A new empirical approach to quark and lepton masses

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We present an alternative to the Koide formula and it's extensions. By introducing two parameters k and α_f derived from charged leptons we are able to construct new empirical formulas that appear to relate all fundamental fermion pole masses. The predicted masses are in excellent agreement with known experimental values and constraints for heavy quarks and neutrinos. For light quarks we predict speculative pole masses of the same order of magnitude as $\mu = 1$ GeV $\overline{\text{MS}}$ masses but higher by a factor of ~ 1.5. The condition where $k^{12} = 3.5$ (exact) is also considered as it would allow ultra high precision predictions.

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One outstanding problem in modern physics is the seemingly unrelated particle masses that exist as free parameters in the standard model. Numerous attempts have been made over the years to derive a systematic mathematical relationship between these masses, most famously by Yoshio Koide with his charged lepton formula [1]

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} \simeq \frac{2}{3}. \tag{1}$$

While this approximation works for charged leptons, and appears to work for some combinations of quarks with running masses [2, 3, 7], it does not work for neutrinos or attempt to relate all fundamental fermions in a consistent manner. It does however hint at the possibility of finding other more fundamental relationships. Indeed, Koide and Nishiura [4, 5], Sumino [6], and others [7, 8] have used it as inspiration for extended models to cover other fermions.

Our goal is not to reverse engineer an extended model to match Koide but to start fresh and look for other empirical relationships that could be derived from the data. After deriving two new parameters from charged leptons we find they enable an interesting set of formulas that relate all fundamental lepton and quark pole (physical) masses. This allows precise predictions of neutrino and quark pole masses and if correct would significantly reduce the number of free parameters in the standard model (extended for neutrinos with mass).

Our first parameter comes from a charged lepton relationship that is close to unity:

$$\frac{m_e m_\tau^2}{m_\mu^3} = 1.36777(18) \simeq 3.5^{1/4}. \tag{2}$$

If we consider this dimensionless residual value to be something peculiar to the muon we can define a second generation normalization factor k

$$k \equiv \frac{(m_e m_\tau^2)^{1/3}}{m_\mu}. \tag{3}$$

With the latest values for m_e, m_μ [9] and m_τ [10] we find $k = 1.110039(50) \simeq 3.5^{1/12}$. This constant allows us to create a charged lepton family equation

$$\frac{m_e m_\tau^2}{(k m_\mu)^3} = 1. \tag{4}$$

Our second parameter is used to relate the widely different scales observed between light and heavy particles. Mac Gregor proposed using the QED coupling constant to explain fermion lifetimes and mass relationships with the zero energy scale $\alpha_{\text{QED}}(0)$ [11, 12]. Instead of the actual QED constant we propose a fermion mass hierarchy parameter α_f with a value similar to an effective QED coupling constant. In our search for suitable values we considered the range $125 < \alpha_f^{-1} < \alpha_{\text{QED}}^{-1}(0)$. While initially targeting $\alpha_{\text{QED}}^{-1}(M_Z^2) = 128.944$ [13] we discovered the useful relationship

$$\alpha_f \equiv 27 \frac{m_e}{m_\tau}. \tag{5}$$

Using the latest values for m_e [9] and m_τ [10] we find $\alpha_f^{-1} = 128.7862 \pm 0.0087$. Despite the similarity in value to an effective QED coupling constant we consider α_f to be a separate parameter related only to mass generation and not electric charge. Regardless of the true physical nature of α_f, we find that it along with k is useful for analyzing fermion mass relationships. With (4) and (5) we can create charged lepton equations

$$m_\mu = 9 \frac{m_e}{k \alpha_f^{2/3}}, \tag{6}$$

$$m_\tau = 27 \frac{m_e}{\alpha_f}. \tag{7}$$

With our choice of definitions for k and α_f we have established an interesting and surprisingly simple charged

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lepton mass hierarchy that could be used to look for similar patterns in other sectors.

