The union is not the limit.

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Abstract: Contrary to the assumptions of transfinite set theory, limit and union of infinite sequences of sets differ. We will show this for the set $\mathbb{N}$ of natural numbers by the newly devised powerful tool of arithmogeometry as well as by scrutinizing the recursive construction. The basic theorem of set theory $\forall n \in \mathbb{N}: n < \aleph_0$ precludes the defined identity $|\mathbb{N}| = \aleph_0$. Further differences of limits in set theory and analysis are discussed.

Arithmogeometry

The sequence of all finite initial segments $F(n) = \{1, 2, 3, ..., n\}$ of the set $\mathbb{N}$ of natural numbers n is represented in the following figure which we call an arithmogeometric figure because it combines features of arithmetic by its elements and features of geometry by its shape:

$$\begin{array}{l} 1 \\ 1, 2 \\ 1, 2, 3 \\ ... \, . \end{array} \tag{1}$$

It has the advantage that its contents can be interpreted as the set F of all finite initial segments

$$\{\{1\}, \{1, 2\}, \{1, 2, 3\}, ...\} = F$$

of $\mathbb{N}$, where only the braces have been replaced by carriage returns, as well as the union $\cup F$, i.e., as the set $\mathbb{N}$ of all natural numbers

$$\{1, 1, 2, 1, 2, 3, ...\} \equiv \{1, 2, 3, ...\} = \mathbb{N} \, .$$

Set theory states

$$|\mathbb{N}| = |\cup F| = \aleph_0 \tag{2}$$

where, by a very basic theorem [1],

$$\forall n \in \mathbb{N}: n < \aleph_0 . \quad (3)$$

Note that this theorem asserts the existence of a cardinal number $\aleph_0$ that is greater than *all* natural numbers *n*, i.e., a feature of *actual infinity*. Of course (3) would be satisfied by (1) when formulated within *potential infinity*

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}: n < m .$$

Clearly *every* row *that we can choose* has only a finite number of predecessors and an infinitude of successors. In that gist the number of elements of the figure is always greater than the number of elements of a chosen row. But that is *not* the meaning of (3). According to Cantor [2], $\aleph_0$ is *a fixed quantity beyond all finite magnitudes*. That means we can consider *all* rows of the figure and nevertheless we find $\aleph_0$ greater than all.

Now, the union $\cup F$ can be accomplished by the following procedure: Every row of (1) is shifted into the first row, maintaining the column of each of its *n* elements. The shifted element replaces the element that exists already in the first row (i.e., the same natural number or a blank). When all rows have been shifted accordingly, then all natural numbers are in the first row. But obviously the first row finally has not $\aleph_0$ elements, if this means more elements than every row can supply, because only finite rows have contributed. How many finite rows is irrelevant in this respect. By the practice of shifting rows we see that the basic theorem (3) of set theory precludes (2).

The contrary claim would be tantamount to the claim that the figure has more elements than all rows of the figure, or that shifting of rows increases their cardinal numbers. Then however translation invariance of mathematical expressions like finite strings would be violated, namely the fact (among others necessary for mathematical discourse) that the place of writing a string must not change the meaning or cardinality of the string.

So we can conclude *that if* (3) *holds*, then $|\mathbb{N}| < \aleph_0$ is *not* a fixed quantity greater than all natural numbers. Since $|\mathbb{N}|$ cannot be equal to a natural number either, the only alternative is potential infinity: $|\mathbb{N}|$ is an always growing, never finished magnitude. It is suitable to denote it by ∞ and to reserve $\aleph_0$ for the cardinal number of the least upper bound ω of the sequence (n) of natural numbers *n* such that $|\omega| = \aleph_0$.

The familiar equality

$$\lim_{n\to\infty} F(n) = \bigcup_{n\in\mathbb{N}} F(n) \quad (4)$$

turns out to be mistaken. Note that also in analysis a strictly monotonously increasing sequence like $(F(n))_{n\in\mathbb{N}}$ does never assume its limit. Since the limit (which is not the set-theoretic limit, see below) of the sequence $(n)_{n\in\mathbb{N}}$ is

$$\lim_{n\to\infty} n = \infty \quad (5)$$

or, as Cantor later wrote, ω, and since all natural numbers n are contained in the finite initial segments $F(n)$ as their last elements, we get

$$\lim_{n\to\infty} F(n) = \lim_{n\to\infty} \{1, 2, 3, ..., n\} = \{1, 2, 3, ..., \omega\} \ .$$

The union of the rows of (1) however does not contain any transfinite number

$$\bigcup_{n\in\mathbb{N}} F(n) = \{1, 2, 3, ...\} = \mathbb{N} \ .$$

Recursion of finite initial segments

The difference between limit and union can also be obtained from the fact that the recursion

$$\forall n \in \mathbb{N}: F(n) = F(n\text{-}1) \cup \{n\} \quad \text{with} \quad F(0) = \{ \}$$

does not include the limit. There are infinitely many (namely ω) unions of singletons and finite initial segments. None of them yields an infinite set, since every result $F(n)$ is finite. The limit is, as every strictly increasing sequence shows, not reached by infinitely many finite steps – and every increase by adding a singleton $\{n\}$ *is* a finite step. But if ω unions *with an always increasing number of elements* do not yield the limit, then another union $\cup F$ (i.e., union number ω + 1) *collecting only what had been collected before* already and this time *not adding anything more*, cannot, by simplest logic, increase the result to reach what before had remained out of reach.

