The Goldbach Conjecture – An Emergence Effect of the Primes

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ABSTRACT

The present paper shows that a principle known as emergence lies beneath the strong Goldbach conjecture. Whereas the traditional approaches focus on the control over the distribution of the primes by means of circle method and sieve theory, we give a proof of the conjecture that involves the constructive properties of the prime numbers, reflecting their multiplicative character within the natural numbers. With an equivalent but more convenient form of the conjecture in mind, we create a structure on the natural numbers which is based on the prime factorization. Then, we realize that the characteristics of this structure immediately imply the conjecture and an even strengthened form of it. Moreover, we can achieve further results by generalizing the structuring. Thus, it turns out that the statement of the strong Goldbach conjecture is the special case of a general principle.

1. INTRODUCTION

In the course of the various attempts to solve the strong and the weak Goldbach conjecture – both formulated by Goldbach and Euler in their correspondence in 1742 – a substantially wrong-headed route was taken, mainly due to the fact that two underlying aspects of the strong (or *binary*) conjecture were overlooked. First, that focusing exclusively on the additive character of the statement does not take into account its real content, and second, that a principle known as *emergence* lies beneath the statement, a principle any existing proof of the conjecture must consider.

Let us discuss some of the most important milestones in the different approaches to the problem.

When a proof could not be achieved even for the sum of three primes (the weak conjecture for odd numbers) without additional assumptions, in the twenties of the previous century mathematicians began to search for the maximum number of primes necessary to represent any natural number greater than 1 as their sum. At the beginning, there were proofs that required hundreds of thousands (!) of primes (L. Schnirelmann [2]). In 1937 the weak conjecture was proven (I. Vinogradov [4]), but only above a constant large enough to make available sufficient primes as summands.

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Almost an entire century passed before the representation for all integers > 1 could be reduced to the maximum of five or six summands of primes, respectively (T. Tao [3]). In 2013 the huge gap of numbers for the weak Goldbach version was closed, using numerical verification combined with a complex estimative proof (H. Helfgott [1]).

The so-called Hardy-Littlewood circle method in combination with sophisticated techniques of sieve theory was employed and constantly improved upon in those approaches. However, these methods do not reflect the primes' actual role in the problem as originally formulated by Goldbach and Euler, by continuously examining 'how many' prime numbers are available as summands. As this does not work for the binary Goldbach conjecture, concern for that original problem has gradually been sidelined up to the present day, even though a solution would definitively resolve the issue of integers represented as the sum of primes.

We will show that the solution lies in the constructive characteristics of the prime numbers and not in their distribution.

2. THE STRONG GOLDBACH CONJECTURE

Theorem 2.1 (Strong Goldbach conjecture (SGB)). *Every even integer greater than 2 can be expressed as the sum of two primes.*

Moreover, we claim

Theorem 2.2 (SSGB). *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3. As usual, $\mathbb{Z}$ is used to denote the set of all integers. Furthermore, we denote the projections from $\mathbb{N}x\mathbb{N}x\mathbb{N}$ onto the i-th factor by π_i, $1 \leq i \leq 3$.

Proof of Theorem 2.1 and 2.2. In order to prove SGB and SSGB, we proceed by contradiction. The basic idea of the proof is as follows: SGB is equivalent to saying that every composite number is the arithmetic mean of two odd primes. Correspondingly, SSGB means that every integer greater than or equal to 4 is the arithmetic mean of two different odd primes. We achieve this result by using the constructive properties of the prime numbers within the natural numbers. Specifically, we provide a structured representation of the natural numbers starting from 3 and we show that this representation leads to a contradiction when we assume that the above equivalent reformulations of SGB and SSGB are not true.

At first, we replace SGB and SSGB with the following equivalent representations:

Every integer greater than 1 is prime or is the arithmetic mean of two different primes, p_1 and p_2.

and

Every integer greater than 3 is the arithmetic mean of two different primes, p_1 and p_2.

$$\text{SGB} \iff \forall\, n \in \mathbb{N}_2 \;\; n \text{ prime} \;\vee\; \exists\, p_1, p_2 \in \mathbb{P}_3 \;\exists\, d \in \mathbb{N} \quad p_1 + d = n = p_2 - d \tag{2.1}$$

$$\text{SSGB} \iff \forall\, n \in \mathbb{N}_4 \;\; \exists\, p_1, p_2 \in \mathbb{P}_3 \;\exists\, d \in \mathbb{N} \quad p_1 + d = n = p_2 - d \tag{2.2}$$

Now, we define

$$S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N};\ p, q \in \mathbb{P}_3,\ p < q;\ m = (p + q)/2 \}$$

and call S_g the g-structure on $\mathbb{N}_3$.

