Goldbach's conjecture

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ABSTRACT

The present paper shows that a principle known as emergence lies beneath the strong Goldbach conjecture. Whereas the traditional approaches focus on the control over the distribution of the primes by means of circle method and sieve theory, we give a proof of the conjecture that involves the constructive properties of the prime numbers, reflecting their multiplicative character within the natural numbers. With an equivalent but more convenient form of the conjecture in mind, we create a structure on the natural numbers which is based on the prime factorization. Then, we realize that the characteristics of this structure immediately imply the conjecture and, in addition, an even strengthened form of it. Moreover, we can achieve further results by generalizing the structuring. Thus, it turns out that the statement of the strong Goldbach conjecture is the special case of a general principle.

1. INTRODUCTION

In the course of the various attempts to solve the strong and the weak Goldbach conjecture – both formulated by Goldbach and Euler in their correspondence in 1742 – a substantially wrong-headed route was taken, mainly due to the fact that two underlying aspects of the strong (or *binary*) conjecture were overlooked. First, that focusing exclusively on the additive character of the statement does not take into account its real content, and second, that a principle known as *emergence* lies beneath the statement, a principle any existing proof of the conjecture must consider.

Let us discuss some of the most important milestones in the different approaches to the problem.

When a proof could not be achieved even for the sum of three primes (the weak conjecture for odd numbers) without additional assumptions, in the twenties of the previous century mathematicians began to search for the maximum number of primes necessary to represent any natural number greater than 1 as their sum. At the beginning, there were proofs that required hundreds of thousands (!) of primes (L. Schnirelmann [2]). In 1937 the weak conjecture was proven (I. Vinogradov [4]), but only above a constant large enough to make available sufficient primes as summands.

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Almost an entire century passed before the representation for all integers > 1 could be reduced to the maximum of five or six summands of primes, respectively (T. Tao [3]). In 2013 the huge gap of numbers for the weak Goldbach version was closed, using numerical verification combined with a complex estimative proof (H. Helfgott [1]).

The so-called Hardy-Littlewood circle method in combination with sophisticated techniques of sieve theory was employed and constantly improved upon in those approaches. However, these methods do not reflect the primes' actual role in the problem as originally formulated by Goldbach and Euler, by continuously examining 'how many' prime numbers are available as summands. As this does not work for the binary Goldbach conjecture, concern for that original problem has gradually been sidelined up to the present day, even though a solution would definitively resolve the issue of integers represented as the sum of primes.

We will show that the solution lies in the constructive characteristics of the prime numbers and not in their distribution.

2. THE STRONG GOLDBACH CONJECTURE

Theorem 2.1 (Strong Goldbach conjecture (SGB)). *Every even integer greater than 2 can be expressed as the sum of two primes.*

Moreover, we claim

Theorem 2.2 (SSGB). *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Proof. In order to prove SGB and SSGB, we proceed by contradiction. The basic idea of the proof is as follows: SGB is equivalent to saying that every composite number is the arithmetic mean of two odd primes. Correspondingly, SSGB means that every integer greater than or equal to 4 is the arithmetic mean of two different odd primes. We achieve this result by using the constructive properties of the prime numbers within the natural numbers. Specifically, we provide a structured representation of the natural numbers starting from 3 and we show that this representation leads to an arithmetic sum formula which becomes contradicted when we assume that the above equivalent reformulation of SGB is not true. In addition, this results in SSGB.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3. As usual, $\mathbb{Z}$ is used to denote the set of all integers. Furthermore, we denote the projections from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ onto the i-th factor by π_i, $1 \leq i \leq 3$.