We will now use the parameters k and α_f to look for similar empirical formulas for left handed neutrinos and predict mass state values for them. While neutrino mass states have not been directly measured yet numerous neutrino oscillation experiments have established increasingly refined limits on neutrino squared mass differences [14]. Cosmological models also put constraints on the sum of all three mass states [15]. The best fits we have found using k, α_f and simple coefficients to the experimental active neutrino squared mass differences are mass states

$$m_1 = \alpha_f^4 m_e = 1.85756 \pm 0.00050 \times 10^{-3}\ \mathrm{eV/c^2}, \tag{8}$$

$$m_2 = 9\frac{\alpha_f^4}{k^6} m_e = 8.9363 \pm 0.0034 \times 10^{-3}\ \mathrm{eV/c^2}, \tag{9}$$

$$m_3 = 27\alpha_f^4 m_e = 5.0154 \pm 0.0013 \times 10^{-2}\ \mathrm{eV/c^2}. \tag{10}$$

These predicted masses give squared mass differences of

$$\Delta m_{21}^2 = 7.6407 \pm 0.0061 \times 10^{-5}\ \mathrm{eV}^2/c^4, \tag{11}$$

$$\Delta m_{31}^2 = 2.5120 \pm 0.0013 \times 10^{-3}\ \mathrm{eV}^2/c^4, \tag{12}$$

$$\Delta m_{32}^2 = 2.4356 \pm 0.0013 \times 10^{-3}\ \mathrm{eV}^2/c^4, \tag{13}$$

and the sum of all three mass states

$$\Sigma m_\nu = 6.0948 \pm 0.0013 \times 10^{-2}\ \mathrm{eV}/c^2.$$

Mass ordering is normal with $m_1 < m_2 < m_3$. These are in excellent agreement with recent global analysis of oscillation experiments [10, 14] and just below cosmological model limits [15]. A comparison of our predicted values to experiments is shown in table I. Remarkably, these equations have the identical integer coefficients (1, 9, 27) as our charged lepton formulas and also satisfy a left handed neutrino family equation

$$\frac{m_1 m_3^2}{(k^6 m_2)^3} = 1, \tag{14}$$

similar to (4) for charged leptons. Right handed sterile neutrinos, if the exist do not yet have reliable constraints established from which we could search for similar relationships.

We will now attempt to apply the techniques we used relating masses in the lepton sector to quarks. While most attempts to find relationships between leptons and quarks use renormalized running masses we have surprisingly found the best overall results using pole masses. The deconfined nature of the t-quark has enabled It's mass to be measured through decay products in particle collision experiments. The best fit we have found using α_f and simple coefficients to the experimental t-quark mass is

$$M_t = \frac{1}{2\pi\alpha_f^3} m_e = 173.719 \pm 0.035\ \mathrm{GeV/c^2}. \tag{15}$$

For quarks we use the notation M_q to denote pole masses and $m_q^{\overline{\mathrm{MS}}}$ for renormalized reference masses. Several different analysis modes exists on data from the ongoing t-quark experiments at the LHC with a comprehensive summary in [16]. Our predicted pole mass is an excellent match for recent analysis including the 2016 PDG average of $M_t = 173.21 \pm 0.51 \pm 0.71$ [10]. We also have a particularly close match to the center value of $M_t = 173.7 \pm 1.5 \pm 1.4^{+1.0}_{-0.5}$ reported in [17].

For the b- and c-quarks we target the pole masses derived from theoretical $\overline{\mathrm{MS}}$ masses. The best fits we have found using k, α_f and simple coefficients for the b- and c-quarks are

$$M_b = \frac{1}{\sqrt{\pi}\alpha_f^2} m_e = 4.78171 \pm 0.00064\ \mathrm{GeV}/c^2, \tag{16}$$

$$M_c = \frac{2}{\pi k^{11}\alpha_f^2} m_e = 1.71127 \pm 0.00088\ \mathrm{GeV}/c^2. \tag{17}$$

These predicted pole masses are in excellent agreement with recent pole mass evaluations of $M_b = 4.78 \pm 0.06$ GeV and $M_c = 1.67 \pm 0.07$ GeV [10].