According to (2.1), SGB is equivalent to saying that all composite numbers $x \in \mathbb{N}_4$ appear as m in a triple component mk of S_g. This is equivalent to saying that for any fixed k all multiples xk, $x \geq 3$, are given by the triple components pk, mk, qk. Correspondingly, SSGB is equivalent to saying that all integers $x \in \mathbb{N}_4$ appear as m in a triple component mk of S_g.

We note that the whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g. This is a simple consequence of prime factorization and is easily verified through the following three cases.

- The primes p in $\mathbb{N}_3$ can be written as components pk with k = 1.

- The composite numbers in $\mathbb{N}_3$, different from the powers of 2, can be written as pk with $p \in \mathbb{P}_3$ and $k \in \mathbb{N}$.

- The powers of 2 in $\mathbb{N}_3$ can be written as mk with m = 4 and k = 1, 2, 4, 8, 16, … .

We call this representation by the components of S_g a '*covering*' or also a '*structuring*' of $\mathbb{N}_3$ and we realize that $\mathbb{P}_3$ is the smallest subset P of odd numbers in $\mathbb{N}_3$ that enables such a complete covering of $\mathbb{N}_3$ through the triples (pk, mk, qk) where p, q $\in$ P and where $\mathbb{N}_3 \setminus$ { powers of 2 } is covered by pk, qk (for a generalization see section 4). The following examples for the number 42 illustrate the redundant character of the covering:

(**42**, 54, 66) = (**7·6**, 9·6, 11·6)

(18, **42**, 66) = (3·6, **7·6**, 11·6)

(30, 36, **42**) = (5·6, 6·6, **7·6**)

(**42**, 70, 98) = (**3·14**, 5·14, 7·14)

(33, **42**, 51) = (11·3, **14·3**, 17·3)

(38, **42**, 46) = (19·2, **21·2**, 23·2)

(41, **42**, 43) = (41·1, **42·1**, 43·1)

(5, **42**, 79) = (5·1, **42·1**, 79·1)

Additionally to the covering of $\mathbb{N}_3$, we will use the following two other properties of S_g in the proof.

Maximality: Actually, for a complete covering of $\mathbb{N}_3$ it would be sufficient if we chose (3k, 4k, 5k) together with triples (pk, mk, qk) in which all other odd primes occur as p, q or m. However, for our purpose we use the structure S_g that is based on all pairs (p, q) of odd primes with p < q. We call this the maximality of the structure S_g.

Equidistance: The successive components in the triples of S_g are always equidistant. So, we call these triples as well as the structure S_g equidistant. We note that the numbers m in the triples are uniquely determined by the pairs (p, q) as the arithmetic mean of p and q.

The structure S_g can be written as a matrix where each row is formed by the triple components $p_i{\cdot}k$, $m_{ij}{\cdot}k$, $q_j{\cdot}k$ with $p_i < q_j$ running through $\mathbb{P}_3$ and $m_{ij} = (p_i + q_j)/2$ for a fixed k ≥ 1. So, we have an infinite matrix indexed by pairs of the form ((i, j), k). It starts as follows:

3·1, 4·1, 5·1,   3·1, 5·1, 7·1 … 5·1, 6·1, 7·1,   5·1, 8·1, 11·1 ...
3·2, 4·2, 5·2,   3·2, 5·2, 7·2 … 5·2, 6·2, 7·2,   5·2, 8·2, 11·2 ...
…
…
…

Written down the complete matrix, what we see is the whole $\mathbb{N}_3$ in structured form. We note that in each k-th row of the matrix every entry $p_i{\cdot}k$ with $p_i \in \mathbb{P}_3$ occurs infinitely many times, each time accompanied by the entries $m_{ij}{\cdot}k$, $q_j{\cdot}k$.

After the initial cases 4 = 2 + 2 and 6 = 3 + 3 for SGB, it suffices to prove SSGB. So, we have to check whether for any fixed k ≥ 1 there exists an nk, n ≥ 4, that is different from all triple components mk of S_g. As we have seen, SSGB is proved if we can show that such an nk does not exist. So, let us assume that nk exists. We want to deduce a contradiction.