At first, we replace SGB and SSGB with the following equivalent representations:

Every integer greater than 1 is prime or is the arithmetic mean of two different primes, p_1 *and* p_2*.*

and

Every integer greater than 3 is the arithmetic mean of two different primes, p_1 *and* p_2*.*

$$\text{SGB} \Leftrightarrow \forall n \in \mathbb{N},\ n > 1 : (\ n \text{ prime} \ \vee\ \exists\, p_1, p_2 \in \mathbb{P}_3\ \exists\, d \in \mathbb{N} \text{ with } p_1 + d = n = p_2 - d\ ) \tag{2.1}$$

$$\text{SSGB} \Leftrightarrow \forall n \in \mathbb{N},\ n > 3 : (\ \exists\, p_1, p_2 \in \mathbb{P}_3\ \exists\, d \in \mathbb{N} \text{ with } p_1 + d = n = p_2 - d\ ) \tag{2.2}$$

Now, we define

$$S_g := \{\ (pk, mk, qk) \mid k, m \in \mathbb{N};\ p, q \in \mathbb{P}_3,\ p < q;\ m = (p + q)/2\ \}$$

and call S_g the g-structure on $\mathbb{N}_3$.

According to (2.1), SGB is equivalent to saying that all composite numbers $n \in \mathbb{N}_4$ occur as arithmetic mean m in a triple of S_g. This is equivalent to saying that for any fixed k all multiples nk, n ≥ 3, are given by the triple components pk, mk, qk. Correspondingly, SSGB is equivalent to saying that all integers $n \in \mathbb{N}_4$ occur as arithmetic mean m in a triple of S_g.

We note that the whole range of $\mathbb{N}_3$ is represented by the triple components of S_g. This is a simple consequence of prime factorization and is easily verified through the following three cases:

The primes p in $\mathbb{N}_3$ are represented by components pk with k = 1; the composite numbers in $\mathbb{N}_3$, different from the powers of 2, are represented by pk with $p \in \mathbb{P}_3$ and $k \in \mathbb{N}$; the powers of 2 in $\mathbb{N}_3$ are represented by mk with m = 4 and k = 1, 2, 4, 8, 16, … .

We call this representation by the components of S_g a '*covering*' or also a '*structuring*' of $\mathbb{N}_3$ (for a generalization see section 4). The following examples for the number 42 illustrate the redundant character of the covering:

(**42**, 54, 66) = (**7·6**, 9·6, 11·6)

(18, **42**, 66) = (3·6, **7·6**, 11·6)

(30, 36, **42**) = (5·6, 6·6, **7·6**)

(**42**, 70, 98) = (**3·14**, 5·14, 7·14)

$(33, \mathbf{42}, 51) = (11 \cdot 3, \mathbf{14 \cdot 3}, 17 \cdot 3)$

$(38, \mathbf{42}, 46) = (19 \cdot 2, \mathbf{21 \cdot 2}, 23 \cdot 2)$

$(41, \mathbf{42}, 43) = (41 \cdot 1, \mathbf{42 \cdot 1}, 43 \cdot 1)$

$(5, \mathbf{42}, 79) \quad = (5 \cdot 1, \mathbf{42 \cdot 1}, 79 \cdot 1)$

Additionally to the covering of $\mathbb{N}_3$, we will use the following two other properties of S_g in the proof.

Equidistance: The successive components in the triples of S_g are always equidistant. So, we call these triples as well as the structure S_g equidistant. We note that the numbers m in the triples are uniquely determined by the pairs (p, q) as the arithmetic mean of p and q.

Maximality: Actually, for a complete covering of $\mathbb{N}_3$ it would be sufficient if we chose (3k, 4k, 5k) together with triples (pk, mk, qk) in which all other odd primes occur as p, q or m. However, for our purpose we use the structure S_g that is based on all pairs (p, q) of odd primes with p < q. We call this the maximality of the structure S_g.

The structure S_g can be written as a matrix where each row is formed by the triple components $p_i \cdot k$, $m_{ij} \cdot k$, $q_j \cdot k$ with $p_i < q_j$ running through $\mathbb{P}_3$ and $m_{ij} = (p_i + q_j)/2$ for a fixed $k \geq 1$. So, we have an infinite matrix indexed by pairs of the form ((i, j), k). It starts as follows:

$(3 \cdot 1, 4 \cdot 1, 5 \cdot 1), (3 \cdot 1, 5 \cdot 1, 7 \cdot 1) \ldots (5 \cdot 1, 6 \cdot 1, 7 \cdot 1), (5 \cdot 1, 8 \cdot 1, 11 \cdot 1) \ldots$
$(3 \cdot 2, 4 \cdot 2, 5 \cdot 2), (3 \cdot 2, 5 \cdot 2, 7 \cdot 2) \ldots (5 \cdot 2, 6 \cdot 2, 7 \cdot 2), (5 \cdot 2, 8 \cdot 2, 11 \cdot 2) \ldots$
…
…
…

Written down the complete matrix, what we see is the whole $\mathbb{N}_3$ in redundant form, i.e. a structured $\mathbb{N}_3$.

