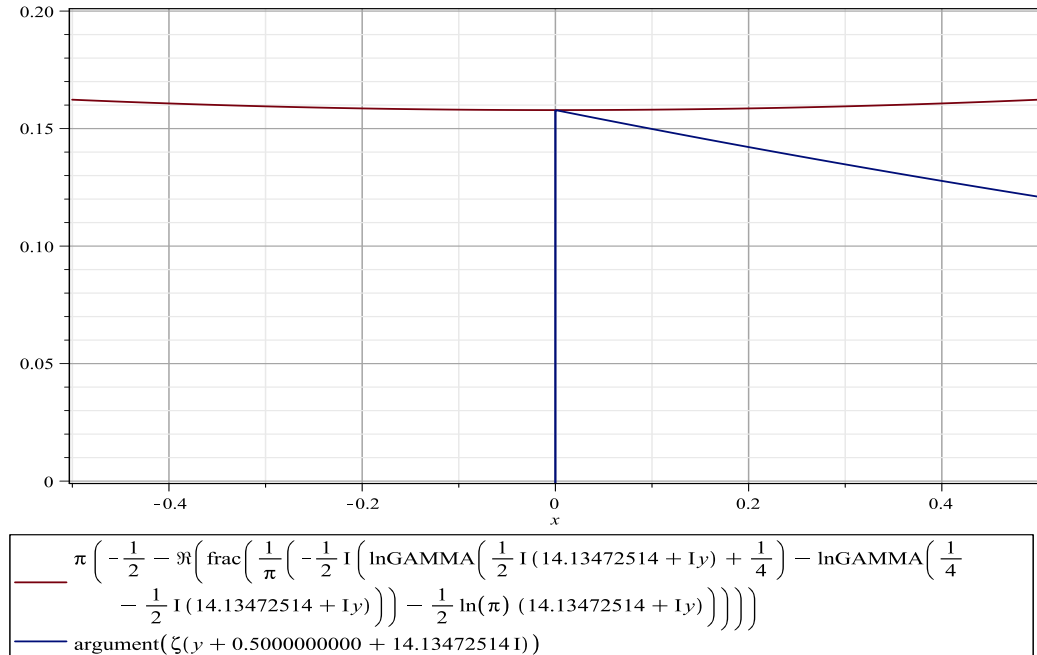


Figure 3. Illustration of convergence around the first zero

2 Appendix

2.1 Transcendental Equations Satisfied By The Nontrivial Riemann Zeros

Definition 7. *The critical line is the line in the complex plane defined by $\text{Re}(t) = \frac{1}{2}$.*

Definition 8. *The critical strip is the strip in the complex plane defined by $0 < \text{Re}(t) < 1$.*

Definition 9. *The asymptotic equation for the n -th zero of the Hardy Z function*

$$\frac{t_n}{2\pi} \ln\left(\frac{t_n}{2\pi t}\right) + S(t_n) = n - \frac{11}{8} \quad (11)$$

[1, Equation 20]

Remark 10. The fact that the exact and asymptotic equations have two solutions when $n = 1$ can be understood by noting that Equations (3) and (11) are derived from the equation

$$n = \tilde{\vartheta}(t) + \frac{\pi}{8} - \frac{5}{8} + S(t) \quad (12)$$

which has a minimum in the interval $(-2, -1)$ and thus $n \geq -1$ so that, in order to follow the convention that the zeros are enumerated by the positive integers, the substitution $n \rightarrow n - 2$ is made in Equation (12) so that

$$n - 2 = \tilde{\vartheta}(t) + \frac{\pi}{8} - \frac{5}{8} + S(t) \quad (13)$$

[1, Equation 12]

Theorem 11. *If the limit*

$$\lim_{\delta \rightarrow 0^+} \arg\left(\zeta\left(\frac{1}{2} + \delta + it\right)\right) \quad (14)$$

exists and is well-defined $\forall t$ then the left-hand side of Equation (11) is well-defined $\forall t$, and due to monotonicity, there must be a unique solution for every $n \in \mathbb{Z}^+$. [1, II.A]

Corollary 12. *The number of solutions of Equation (11) over the interval $[0, t]$ is given by*

$$N_0(t) = \frac{t}{2\pi} \ln\left(\frac{t}{2\pi e}\right) + \frac{7}{8} + S(t) + O(t^{-1}) \quad (15)$$

which counts the number of zeros on the critical line.

Conjecture 13. *(The Riemann hypothesis) All solutions t of the equation*

$$\zeta(t) = 0 \quad (16)$$

besides the trivial solutions $t = -2n$ with $n \in \mathbb{Z}^+$ have real-part $\frac{1}{2}$, that is, $\text{Re}(t) = \frac{1}{2}$ when $\zeta(t) = 0$ and $t \neq -2n$.

Definition 14. *The Riemann-von-Mangoldt formula makes use of Cauchy's argument principle to count the number of zeros inside the critical strip $0 < \text{Im}(\rho_n) < t$ where $\zeta(\sigma + i\rho_n)$ with $0 < \sigma < 1$*

$$N(t) = \frac{t}{2\pi} \ln\left(\frac{t}{2\pi e}\right) + \frac{7}{8} + S(t) + O(t^{-1}) \quad (17)$$

and this definition does not depend on the Riemann hypothesis (Conjecture 13). This equation has exactly the same form as the asymptotic Equation 11. [1, Equation 15]

Lemma 15. *If the exact Equation (3) has a unique solution for each $n \in \mathbb{Z}^+$ then Conjecture 13, the Riemann hypothesis, follows.*

Proof. If the exact equation has a unique solution for each n , then the zeros obtained from its solutions on the **critical line** can be counted since they are enumerated by the integer n , leading to the counting function $N_0(t)$ in Equation (15). The number of solutions obtained on the **critical line** would saturate counting function of the number of solutions on the **critical strip** so that $N(t) = N_0(t)$ and thus all of the non-trivial zeros of ζ would be enumerated in this manner. If there are zeros off of the critical line, or zeros with multiplicity $m \geq 2$, then the exact Equation (3) would fail to capture all the zeros on the critical strip which would mean $N_0(t) < N(t)$. [1, IX] \square

Bibliography

- [1] Guilherme França and André LeClair. Transcendental equations satisfied by the individual zeros of riemann ζ , dirichlet and modular l-functions. *Communications in Number Theory and Physics*, 2015.