Due to the covering of $\mathbb{N}_3$ through the S_g matrix, it is not possible that this nk lies in a subset of $\mathbb{N}_3$ which is not covered by the matrix.

Due to the maximality of S_g, it is also not possible that this nk equals some mk in a triple (pk, mk, qk) that is based on a pair of primes (p, q) not used in S_g.

As all triples in S_g are equidistant, in each k-th row of the S_g matrix we have an additional nk with nk < mk or nk > mk for all of the mk.

For each k ≥ 1, nk can be written as some pk when n is prime, as some pk' when n is composite and not a power of 2, or as 4k' when n is a power of 2; $p \in \mathbb{P}_3$; $k, k' \in \mathbb{N}$.

The expression pk' for nk with k' = k or k' ≠ k is a first component of S_g triples and the expression 4k' for nk is component of the triple (3k', 4k', 5k'). So, since nk equals some triple component pk' or 4k' that exists by definition of S_g and since the S_g triples are generated by the first and third components, the triples are the same in the case nk exists and in the case nk does not exist. This means that the set S_g stays as it is defined in the case ¬SSGB holds and in the case SSGB holds. We can formalize this as follows.

For the assumed nk, we have shown

(C): $\forall\, k \in \mathbb{N} \quad \exists\, (pk', mk', qk') \in S_g \quad nk = pk' \;\vee\; nk = mk'$

Therefore, using that the difference between case SSGB and case ¬SSGB is just the (non-) existence of nk for each k ≥ 1, we can state:

$\forall$ sets S, S'

((SSGB => S_g = S) and (¬SSGB => S_g = S'))

=> ((S_g = S) and ((C) => S_g = S')), since S_g stays as it is defined if there is no nk and since in case of ¬SSGB S_g stays as it is defined if all nk equal some S_g triple component.

=> ((S_g = S) and (S_g = S')), since (C) is true for any n given by ¬SSGB.

=> S = S'

So, we have that the set S_g remains the same, regardless of whether the assumed nk actually exists or not. Hence, we get: ¬SSGB => SSGB. This is a contradiction to the assumed existence of nk and therefore SSGB is proved.

□

Note. We have seen that the three properties, covering, maximality and equidistance, that we identified in our structure S_g, lead to the following consequence: The multiples in the triple form (pk, mk, qk) already represent all multiples xk, x ≥ 3, of a fixed k ≥ 1. More specifically, the triples are divided into two types: First, all triples (pk, mk, qk) where m is composite, and second, all remaining triples (pk, mk, qk) where m is prime. The first type yields SGB and the second type, together with the first, implies SSGB.

The structure S_g reveals that a principle known as *emergence* lies beneath the Goldbach statement: For a given nk, n ≥ 4, k ≥ 1, the existence of two odd primes p, q such that nk is the arithmetic mean of pk and qk becomes visible only when we consider all odd primes and all k simultaneously. The triple form (pk, mk, qk) for all multiples xk, x ≥ 3, of a fixed k ≥ 1 is an effect that emerges from the interaction of all such triples when k runs through $\mathbb{N}$. In other words: Goldbach's conjecture is true owing to an emergence effect of the prime numbers. See also the Remark 5.3.

3. EXAMPLES FOR SGB AND SSGB

In the previous section we have seen that the multiples of numbers k ≥ 1 in $\mathbb{N}_3$ are strictly set by our structure S_g. Let us call these multiples the occurrences of k within the structure. The representation of a nk, where n > 5 is composite, as nk = n'k', n' $\in \mathbb{P}_3$ or n' = 4, k' ≠ k, constitutes two distinct occurrences, i.e. one of the number k and another of the number k'. The occurrences of both, k and k', are ruled by the triples separately.

3.1. n = 14 and k = 3:
Let us assume that n is not the arithmetic mean of two primes. For nk = 42, we find for example (pk', mk', qk') = (3·6, 5·6, **7·6**), which is part of the occurrence of 6 in the structure. But there is no triple (p·3, m·3, q·3) that contains n·3. Thus, n·3 violates the occurrence of 3 in the structure.

This contradiction can be resolved only if n = m, that is, n must be the arithmetic mean of, for example, 11 and 17.

