An Expression For The Argument of ζ at Zeros on the Critical Line

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Abstract

It is conjectured that $\arg\zeta\left(\frac{1}{2}+it_n\right)=\pi\left(\frac{1}{2}-\operatorname{frac}\left(\frac{\vartheta(t_n)}{\pi}\right)-(\lfloor\tilde{g}^{-1}(n)\rfloor-n+1)\right)\forall n\geqslant 2$ where $\tilde{g}^{-1}(n)=\frac{t\ln\left(\frac{t}{2\pi e}\right)}{2\pi}+\frac{7}{8}$ is the inverse of $\tilde{g}(n)=\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}$ which accurately approximates the Gram points $g(n)$ and that all of the non-trivial zeros of ζ, enumerated by n, are on the critical line. Therefore, the exact transcendental equation for the Riemann zeros has a solution for each positive integer n which proves that Riemann's hypothesis is true since the counting function for zeros on the critical line is equal to the counting function for zeros on the critical strip if the transcendental equation has a solution for each n.

Let $g(n)=\{t\colon\vartheta(t)-(n-1)\,\pi=0\}$ be the n-th Gram point where

$$\vartheta(t)=-\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)-\ln\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\right)-\frac{\ln(\pi)\,t}{2}$$

is the Riemann-Siegel ϑ function. Let

$$\tilde{\vartheta}(t)=\frac{t\ln\left(\frac{t}{2\pi e}\right)}{2}-\frac{\pi}{8}$$

be the approximate ϑ function where the Γ function has been replaced with its Stirling approximation. A very accurate approximation to $g(n)$ is found by inverting $\tilde{\vartheta}(t)$ to get

$$\begin{aligned}\tilde{g}(n) &=\{t\colon\vartheta(t)-(n-1)\pi=0\}\\ &=\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}\end{aligned}$$

where W is the Lambert W function. The inverse of $\tilde{g}(n)$ is given by

$$\begin{aligned}\tilde{g}^{-1}(n) &=\{t\colon\tilde{g}(n)=0\}\\ &=\frac{t\ln\left(\frac{t}{2\pi e}\right)}{2\pi}+\frac{7}{8}\end{aligned}$$

Now define

$$\begin{aligned}T_n(t) &=\lfloor\tilde{g}^{-1}(n)\rfloor-n+1\\ &=\left\lfloor\frac{t\ln\left(\frac{t}{2\pi e}\right)}{2\pi}+\frac{7}{8}\right\rfloor-n+1\end{aligned}$$

Let $\operatorname{frac}(x)=\begin{cases}x-\lfloor x\rfloor & x\geqslant 0\\ x-\lceil x\rceil & x<0\end{cases}\forall x\in\mathbb{R}$ be the function which gives the fractional part of a real number by subtracting either the floor $\lfloor x\rfloor$ or the ceiling $\lceil x\rceil$ of x from x, depending upon its sign.

Conjecture 1. *The argument of* ζ *at all non-trivial zeros of* $\zeta\left(\frac{1}{2}+it_n\right)$ *with imaginary part* t_n *with* $n\geqslant 2$ *is equal to*

$$\begin{aligned} S(t_n) &= \pi\left(\frac{1}{2}-\operatorname{frac}\left(\frac{\vartheta(t)}{\pi}\right)-T_n(t)\right) \\ &= \pi\left(\frac{1}{2}-\operatorname{frac}\left(\frac{\vartheta(t)}{\pi}\right)-\left(\left\lfloor\frac{t\ln\left(\frac{t}{2\pi e}\right)}{2\pi}+\frac{7}{8}\right\rfloor-n+1\right)\right)\end{aligned}$$

Remark 2. The argument $S(t_1)=\arg\zeta\left(\frac{1}{2}+it_1\right)$ of $\zeta(\rho_1)$ is equal to

$$S(t_1)=\pi\left(-\frac{1}{2}-\operatorname{frac}\left(\frac{\vartheta(t_1)}{\pi}\right)\right)=0.1578739219\ldots$$