Now, we will check if for any fixed k ≥ 1 there exists an additional nk, n ≥ 3, that does not appear in the k-th row of the matrix. As we have seen, SGB is proved if we can show that such an nk does not exist.

Due to the covering of $\mathbb{N}_3$ through the S_g matrix, it is not possible that this nk lies in a subset of $\mathbb{N}_3$ which is not covered by the matrix.

Note: Based on the covering, for a composite n > 5 a representation of nk as nk = n'k', k' ≠ k, in the matrix (with $n' \in \mathbb{P}_3$ when nk not a power of 2 and with n' = 4 when nk a power of 2) is always possible. So, when we say that nk does not appear in the k-th row of the matrix then it refers to the fixed factor k in the decomposition nk of this number.

Due to the maximality of S_g, it is also not possible that this nk equals some mk in a triple (pk, mk, qk) that is based on a pair of primes (p, q) not used in S_g.

Since all triples in S_g are equidistant, this implies: For every n, if nk does not appear in the k-th row of the matrix, then for every pair of primes p, q with p < q we have n < m or n > m, where m is the arithmetic mean of p and q.

So, as the pairs (p, q) are the exclusive parameters in the triples for each k, the construction of S_g would lead to the above consequence if we added that decomposition nk with fixed factor k to the k-th row of the matrix. As we shall now see, this implies that such an additional nk can not exist. In other words: As the matrix represents $\mathbb{N}_3$, the structure S_g leaves no space in $\mathbb{N}_3$ for such an nk.

We can express our reasoning arithmetically by using the following sum.

$$\mathrm{sum}_g := \sum_{k \in \mathbb{N}} k \left(\sum_{\substack{p,q \in \mathbb{P}_3 \\ p < q \\ m=(p+q)/2}} (q-m) - (m-p) \right) = \sum_{k \in \mathbb{N}} \sum_{\substack{p,q \in \mathbb{P}_3 \\ p < q \\ m=(p+q)/2}} (pk + qk - 2mk)$$

That is, over all rows and over all pairs (p, q) of the matrix we sum up the difference of the distances between the successive triple components. Due to the equidistance of all triples (p, m, q), trivially we have $\mathrm{sum}_g = 0$.

As the term in the inner sum of sum_g has a constant value of zero for each pair (p, q), p < q, and so sum_g is absolutely convergent, we are allowed to rearrange it. Any rearrangement of sum_g is exclusively based on the summands pk, qk, 2mk. Therefore, it does not depend on whether there exists a number $x \in \mathbb{N}_5$ with $x \neq m$ for all m, because such an x would already be contained in the sum as pk when it is not a power of 2, or as (p+q)k when it is a power of 2. That is, without changing the sum value zero, sum_g can be rearranged as if such an $x \geq 5$ with $x \neq m$ for all m does not exist.

Since we build the sum over all pairs of odd primes, this implies: For every $y \in \mathbb{N}_5$, in sum_g represented by y = pk or by y = qk when it is not a power of 2 and represented by y = (p+q)k when it is a power of 2, and for all decompositions y = y'k' with $y' \geq 4$ and $k' \geq 1$, there are arithmetic means m of two odd primes such that y' = m.

Let us assume now that for any fixed $k \geq 1$ there exists an nk, $n \geq 3$, that does not appear in the k-th row of the S_g matrix. We have seen that this leads to an $n \in \mathbb{N}_3$ that is additional to all p, q, m and that must be located either between p and m or between m and q for every pair of primes p, q with p < q. The case n = 4 is covered by m = (3+5) ⁄ 2 so that we consider n > 5.