3.2. n = 9 and k = 3:
Let us assume that n is not the arithmetic mean of two primes. For nk = 27, we only find (pk', mk', qk') = (**3·9**, m·9, q·9), which is part of the occurrence of 9 in the structure. But there is no triple (p·3, m·3, q·3) that contains n·3. Thus, n·3 violates the occurrence of 3 in the structure.

This contradiction can be resolved only if n = m, that is, n must be the arithmetic mean of, for example, 7 and 11.

3.3. n = 19 and k = 3:
Let us assume that n is not the arithmetic mean of two primes. For nk = 57, we find for example (p'k, m'k, q'k) = (17·3, 18·3, **19·3**), which is already part of the occurrence of 3 in the structure. But there is no triple (p·3, m·3, q·3) with p < 19 < q that contains n·3. Thus, n·3 violates the occurrence of 3 in the structure.

This contradiction can be resolved only if n = m, that is, n must be the arithmetic mean of 7 and 31.

4. GENERALIZATION AND FURTHER RESULTS

We will now embed the structure S_g from section 2 into a general concept and, on this basis, we will obtain a generalization of SSGB. We make the following definitions.

Definition 4.1. *Let T be a non-empty subset of* $\mathbb{N}_3 \times \mathbb{N}_3 \times \mathbb{N}_3$. *A triple structure, or simply,* ***structure*** *S in* $\mathbb{N}_3$ *is a set defined by* $S := \{ (t_1 \cdot k, t_2 \cdot k, t_3 \cdot k) \mid (t_1, t_2, t_3) \in T; k \in \mathbb{N} \}$.

Definition 4.2. *Let S be a structure in* $\mathbb{N}_3$, *given by the triples* (s_1, s_2, s_3). *Then, a set* $N \subseteq \mathbb{N}_3$ *is* ***covered*** *by the structure S if every* $n \in N$ *can be represented by at least one* s_i, $1 \le i \le 3$; *that is,* $\forall n \in N \ \exists s_i, 1 \le i \le 3$, *such that* $n = s_i$. *We say that the structure S provides a* ***covering*** *of N.*

Based on these definitions, we can make the following elementary statement.

Lemma 4.3. *Let S be a structure based on the set T. Then,* $\mathbb{N}_3$ *is covered by S if and only if*

$$\mathbb{P}_3 \cup \{ 4 \} \subseteq \bigcup_{1 \le i \le 3} \pi_i (T).$$

Proof. Let the union of the sets $\pi_i (T)$, $1 \le i \le 3$, contain all odd primes and the number 4.

Then, every prime number in $\mathbb{N}_3$ is represented by a component $t_i \cdot k$ with $t_i \in \mathbb{P}_3$ and $k = 1$.

Furthermore, every composite number n, different from the powers of 2, has a prime decomposition $n = pk$ with $p \in \mathbb{P}_3$ and $k \in \mathbb{N}$, and as such, is represented by a triple component $t_i \cdot k$ of S.

The powers of 2 are represented by $t_i \cdot k$ with $t_i = 4$ and $k = 1, 2, 4, 8, 16, \dots$.

So, the whole range of $\mathbb{N}_3$ is covered by S. On the other hand, if any odd prime or the number 4 is missing in the union of the sets $\pi_i (T)$, $1 \le i \le 3$, at least one of the representations described above is no longer possible.

□

For the structure S_g, we have $\pi_1 (T_g) = \mathbb{P}_3$ and $4 \in \pi_2 (T_g)$. Based on the above definitions, we can generalize S_g in the following manner.

Let P be a subset of the set of all odd numbers in $\mathbb{N}_3$ with at least two elements. For a subset $T_P \subseteq P \times \mathbb{N}_3 \times P$, where $p < m < q$ for all $(p, m, q) \in T_P$, we then define the structure $S_P := \{ (pk, mk, qk) \mid k \in \mathbb{N}; (p, m, q) \in T_P \}$. We call the structure S_P maximal if all pairs $(p, q) \in P \times P$ with $p < q$ are used in T_P. Furthermore, we call the structure S_P distance-preserving if for all $(p, m, q) \in T_P$: $(q – m) – (m – p) = c$ with a constant c. Specifically, we call S_P equidistant if $c = 0$. We note that in a distance-preserving S_P the component mk is uniquely determined by pk, qk. In the case of an equidistant S_P, we obtain the arithmetic mean for m.