Theorem 3. $\arg\left(\zeta\left(\frac{1}{2}+ig(n)\right)\right)=0\forall n\in\mathbb{Z}^+$

Proof. The argument of any positive number x with $\operatorname{Im}(x)=0$ is equal to 0 and $\operatorname{Im}\left(\zeta\left(\frac{1}{2}+ig(n)\right)\right)=0$ □

Define $f(t)$ to be the fractional part of $\frac{\vartheta(t)}{\pi}$

$$f_p(t)=p-\left\lfloor\frac{\vartheta(t)}{\pi}\right\rfloor$$

and let

$$S(t)=\arg\left(\zeta\left(\frac{1}{2}+\mathrm{it}\right)\right)=\lim_{\varepsilon\to 0}\frac{1}{2}((S(\rho+i\varepsilon)-S(\rho-i\varepsilon))$$

be the normalized argument of ζ on the critical line.

Conjecture 4. $S(t)-f_0(t)\in\{-1,0,1\}\forall t\in\mathbb{R}$*. That is*

$$\operatorname{frac}\left(\frac{\vartheta(t)}{\pi}\right)+\frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2}+\mathrm{it}\right)\right)\in\{-1,0,1\}\forall 0<t\in\mathbb{R}$$

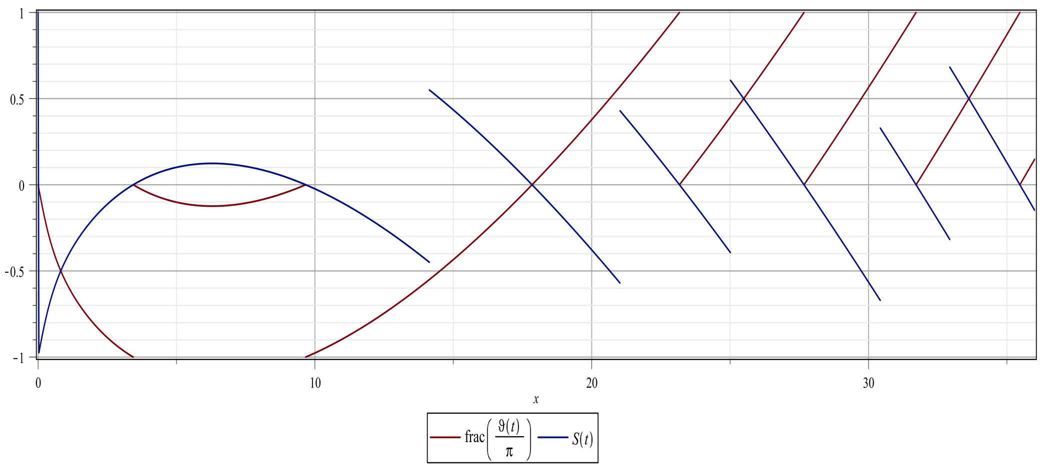

Figure 1. $f(t)$ and $S(t)$

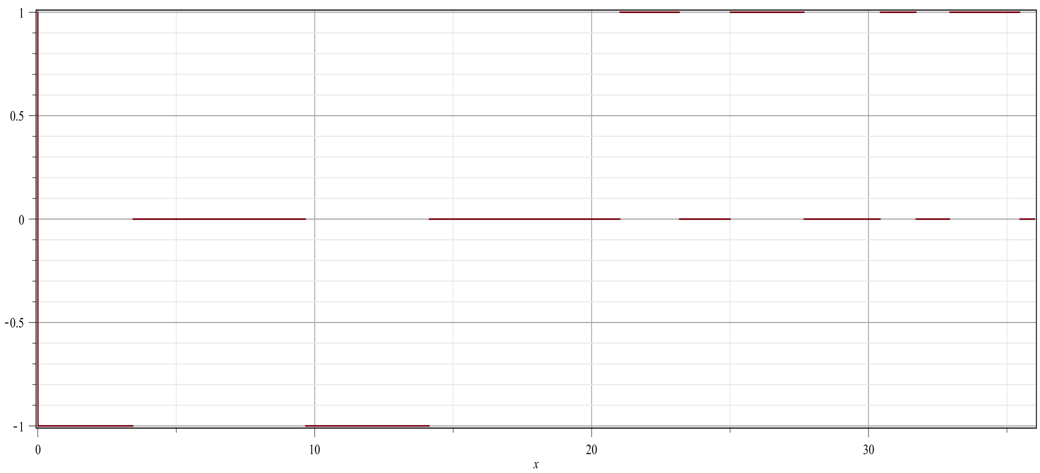

Figure 2. $f(t) + S(t)$

Theorem 5. *The exact transcendental equation for the Riemann zeros Equation 1*

$$\vartheta(y_n) + S(y_n) = \left(n - \frac{3}{2}\right)\pi$$

has a solution for each positive integer n.

Proof. $S(y_n)$ exists and is well-defined due to Theorem 5. ☐

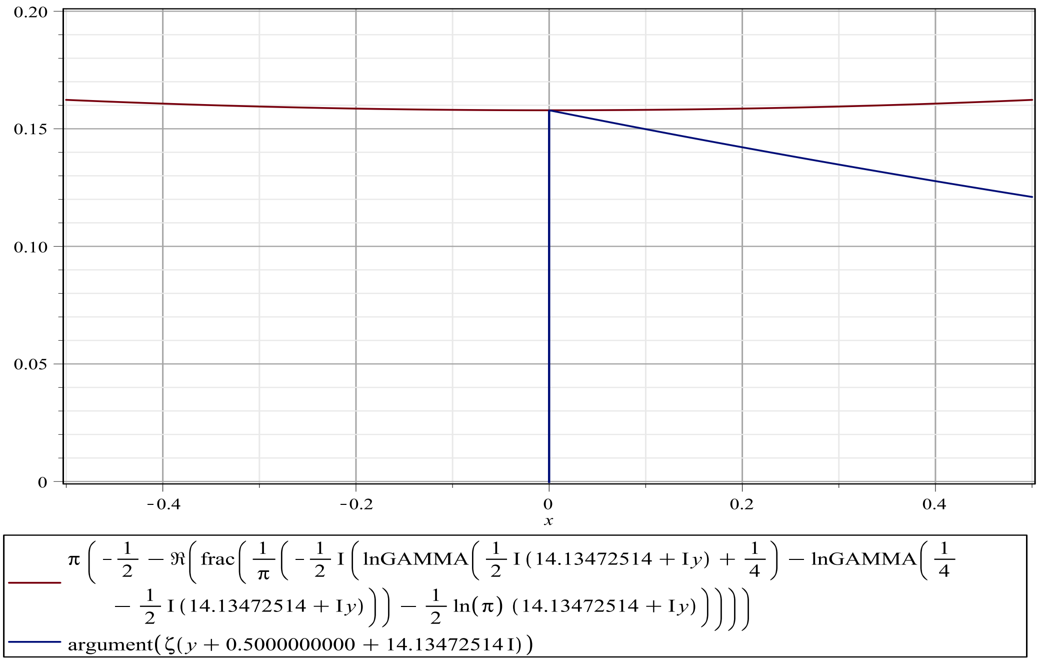

Figure 3. Illustration of convergence around the first zero

1 Appendix

1.1 Transcendental Equations Satisifed By The Nontrivial Riemann Zeros

Definition 6. *The* `critical line` *is the line in the complex plane defined by* $\mathrm{Re}(t) = \frac{1}{2}$.

Definition 7. *The* `critical strip` *is the strip in the complex plane defined by* $0 < \mathrm{Re}(t) < 1$.