As we have seen above, on the one hand sum_g is clearly zero, but on the other hand, using the assumption, we can now recalculate sum_g in a different way.

The assumed n is represented by some pk or by some (p+q)k in sum_g, but since n is different from all p, q, m, with respect to the equalities described above, for n the corresponding negative part mk' with k' = 1 is missing in the sum. So, assuming such an n we can split up sum_g in the following manner.

$$sum_g = \sum_{k \in \mathbb{N}} kn \quad + \sum_{k \in \mathbb{N}} \quad \sum_{\substack{p,q \in \mathbb{P}_3 \\ p < q \\ m=(p+q)/2}} (pk + qk - 2mk)$$

Therefore, under the assumption about n, we obtain

$$sum_g = \sum_{k \in \mathbb{N}} kn = +\infty$$

This is a contradiction to the initial calculation $sum_g = 0$. In case of more than one n with the assumed condition, the same contradiction applies because for each n the corresponding changes on sum_g would be additive.

So, we realize that the three properties, covering, equidistance and maximality, that we identified in our structure S_g, lead to the following consequence:

The multiples in the triple form (pk, mk, qk) already represent all multiples nk, n ≥ 3, of a fixed k ≥ 1. More specifically, the triples are divided into two types: First, all triples (pk, mk, qk) where m is composite, and second, all remaining triples (pk, mk, qk) where m is prime. The first type yields SGB and the second type, together with the first, implies SSGB.

□

Note. The structure S_g reveals that a principle known as *emergence* lies beneath the Goldbach statement: For a given nk, n ≥ 4, k ≥ 1, the existence of two odd primes p, q such that nk is the arithmetic mean of pk and qk becomes visible only when we consider all odd primes and all k simultaneously. The triple form (pk, mk, qk) for all multiples nk, n ≥ 3, of a fixed k ≥ 1 is an effect that emerges from the interaction of all such triples when k runs through $\mathbb{N}$. See also the Remark 5.3.

3. EXAMPLES FOR SGB AND SSGB

In the previous chapter we have seen that the multiples of the numbers k in $\mathbb{N}_3$ are strictly set by our structure S_g. Let us call these multiples the occurrences of k within the structure. In the proof it was essential to understand that the representation of a nk, where n > 5 is composite, as nk = n'k', n' $\in \mathbb{P}_3$ or n' = 4, k' ≠ k, constitutes two distinct occurrences, i.e. one of the number k and another of the number k'. The occurrences of both, k and k', are ruled by the triples separately.

3.1. n = 14 and k = 3:
Let us assume that n is not the arithmetic mean of two primes. For nk = 42, we find for example (pk', mk', qk') = (3·6, 5·6, **7·6**), which is part of the occurrence of 6 in the structure. But there is no triple (p·3, m·3, q·3) that contains n·3. Thus, n·3 violates the occurrence of 3 in the structure.

This contradiction can be resolved only if n = m, that is, n must be the arithmetic mean of, for example, 11 and 17.

3.2. n = 9 and k = 3:
Let us assume that n is not the arithmetic mean of two primes. For nk = 27, we only find (pk', mk', qk') = (**3·9**, m·9, q·9), which is part of the occurrence of 9 in the structure. But there is no triple (p·3, m·3, q·3) that contains n·3. Thus, n·3 violates the occurrence of 3 in the structure.

This contradiction can be resolved only if n = m, that is, n must be the arithmetic mean of, for example, 7 and 11.

3.3. n = 19 and k = 3:
Let us assume that n is not the arithmetic mean of two primes. For nk = 57, we find for example (p'k, m'k, q'k) = (17·3, 18·3, **19·3**), which is part of the occurrence of 3 in the structure. But there is no triple (p·3, m·3, q·3) with p < 19 < q that contains n·3. Thus, n·3 violates the occurrence of 3 in the structure.

This contradiction can be resolved only if n = m, that is, n must be the arithmetic mean of 7 and 31.