In order to get a covering of $\mathbb{N}_3 \setminus$ { powers of 2 } through the components pk, qk of S_P, P must contain all odd primes. In this case, due to the construction of S_P, a maximal and distance-preserving S_P covers $\mathbb{N}_3$ and is equidistant because the triples (3k, 4k, 5k) are contained.

Now, let S_P be a structure that provides a covering of $\mathbb{N}_3$ where $\mathbb{N}_3 \setminus$ { powers of 2 } is covered by the components pk, qk. That is, we only know that at least all odd primes as p or q and the number m = 4 must be used in S_P. Then, we state the following.

$\exists\, n \in \mathbb{N}_3 \;\; \exists\, k \in \mathbb{N} \quad nk \neq pk \;\wedge\; nk \neq mk \;\wedge\; nk \neq qk \quad$ for all $(pk, mk, qk) \in S_P$

=>

(i) S_P is not maximal

$\vee$

(ii) S_P is not distance-preserving.

It suffices to verify this for $P = \mathbb{P}_3$ because, due to (i), it applies to larger sets P by implication. So, when $P = \mathbb{P}_3$ we have $S_P = S_g$ and in section 2 we have shown that the above statement is true.

For a generalization of the arithmetic mean m used for SSGB, we'll now consider functions $f : \mathbb{P}_3 \times \mathbb{P}_3 \rightarrow \mathbb{Z}$. In order to achieve useful results, we define the following restrictions on f.

First, we restrict f with the condition (f1): For all pairs $(p, q) \in \mathbb{P}_3 \times \mathbb{P}_3$, $p < q$, the triples (p, q, f(p,q)) have the same numerical ordering and the difference between the two distances of successive components remains constant. I.e., the resulting triples (t_1, t_2, t_3) satisfy: $t_1 < t_2 < t_3$ and $(t_3 – t_2) – (t_2 – t_1) = c$ with a constant c.

Secondly, we set the condition (f2): There is a pair $(p, q) \in \mathbb{P}_3 \times \mathbb{P}_3$, $p < q$, with $f(p,q) = 4$.
So, the powers of 2 are contained when we consider the triples (pk, qk, f(p,q)k) for all $k \geq 1$. Therefore, according to Lemma 4.3, the whole range of $\mathbb{N}_3$ is covered by the components of these triples.

Let f be a function with the conditions (f1), (f2). We then define a f-specific, distance-preserving, structure which covers $\mathbb{N}_3$ by

$$S_f := \{ (pk, qk, f(p,q)k) \mid k \in \mathbb{N};\ p, q \in \mathbb{P}_3,\ p < q;\ f(p,q) \in \mathbb{N}_3 \}$$

and call it the f-structure. Also here we use the maximality considering all pairs (p, q) of odd primes with p < q. And again the pairs (p, q) are the exclusive parameters in the triples (pk, qk, f(p,q)k) for each k. Moreover, for each fixed k the triples (pk, qk, f(p,q)k) distribute their components uniformly in accordance with (f1).

From this we obtain the following:

Any function f as above generates exactly one of three possible classes of numbers: only even numbers or only odd numbers or both. We call this the f-class. In case of odd numbers, f cannot satisfy (f2) so that the f-structure would not yield a complete covering of $\mathbb{N}_3$. So, f is restricted to be a function which generates either only even numbers or both even and odd numbers. Furthermore, we notice that a f-structure provides a distribution exclusively for f(p,q)k by means of the triples (pk, qk, f(p,q)k) for each k. If, for example, f produces only even numbers, only such even multiples of k are being distributed through the structure S_f. In this case, we would have no information regarding the odd multiples which are not prime.

A few observations on the special case when f is the arithmetic mean:

In the proof of SSGB we used the function f = g that determines the arithmetic mean g(p,q) = (p + q) ⁄ 2 = m. g generates even and odd numbers and satisfies the conditions (f1) and (f2) for building the g-structure on $\mathbb{N}_3$. In this case, c = 0 so that distance-preserving means equidistance. The pairs (pk, qk) are expanded into triples (pk, mk, qk) including the powers of 2 through (3k, 4k, 5k). As is easily verified, the arithmetic mean is the only function fulfilling (f1) and (f2) that has its values in the middle component of the ordered triples and that generates even and odd numbers.

For the definition of S_f we replaced the arithmetic mean m used in the structure S_g by numbers f(p,q) $\in \mathbb{N}_3$ determined by a function f with the conditions (f1), (f2). Now, we apply the proof of SSGB to the triples (pk, qk, f(p,q)k) generalizing the equidistance by the condition (f1), where the parameter n used for the multiples nk in that proof is now of the f-class. Then, based on the f-structure S_f, we obtain the following property as a generalization of SSGB.