Definition 8. *The* `exact equation` *for the n-th zero of the Hardy Z function* y_n *is given by [?, Equation 20]*

$$\vartheta(y_n) + \pi S(y_n) = \left(n - \frac{3}{2}\right)\pi \tag{1}$$

where y_n *enumerate the zeros of* Z *on the real line and the zeros of* ζ *on the critical line*

$$\mathrm{Z}(y_n) = 0 \text{ and } \zeta\left(\frac{1}{2} + i y_n\right) \forall n \in \mathbb{Z}^+ \tag{2}$$

where $\mathbb{Z}^+$ *denotes the positive integers. [?, Equation 14]*

By replacing the $\ln\Gamma$ function in (?) with Stirling's asymptotic expansion as in [?, Equation 13] we get

$$\tilde{\vartheta}(t) = \frac{t}{2}\ln\left(\frac{t}{2\pi e}\right) - \frac{\pi}{8} + O(t^{-1}) \tag{3}$$

and substitute $\vartheta(t)$ with $\tilde{\vartheta}(t)$ in Equation 1 which leads to

Definition 9. *The* `asymptotic equation` *for the n-th zero of the Hardy Z function*

$$\frac{t_n}{2\pi}\ln\left(\frac{t_n}{2\pi t}\right) + S(t_n) = n - \frac{11}{8} \tag{4}$$

[?, Equation 20]

Remark 10. The fact that the exact and asymptotic equations have two solutions when $n = 1$ can be understood by noting that Equations (1) and (4) are derived from the equation

$$n = \tilde{\vartheta}(t) + \frac{\pi}{8} - \frac{5}{8} + S(t) \tag{5}$$

which has a minimum in the interval $(-2, -1)$ and thus $n \geqslant -1$ so that, in order to follow the convention that the zeros are enumerated by the positive integers, the substituion $n \to n - 2$ is made in Equation (5) so that

$$n - 2 = \tilde{\vartheta}(t) + \frac{\pi}{8} - \frac{5}{8} + S(t) \tag{6}$$

[?, Equation 12]

Theorem 11. *If the limit*

$$\lim_{\delta \to 0^+} \arg\left(\zeta\left(\frac{1}{2} + \delta + i t\right)\right) \tag{7}$$

is exists and is well-defined $\forall t$ *then the left-hand side of Equation (4) is well-defined* $\forall t$*, and due to monotonicity, there must be a unique solution for every* $n \in \mathbb{Z}^+$*. [?, II.A]*

Corollary 12. *The number of solutions of Equation (4) over the interval* $[0, t]$ *is given by*

$$N_0(t) = \frac{t}{2\pi}\ln\left(\frac{t}{2\pi e}\right) + \frac{7}{8} + S(t) + O(t^{-1}) \tag{8}$$

which counts the number of zeros `on the critical line`.

Conjecture 13. *(The Riemann hypothesis) All solutions t of the equation*

$$\zeta(t) = 0 \tag{9}$$

besides the trivial solutions $t = -2n$ with $n \in \mathbb{Z}^+$ have real-part $\frac{1}{2}$, that is, $\operatorname{Re}(t) = \frac{1}{2}$ when $\zeta(t) = 0$ and $t \neq -2n$.

Definition 14. *The Riemann-von-Mangoldt formula makes use of Cauchy's argument principle to count the number of zeros* `inside the critical strip` *$0 < \operatorname{Im}(\rho_n) < t$ where $\zeta(\sigma + i\rho_n)$ with $0 < \sigma < 1$*

$$N(t) = \frac{t}{2\pi}\ln\left(\frac{t}{2\pi e}\right) + \frac{7}{8} + S(t) + O(t^{-1}) \tag{10}$$

and this definition does not depend on the Riemann hypothesis(Conjecture 13). This equation has exactly the same form as the asymptotic Equation 4. [?, Equation 15]

Lemma 15. *If the exact Equation (1) has a unique solution for each $n \in \mathbb{Z}^+$ then Conjecture 13, the Riemann hypothesis, follows.*

Proof. If the exact equation has a unique solution for each n, then the zeros obtained from its solutions on the `critical line` can be counted since they are enumerated by the integer n, leading to the counting function $N_0(t)$ in Equation (8). The number of solutions obtained on the `critical line` would saturate counting function of the number of solutions on the `critical strip` so that $N(t) = N_0(t)$ and thus all of the non-trivial zeros of ζ would be enumerated in this manner. If there are zeros off of the critical line, or zeros with multiplicity $m \geqslant 2$, then the exact Equation (1) would fail to capture all the zeros on the critical strip which would mean $N_0(t) < N(t)$. [?, IX] □

Bibliography