For the light quarks perturbative QCD does not provide reliable pole masses so we are left to extrapolate relationships from the heavy quarks. For the purpose of extrapolation we assume that two quark family equations exist similar to (4) and (14) for the leptons. This is straightforward for the u-quark using (15), (17) and (21). For the d- and s-quarks we have made our best attempt to find equations of similar form as the b- quark with resulting masses of the same order of magnitude but higher than the $\mu = 1$ GeV $\overline{\mathrm{MS}}$ mass. It must be emphasized that our light quark formulas are speculative and are only shown to demonstrate the ability to predict masses of expected magnitude while satisfying the proposed family equations. The best fits we have found using k, α_f and simple coefficients for the u-, d- and s-quarks are

$$M_u = \frac{32}{\pi} m_e = 5.204992524(32)\ \mathrm{MeV}/c^2, \tag{18}$$

$$M_d = \frac{27}{\sqrt{\pi}} m_e = 7.784107630(47)\ \mathrm{MeV}/c^2, \tag{19}$$

$$M_s = \frac{3}{\sqrt{\pi}k^{12}\alpha_f^{4/3}} m_e = 160.719 \pm 0.088\ \mathrm{MeV}/c^2. \tag{20}$$

The quark family equations are then given by

$$\frac{M_u M_t^2}{(k^{11} M_c)^3} = 1, \tag{21}$$

$$\frac{M_d M_b^2}{(k^{12} M_s)^3} = 1. \tag{22}$$

These predicted pole masses are the same order of magnitude but higher than recent evaluations at $\mu = 1$ GeV $\overline{\mathrm{MS}}$. We compare them to Aoki [18] $\mu = 2$ GeV $\overline{\mathrm{MS}}$

masses scaled by a factor of 1.35 to $m_u^{\overline{\rm MS}}(1{\rm GeV}) = 3.47$ MeV, $m_d^{\overline{\rm MS}}(1{\rm GeV}) = 4.97$ MeV, and $m_s^{\overline{\rm MS}}(1{\rm GeV}) = 112.86$ MeV. The PDG review [10] notes that u- and d-quark masses are not without controversy and we see there is significant variance in results for the u-quark. Using Aoki for light quarks scaled by 1.35 to 1GeV the ratio of our predicted pole light quark masses to the corresponding $\mu = 1$ GeV $\overline{\rm MS}$ reference masses is ~ 1.5 for the u- and d-quarks and 1.43 for the s-quark. Our predicted mass ratio $M_u/M_d \simeq 0.669$ is higher than the 2016 PDG evaluation of 0.38-0.58 but consistent with Aoki of 0.698 ± 0.051 used in the PDG evaluation.

While the preceding formulas in terms of m_e are convenient for calculating precision predicted values it is more interesting to analyze them together in terms of the Higgs vacuum expectation value $v \simeq 246.22$ GeV [10]. With the top quark Yukawa coupling $y_t = \sqrt{2}M_t/v \simeq 0.99779$ we can rewrite all of our proposed formulas in terms of y_t and v:

$$\begin{aligned}
&m_1 = \sqrt{2}\pi\alpha_f^7 y_t v, && m_e = \sqrt{2}\pi\alpha_f^3 y_t v, && M_d = 27\sqrt{2\pi}\alpha_f^3 y_t v, && M_u = \frac{64}{\sqrt{2}}\alpha_f^3 y_t v,\\
&m_2 = 9\sqrt{2}\pi\frac{\alpha_f^7}{k^6} y_t v, && m_\mu = 9\sqrt{2}\pi\frac{\alpha_f^{7/3}}{k} y_t v, && M_s = 3\sqrt{2\pi}\frac{\alpha_f^{5/3}}{k^{12}} y_t v, && M_c = \frac{4}{\sqrt{2}}\frac{\alpha_f}{k^{11}} y_t v,\\
&m_3 = 27\sqrt{2}\pi\alpha_f^7 y_t v, && m_\tau = 27\sqrt{2}\pi\alpha_f^2 y_t v, && M_b = \sqrt{2\pi}\alpha_f y_t v, && M_t = \frac{1}{\sqrt{2}} y_t v,\\
&\frac{m_1 m_3^2}{(k^6 m_2)^3} = 1, && \frac{m_e m_\tau^2}{(k m_\mu)^3} = 1, && \frac{M_d M_b^2}{(k^{12} M_s)^3} = 1, && \frac{M_u M_t^2}{(k^{11} M_c)^3} = 1.
\end{aligned} \tag{23}$$