> **(F)** *For each fixed k ≥ 1 the triples (pk, qk, f(p,q)k) form a distribution of all nk, n ≥ 4, n of the f-class, with respect to pk, qk that is determined by (f1).*

Let us now consider functions f which satisfy the conditions (f1) and (f2) and build a f-structure on $\mathbb{N}_3$, with the outcome that f(p,q) represents all even integers greater than 2. Due to (f2), in this case f(p,q) is always the first component in the ordered triples.

For example, we can state

Corollary 4.4. *All even positive integers are of the form 2p – q + 1 with odd primes p < q.*

Proof. For the number 2 we have: $2 = 2 \cdot 3 - 5 + 1$. For all even numbers in $\mathbb{N}_3$ we apply our concept of the f-structure.

As is easily verified, $f(p,q) = 2p - q + 1$ satisfies the conditions (f1) and (f2) for building a f-structure on $\mathbb{N}_3$. We consider only those $f(p,q)$ which lie in $\mathbb{N}_3$ and we note that by the Bertrand-Chebyshev theorem: $\forall p \in \mathbb{P}_3$, $p > 3$, $\exists\, q \in \mathbb{P}_3$, $q > p$, such that $f(p,q) \in \mathbb{N}_3$.

We now assume that there is an even integer $n > 2$ which is not of the form $n = 2p - q + 1$ with two odd primes p, q. We then consider the multiple nk for any $k \geq 1$ and note that nk belongs to none of the triples $((2p - q + 1)k, pk, qk)$. This causes a contradiction to (F) and proves the corollary.

□

Note. If we interchange the primes p, q and consider $f\,'(p,q) = 2q - p + 1$, then f ' also satisfies the condition (f1) for building a f-structure. But for a complete covering of $\mathbb{N}_3$ the number 4 is missing, and we can easily verify that there are other even numbers in $\mathbb{N}_3$ which cannot be represented by $f\,'(p,q)$.

Another interesting example is $f(p,q) = 2p - q - 3$ versus $f\,'(p,q) = 2q - p - 3$. f satisfies all conditions, including the covering, and therefore represents all even numbers, whereas f ' satisfies the covering because of $f\,'(3,5) = 4$, but it violates numerical ordering and distance-preserving. There are even numbers in $\mathbb{N}_3$ which cannot be represented by $f\,'(p,q)$.

5. REMARKS

5.1. Due to the unpredictable way that the primes are distributed, all studies on the representation of natural numbers as the sum of primes are problematic when they use approaches based on the distribution of the primes.

Despite tremendous efforts over the centuries, the best result by 2012 was five summands. I was always convinced that the solution must lie in the constructive characteristics of the prime numbers and not in their distribution.

5.2. The main element in the proof is the symmetric structure (pk, mk, qk) where the multiplicative property of the prime numbers ensures the complete covering of $\mathbb{N}_3$ through this structure. As we have shown, the binary Goldbach conjecture actually is a specific case of a general distribution principle within the natural numbers.

In order to discard the usual interpretation of the conjecture that focuses on the sums of primes and thus opposes their multiplicative character, we have tackled the problem differently after shifting to the triple form: Instead of searching for primes which determine the needed arithmetic mean equal to a given n, we have approached the issue from the opposite direction. Based on the multiplicative prime decomposition, we identify nk as the component of a structure, in our case determined by the arithmetic mean.

In the proof, there is a dual role of the numbers k: As multiplier they generate composite numbers while their own multiples in $\mathbb{N}_3$ are strictly set by the used structure.

5.3. In other subject areas, the effect of the formation of new properties after the transition from single items to a whole system is called *emergence* (‘*The whole is more than the sum of its parts.*’). The structure S_g reveals that such principle lies beneath the Goldbach statement: For a given nk, n ≥ 4, k ≥ 1, the existence of two odd primes p, q such that nk is the arithmetic mean of pk and qk becomes visible only when we consider all odd primes and all k simultaneously. There is a remarkable aspect of this *emergence*: The two primes which form the so-called Goldbach partition of a given even number 2n are located before 2n, however, the reason for the existence of that partition also involves the primes beyond 2n.

It can be expected that also other questions in number theory own a solution based on this underlying principle.

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