On the Firoozbakht's and Cramér's conjectures as $n \to \infty$

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Abstract. Let p_n be the nth prime number. We prove that $p_{n+1} < p_n^{\frac{n+1}{n}\left(\frac{\log p_{n+1}}{\log p_n}\right)}$ for every $n \geq 1$. This inequality is weaker than the Firoozbakht's conjecture $p_{n+1} < p_n^{\frac{n+1}{n}}$.Afterward we prove that the new inequality is equivalent to Firoozbakht's conjecture, as $n \to \infty$, and hence the Cramér's conjecture $p_{n+1} - p_n = O(\log^2 p_n)$ to be hold, because the Firoozbakht's conjecture is stronger than the Cramér's conjecture, see [5].

Key words: Cramér's conjecture, Firoozbakht's conjecture, primes.

MSC (2010): primary 11A41, secondary 11N05.

1. Introduction

Let $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$, be the successive primes in their natural ordering, and in general, let p_n be the nth prime. There are many conjectures on consecutive primes p_n and p_{n+1} , that some interesting of these conjectures are Cramér's and Firoozbakht's conjectures. These unsolved problems are as follows:

Cramér's conjecture. This conjecture presented by the Swedish mathematician Harald Cramér in 1936 [2]. The Cramér's conjecture assert that $p_{n+1} - p_n = O(\log^2 p_n)$, or in other word

$$\lim_{n\to\infty} sup \frac{p_{n+1} - p_n}{\log^2 p_n} = 1.$$

In fact already in 1920, as a weaker statement, Cramér under the Riemann Hypothesis proved that $p_{n+1} - p_n = O(\sqrt{p_n} \log p_n)$ [1]. Afterward, in the mid 1930's, Cramér proposed a probabilistic model for primes that leads to very precise predictions of the asymptotic properties of primes [2,6]. Using his probabilistic model, Cramér showed that the probability that a given integer *m* should be a prime is approximately $\frac{1}{\log m}$. Also, Cramér proved that for random primes P_n, with probability 1, we have $\lim_{n\to\infty} sup \frac{P_{n+1} - P_n}{\log^2 P_n} = 1$. This result, restated for true primes p_n and constitutes the above well-known Cramér's conjecture. Further, tow other similar conjectures are shank's conjecture [10] and Granville's (or Cramér-Granville) conjecture [3]. We believe that if Cramér's conjecture be hold, then prove of shank's and Granville's conjectures is possible.

Firoozbakht's conjecture. This conjecture presented by the Iranian mathematician Farideh Firoozbakht in 1982 [8]. The Firoozbakht's conjecture states that sequence of $\left\{p_n^{\frac{1}{n}}\right\}_{n\geq 1}$ is strictly decreasing, that this means that for every $n \geq 1$, always $p_{n+1}^{\frac{1}{n+1}} < p_n^{\frac{1}{n}}$ or $p_{n+1} < p_n^{\frac{n+1}{n}}$. The Firoozbakht's conjecture was verified for all primes below 4×10^{18}, see [4]. Also, if Firoozbakht's conjecture be hold, then $p_{n+1} - p_n < \log^2 p_n - \log p_n - 1$ for $n > 9$, see [5, Theorem 1], and hence the Cramér's conjecture also is true. The Firoozbakht's conjecture is one of the strongest conjectures in number theory, but there are some other conjectures that are stronger than the Firoozbakht's conjecture. For example, the Farhadian's conjecture is one of these conjectures. This conjecture presented by the author in 2016 [9]. The Farhadian's conjecture states that for every $n > 4$, if $\psi_n = \left(\frac{p_{n+1}}{p_n}\right)^n$, then $p_n^{\psi_n} < n^{p_n}$.

However, above unsolved problems appears difficult, but we will argue that this difficulty is not perennial, because we want to prove that these conjectures are hold as $n \to \infty$. Further, we will to present a new inequality consist of p_n and p_{n+1}, that this inequality is equivalent to conjectural inequality by the Firoozbakht's conjecture as $n \to \infty$, and hence the Cramér's conjecture to be hold (In fact if Firoozbakht's conjecture be hold as $n \to \infty$, this is sufficient to prove the Cramér's conjecture, see [5]). However, this work seems likely, but we can suggest a method for it. Thus, we know the Firoozbakht's conjecture assert that $p_{n+1} < p_n^{\frac{n+1}{n}}$. Now we will to suggest a new inequality by the form of $p_{n+1} < p_n^{\frac{n+1}{n} f(n)}$, where $f(n) > 1$ and $\lim_{n\to\infty} f(n) = 1$. We know this inequality is similar to the Firoozbakht's conjecture, but is weaker than it, while, both are equivalent as $n \to \infty$. In the next section, we prove this suggestion for $f(n) = \frac{\log p_{n+1}}{\log p_n}$.

2. Main Results

In this section we want to present the main results. Firstly, consider the following lemma.