There are several interesting patterns in these expressions, only some of which are enforced by the family equations. Despite these patterns there does not appear to be any relationship between them and electric charge or any other properties besides generation, family and mass. The lack of obvious equations for each row relating all fermions in a specific generation leaves open the question of whether we are missing families such as right handed neutrinos or superpartners. It is however possible to write several combinations of the family equations that cancel factors of k, for example:

$$\frac{M_s^3 M_t^2 m_\tau^2 M_u m_e}{M_c^3 m_\mu^3 M_b^2 M_d} = 1, \tag{24}$$

$$\frac{m_2^{12} M_t^2 M_b^2 m_\tau^2 M_d M_u m_e}{m_3^8 m_1^4 M_c^3 M_s^3 m_\mu^3} = 1, \tag{25}$$

$$\frac{m_2^6 M_b^2 M_d}{m_3^4 m_1^2 M_s^3} = 1. \tag{26}$$

In table I we provide a summary of the predicted masses and parameters derived from these equations.

The precision of the predicted values in table I is limited by the relative uncertainty in the tau mass of 6.8×10^{-5} [10]. We noted in (3) that $k \simeq 3.5^{1/12}$. In fact k differs from $3.5^{1/12}$ by only 2.0×10^{-6} which is significantly less than the relative uncertainty in the tau. If we assume that $k^{12} = 3.5$ (exact) we can reduce the relative uncertainty to be comparable to the muon. For example we could then predict the tau mass to a relative uncertainty of 3.4×10^{-8}:

$$m_\tau = \left(\frac{(k m_\mu)^3}{me}\right)^{\frac{1}{2}} = 1776.864828(61)\ {\rm MeV}/c^2. \tag{27}$$

This would then allow a higher precision value of α_f and the resulting predicted masses. While there is no known reason for the significance of the number 3.5 to justify this assumption, we provide a list of the ultra high precision calculations in table II for reference.

In conclusion, we have found new empirical fermion mass relationships that are surprising in several ways. Instead of the conventional expectation that such relationships would only appear at higher energy scales using renormalized masses, these formulas appear to relate pole (physical) masses across all sectors. They relate individual fermion masses to each other across families while also satisfying an equation for each family. The patterns in the coefficients and exponents suggest that they could lead to a better understanding of the mass hierarchy. Finally, they predict masses in excellent agreement with known experimental values and constraints. However,