4. GENERALIZATION AND FURTHER RESULTS

First, we will embed the structure used in the proof of SSGB in a general concept. We give the following definitions:

Definition 4.1. *Let T be a non-empty subset of $\mathbb{N}_3$ x $\mathbb{N}_3$ x $\mathbb{N}_3$. A triple structure, or simply,* ***structure*** *S in $\mathbb{N}_3$ is a set defined by S := { $(t_1 \cdot k, t_2 \cdot k, t_3 \cdot k)$* **|** *$(t_1, t_2, t_3) \in T$; $k \in \mathbb{N}$ }.*

Definition 4.2. *Let S be a structure in $\mathbb{N}_3$, given by the triples (s_1, s_2, s_3). Then, a set $N \subseteq \mathbb{N}_3$ is* ***covered*** *by the structure S if every $n \in N$ can be represented by at least one s_i, $1 \le i \le 3$; that is, $\forall n \in N$ $\exists s_i$, $1 \le i \le 3$, such that $n = s_i$. We say that the structure S provides a* ***covering*** *of N.*

Based on these definitions, we can make the following elementary statement:

Lemma 4.3. *Let S be a structure based on the set T. Then,* $\mathbb{N}_3$ *is covered by S if and only if* $\mathbb{P}_3 \cup \{ 4 \} \subseteq \bigcup_{1 \le i \le 3} \pi_i(T)$.

Proof. Let the union of the sets $\pi_i(T)$, $1 \le i \le 3$, contain all odd primes and the number 4.

Then, every prime number in $\mathbb{N}_3$ is represented by a component $t_i{\cdot}k$ with $t_i \in \mathbb{P}_3$ and $k = 1$.

Furthermore, every composite number n, different from the powers of 2, has a prime decomposition $n = pk$ with $p \in \mathbb{P}_3$ and $k \in \mathbb{N}$, and as such, is represented by a triple component $t_i{\cdot}k$ of S.

The powers of 2 are represented by $t_i{\cdot}k$ with $t_i = 4$ and $k = 1, 2, 4, 8, 16, \ldots$.

So, the whole range of $\mathbb{N}_3$ is covered by S. On the other hand, if any odd prime or the number 4 is missing in the union of the sets $\pi_i(T)$, $1 \le i \le 3$, at least one of the representations described above is no longer possible.

□

For the structure S_g, used in chapter 2, that covers $\mathbb{N}_3$ we have $\pi_1(T_g) = \mathbb{P}_3$ and $4 \in \pi_2(T_g)$. Based on the above definitions, we can generalize S_g in the following manner:

Let P be a subset of the set of all odd numbers in $\mathbb{N}_3$ with at least two elements. For a subset $T_P \subseteq P \times \mathbb{N}_3 \times P$, where $p < m < q$ for all $(p, m, q) \in T_P$, we then define the structure $S_P := \{ (pk, mk, qk) \mid k \in \mathbb{N}; (p, m, q) \in T_P \}$. We call the structure S_P maximal if all pairs $(p, q) \in P \times P$ with $p < q$ are used in T_P. Furthermore, we call the structure S_P distance-preserving if for all $(p, m, q) \in T_P : (q – m) – (m – p) = c$ with a constant c. Specifically, we call S_P equidistant if $c = 0$. We note that in a distance-preserving S_P the component mk is uniquely determined by pk, qk. In the case of an equidistant S_P, we obtain the arithmetic mean for m.

In order to get a covering of $\mathbb{N}_3 \setminus \{$ powers of 2 $\}$ through the components pk, qk of S_P, P must contain all odd primes. In this case, due to the construction of S_P, a maximal and distance-preserving S_P covers $\mathbb{N}_3$ and is equidistant because the triples (3k, 4k, 5k) are contained. We then obtain the structure S_g by setting $P = \mathbb{P}_3$ and we realize that $\mathbb{P}_3$ is the smallest subset of odd numbers in $\mathbb{N}_3$ that enables such a complete covering of $\mathbb{N}_3$ through the triples (pk, mk, qk) with $p, q \in \mathbb{P}_3$.