Lemma 1. *Let p_n be the nth prime. Then for every $n \geq 1$, we have*

$$p_{n+1} < p_n^{\frac{n+1}{n}\left(\frac{\log p_{n+1}}{\log p_n}\right)}. \qquad (1)$$

Proof. We know

$$\frac{n\log p_{n+1}}{n\log p_n + \log p_n} < \frac{n\log p_{n+1}}{n\log p_n} = \frac{\log p_{n+1}}{\log p_n}, \qquad \forall n \geq 1$$

Hence

$$n\log p_{n+1} < \frac{\log p_{n+1}}{\log p_n}(n\log p_n + \log p_n), \qquad \forall n \geq 1 \qquad (2)$$

We take the exponential of the inequality (2), and we obtain ${p_{n+1}}^{n} < ({p_n}^{n+1})^{\frac{\log p_{n+1}}{\log p_n}}$, $\forall n \geq 1$, or in other word $p_{n+1} < {p_n}^{\frac{n+1}{n}\left(\frac{\log p_{n+1}}{\log p_n}\right)}$.

Note that the inequality (1) as well as slightly weaker than the Firoozbakht's conjectures, when n be very large. Because, by Firoozbakht's conjecture $p_{n+1} < {p_n}^{\frac{n+1}{n}}$, and by inequality (1), we know $p_{n+1} < {p_n}^{\frac{n+1}{n}\left(\frac{\log p_{n+1}}{\log p_n}\right)}$, where $\frac{\log p_{n+1}}{\log p_n} > 1$.

Corollary 1. *The sequence of* $\left\{{p_n}^{\frac{1}{n \log p_n}}\right\}_{n \geq 1}$ *is strictly decreasing.*

Proof. By lemma 1, proof is immediacy.

Theorem. *For* $\gg 1$,

$$p_{n+1} < {p_n}^{\frac{n+1}{n}}. \tag{3}$$

Or in other word, Firoozbakht's conjecture is true as $n \to \infty$.

Proof. By inequality (1) from lemma 1, we know $p_{n+1} < {p_n}^{\frac{n+1}{n}\left(\frac{\log p_{n+1}}{\log p_n}\right)}, \forall n \geq 1$. Hence, we prove that where as $n \to \infty$,

$${p_n}^{\frac{n+1}{n}\left(\frac{\log p_{n+1}}{\log p_n}\right)} \sim {p_n}^{\frac{n+1}{n}}$$

Then as $n \to \infty$,

$$p_{n+1} < {p_n}^{\frac{n+1}{n}\left(\frac{\log p_{n+1}}{\log p_n}\right)} \sim {p_n}^{\frac{n+1}{n}}$$

So, we have

$$\lim_{n\to\infty} \frac{{p_n}^{\frac{n+1}{n}\left(\frac{\log p_{n+1}}{\log p_n}\right)}}{{p_n}^{\frac{n+1}{n}}} = \lim_{n\to\infty} \frac{\exp\left(\log\left({p_n}^{\frac{n+1}{n}\left(\frac{\log p_{n+1}}{\log p_n}\right)}\right)\right)}{{p_n}^{\frac{n+1}{n}}} = \lim_{n\to\infty} \frac{\exp\left(\frac{n+1}{n} \times \frac{\log p_{n+1}}{\log p_n} \times \log p_n\right)}{{p_n}^{\frac{n+1}{n}}}$$

$$= \lim_{n\to\infty} \frac{\exp\left(\frac{n+1}{n} \times \log p_{n+1}\right)}{{p_n}^{\frac{n+1}{n}}} = \lim_{n\to\infty} \frac{{p_{n+1}}^{\frac{n+1}{n}}}{{p_n}^{\frac{n+1}{n}}} = \lim_{n\to\infty} \left(\frac{p_{n+1}}{p_n}\right)^{\frac{n+1}{n}}$$

$$= \lim_{n\to\infty} \exp\left(\frac{n+1}{n} \times \log \frac{p_{n+1}}{p_n}\right) = \exp\left(\overbrace{\lim_{n\to\infty} \frac{n+1}{n}}^{1} \times \lim_{n\to\infty} \log \frac{p_{n+1}}{p_n}\right) = \exp\left(\log\left(\overbrace{\lim_{n\to\infty} \frac{p_{n+1}}{p_n}}^{1}\right)\right)$$

$$= \exp(0) = 1.$$

Corollary 2. *If p_n be the nth prime, then*

$$p_{n+1} - p_n < \log^2 p_n - \log p_n - 1 + O(1) \quad \text{as } n \to \infty \tag{4}$$

Proof. We know if Firoozbakht's conjecture be hold, then

$$p_{n+1} - p_n < {p_n}^{1+\frac{1}{n}} - p_n, \tag{5}$$

Also by [5, Theorem 5], we know if $f_n = {p_n}^{1+\frac{1}{n}} - p_n$, then

$$f_n = \log^2 p_n - \log p_n - 1 + O(1) \quad \text{as } n \to \infty \tag{6}$$

On the other hand, by theorem 1, we know Firoozbakht's conjecture is hold as $n \to \infty$. Hence, by (5) and (6), we have $p_{n+1} - p_n < \log^2 p_n - \log p_n - 1 + O(1)$ as $n \to \infty$.

Corollary 3. *Cramér's conjecture is true.*

Proof. We know, the inequality (4) is sharper than Cramér's conjecture and hence the Cramér's conjecture is true.

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