Parameter	THIS WORK	Reference Value	Ref.
m_e	Used as input	0.5109989461(31) MeV	CODATA 2014 [9]
m_μ	Used as input	105.6583745(24) MeV	CODATA 2014 [9]
m_τ	Used as input	1776.86 ± 0.12 MeV	PDG 2016 [10]
k	1.110039(50)		
α_f	0.00776481(52)		
α_f^{-1}	128.7862 ± 0.0087		
m_1	$1.85756 \pm 0.00050 \times 10^{-3}$ eV		
m_2	$8.9363 \pm 0.0034 \times 10^{-3}$ eV		
m_3	$5.0154 \pm 0.0013 \times 10^{-2}$ eV		
Δm^2_{21}	$7.6407 \pm 0.0061 \times 10^{-5}$ eV^2	$7.53 \pm 0.18 \times 10^{-5}$ eV^2	PDG 2016 [10]
Δm^2_{31}	$2.5120 \pm 0.0013 \times 10^{-3}$ eV^2	$2.524^{+0.038}_{-0.041} \times 10^{-3}$ eV^2	Esteban 2016 [14]
Δm^2_{32}	$2.4356 \pm 0.0013 \times 10^{-3}$ eV^2	$2.44 \pm 0.06 \times 10^{-3}$ eV^2	PDG 2016 [10]
$\sum m_\nu$	$6.0948 \pm 0.0013 \times 10^{-2}$ eV	$< 9.68 \times 10^{-2}$ eV	Giusarma 2016 [15]
M_c	1.71127 ± 0.00088 GeV	$M_c = 1.67 \pm 0.07$ GeV	PDG 2016 [10]
M_b	4.78171 ± 0.00064 GeV	$M_b = 4.78 \pm 0.06$ GeV	PDG 2016 [10]
M_t	173.719 ± 0.035 GeV	$M_t = 173.21 \pm 0.51 \pm 0.71$ GeV	PDG 2016 [10]
y_t	0.99779 ± 0.000020		
M_u	5.204992524(32) MeV	$m_u^{\overline{\rm MS}}(1\rm GeV) = 3.47 \pm 0.35$	AOKI 2012 [18]
M_d	7.784107630(47) MeV	$m_d^{\overline{\rm MS}}(1\rm GeV) = 4.97 \pm 0.39$	AOKI 2012 [18]
M_s	160.719 ± 0.088 MeV	$m_s^{\overline{\rm MS}}(1\rm GeV) = 112.86 \pm 0.78$	AOKI 2012 [18]
$M_u/M_d = 32/(27\sqrt{\pi})$	0.66866913...	0.698 ± 0.051	AOKI 2012 [18]
$M_u/m_u^{\overline{\rm MS}}(1\rm GeV)$	1.50 ± 0.17		
$M_d/m_d^{\overline{\rm MS}}(1\rm GeV)$	1.57 ± 0.12		
$M_s/m_s^{\overline{\rm MS}}(1\rm GeV)$	1.4241 ± 0.0099		

Table I. Predicted pole masses and derived values. Light quark $\mu = 2$ GeV $\overline{\rm MS}$ reference masses have been rescaled to $\mu = 1$GeV by multiplying by 1.35.

as with any empirical formulas they could just be coincidental or approximations of more complex underlying phenomena. The most exciting area of experimental confirmation would be refinement of the neutrino oscillation parameters and especially measurement of the neutrino mass states directly. The KATRIN experiment [19] will search for neutrino mass signatures in decay from tritium β-decay but has a minimum sensitivity threshold of 0.2 eV, 100 times higher than our predicted m_1 and 25 times higher than our predicted m_3. Further refinement of the t-quark mass is another important confirmation opportunity. Further refinement of the Tau mass could provide support for the $k^{12} = 3.5$ (exact) condition which would allow ultra high precision predictions of other masses as shown in Table II.

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Parameter	THIS WORK if $k^{12} = 3.5$ (exact)
k^{12}	3.5 (exact)
k	1.110040958331563...
α_f	0.00776478398(27)
α_f^{-1}	128.7865835(45)
m_τ	1776.864828(61) MeV
m_1	$1.85753718(26) \times 10^{-3}$ eV
m_2	$8.9360585(12) \times 10^{-3}$ eV
m_3	$5.01535040(70) \times 10^{-2}$ eV
Δm^2_{21}	$7.6402697(21) \times 10^{-5}$ eV^2
Δm^2_{31}	$2.5119235(70) \times 10^{-3}$ eV^2
Δm^2_{32}	$2.4355208(70) \times 10^{-3}$ eV^2
$\sum m_\upsilon$	$6.0947100(70) \times 10^{-2}$ eV
M_c	1.71124555(12) GeV
M_b	4.78174390(33) GeV
M_t	173.720873(18) GeV
y_t	0.99780181(28) GeV
M_u	5.204992524(32) MeV
M_d	7.784107630(47) MeV
M_s	160.7159974(75) MeV

Table II. Ultra high precision Predicted pole masses and derived values assuming $k^{12} = 3.5$(exact).

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