Consequently $\aleph_0 = |\omega|$ is not the cardinality of the set $\mathbb{N}$. The set $\mathbb{N}$ is infinite with no doubt, but it is only potentially infinite, having not a cardinal number, i.e., a fixed quantity [2], because a finite cardinal number cannot be attached to it and the first infinite cardinal number $\aleph_0$ has been excluded as too large.

Failure of set theoretic limit

Finally, in order to see that for sequences of sets in general the set theoretic limit, henceforth denoted by Lim, differs from the analytical limit, denoted by lim

$$\text{Lim } S(n) \neq \lim S(n)$$

we consider the sequence of closed real intervals $[1/n, 1]$.

In set theory a sequence (M_n) of sets M_n has the limit Lim M_n if and only if [3]

$$\text{Lim } M_n = \text{LimSup } M_n = \text{LimInf } M_n \quad (6)$$

where

$$\text{LimSup}\, M_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} M_k$$

and

$$\text{LimInf}\, M_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} M_k \, .$$

According to this formalism the limit of the sequence of closed real intervals, Lim [1/*n*, 1], is their union, namely the half-open real interval (0, 1]. In analysis however, we find from

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

the analytical limit

$$\lim_{n\to\infty} [\frac{1}{n},1] = [\lim_{n\to\infty} \frac{1}{n},1] = [0,1]$$

because the limit of the sequence $(1/n)_{n\in\mathbb{N}}$ of fractions 1/*n* cannot depend on the question whether they represent singular points or edges of intervals extending to the right-hand side which even is opposite to the limit-point 0 on the left-hand side.

To show the general inappropriateness of the set theoretical limit and its being in contradiction with analytical results we will briefly discuss three further simple examples.

• First consider the sequence of real intervals $[\frac{(-1)^n}{n},1]$ which has the analytical limit

$$\lim_{n\to\infty} [\frac{(-1)^n}{n},1] = [\lim_{n\to\infty} \frac{(-1)^n}{n},1] = [0,1] \, .$$

In set theory this sequence has no limit because every supremum

$$\text{Sup}\, M_n = \bigcup_{k=n}^{\infty} M_k = [0,1]$$

contains 0 and so does the intersection of all suprema, but no infinimum

$$\text{Inf}\, M_n = \bigcap_{k=n}^{\infty} M_k = (0,1]$$

contains 0, so that 0 is also lacking in the union of all infima. According to (6) there is no limit because $\text{LimSup}\, M_n \neq \text{LimInf}\, M_n$.

• As second example we consider the limit of the sequence 0, 1, 2, 3, ... of finite cardinal numbers $\mathbb{N}_0$. This sequence can be represented in many different ways using Arabic digits, circles as unary symbols, Zermelo's system, or von Neumann's system

$$
\begin{array}{lllll}
\{0\}, & \{1\}, & \{2\}, & \{3\}, & ... \to \{\ \} \\
\{\ \}, & \{o\}, & \{oo\}, & \{ooo\}, & ... \to \{ooo...\} \\
\{\ \}, & \{0\}, & \{1\}, & \{2\}, & ... \to \{\ \} \\
\{\ \}, & \{0\}, & \{0, 1\}, & \{0, 1, 2\}, & ... \to \{0, 1, 2, ...\} = \mathbb{N}_0 .
\end{array}
$$

All symbols in the same column, looking very differently though, denote one and the same cardinal number *n*. The limits however have not only different shape but denote completely different objects.

• The last example is unsurpassable in showing the uselessness of the set theoretic limit.

Scrooge McDuck earns 10 $ and spends 1 $ every day. Since a comic character lives forever his wealth grows immeasurably – at least when applying the analytical limit of the sequence $(9n)_{n\in\mathbb{N}}$. However if he issues always the dollars received first and if he applies (6), then he will go bankrupt since the sequence of sets $\{n+1, n+2, n+3, ..., 10n\}$ has the empty set as its limit. Not necessary to mention that this result is inacceptable, in particular because it depends on the *indices* of the spent dollars. Such a dependence cannot be accepted in any scientific theory.

It has to be mentioned however that this very limit, transformed from the time axis to the spatial axis (the points of which, according to Cantor's axiom, are eqivalent to the real numbers) and with the ratio *infinite* instead of 10, is necessary to enumerate all rational numbers by natural numbers such that the set of not enumerated rational numbers, just like the set of dollars kept by McDuck, is empty. [4]

References

[1] K. Hrbacek, T. Jech: "Introduction to Set Theory", 2nd ed., Marcel Dekker, New York (1984) p. 83.
[2] E. Zermelo (ed.): "Georg Cantor, Gesammelte Abhandlungen mathematischen und philosophischen Inhalts", Springer, Berlin (1932) p. 374.
[3] S.I. Resnick: "A probability path", Birkhäuser, Boston (1998) p. 6.
[4] W. Mückenheim: "Does Set Theory Cause Perceptual Problems?", viXra 1702.0280 (2017).