Now, for a generalization in terms of the numbers m we consider functions $f : \mathbb{P}_3 \times \mathbb{P}_3 \rightarrow \mathbb{Z}$. To achieve useful results, we define the following restrictions on f:

First, we restrict f with the condition (f1): For all pairs $(p, q) \in \mathbb{P}_3 \times \mathbb{P}_3$ with $p < q$ the triples $(p, q, f(p,q))$ have the same numerical ordering and the difference between the two distances of successive components remains constant. I.e., the resulting triples (t_1, t_2, t_3) satisfy: $t_1 < t_2 < t_3$ and $(t_3 - t_2) - (t_2 - t_1) = c$ with a constant c.

Additionally, we set the condition (f2): $\exists$ (p, q) $\in \mathbb{P}_3 \times \mathbb{P}_3$, p < q, with f(p,q) = 4. So, the powers of 2 are contained when we consider the triples (pk, qk, f(p,q)k) for all k ≥ 1. Therefore, according to Lemma 4.3, the whole range of $\mathbb{N}_3$ is covered by the components of these triples.

For the function f with the conditions (f1), (f2) we then define a f-specific, distance-preserving, structure which covers $\mathbb{N}_3$ by

$$S_f := \{ (pk, qk, f(p,q)k) \mid k \in \mathbb{N};\ p, q \in \mathbb{P}_3,\ p < q;\ f(p,q) \in \mathbb{N}_3 \}$$

and call it the f-structure. Also here we use the maximality considering all pairs (p, q) of odd primes with p < q. And again the pairs (p, q) are the exclusive parameters in the triples (pk, qk, f(p,q)k) for each k. Moreover, for each fixed k the triples (pk, qk, f(p,q)k) distribute their components uniformly in accordance with (f1).

From this we obtain the following:

Any function f as above generates exactly one of three possible classes of numbers: only even numbers or only odd numbers or both. We call this the f-class. In case of odd numbers, f cannot satisfy (f2) so that the f-structure would not yield a complete covering of $\mathbb{N}_3$. So, f is restricted to be a function which generates either only even numbers or both even and odd numbers. Furthermore, we notice that a f-structure provides a distribution exclusively for f(p,q)k by means of the triples (pk, qk, f(p,q)k) for each k. If, for example, f produces only even numbers, only such even multiples of k are being distributed through the structure S_f. In this case, we would have no information regarding the odd multiples which are not prime.

A few observations on the special case when f is the arithmetic mean:

In the proof of SSGB we used the function f = g that determines the arithmetic mean $g(p,q) = (p + q) / 2 = m$. g generates even and odd numbers and satisfies the conditions (f1) and (f2) for building the g-structure on $\mathbb{N}_3$. In this case, c = 0 so that distance-preserving means equidistance. The pairs (pk, qk) are expanded into triples (pk, mk, qk) including the powers of 2 through (3k, 4k, 5k). As is easily verified, the arithmetic mean is the only function fulfilling (f1) and (f2) that has its values in the middle component of the ordered triples and that generates even and odd numbers.

For the definition of S_f we replaced the arithmetic mean m used in the structure S_g by numbers f(p,q) $\in \mathbb{N}_3$ determined by a function f with the conditions (f1), (f2). Now, we apply the proof of SSGB to the triples (pk, qk, f(p,q)k) generalizing the argument of equidistance by the condition (f1), where the parameter n used for the multiples nk in that proof is now of the f-class. Then, based on the f-structure S_f, we obtain the following property as a generalization of SSGB:

(F) *For each fixed k ≥ 1 the triples (pk, qk, f(p,q)k) form a distribution of all nk, n ≥ 4, n of the f-class, with respect to pk, qk that is determined by (f1).*

Let us now consider other functions f which satisfy the conditions (f1) and (f2) and build a f-structure on $\mathbb{N}_3$, with the outcome that f(p,q) represents all even integers greater than 2. Due to (f2), in this case f(p,q) is always the first component in the ordered triples.

For example, we can state

Corollary 4.4. *All even positive integers are of the form 2p – q + 1 with odd primes p < q.*

Proof. For the number 2 we have: $2 = 2 \cdot 3 - 5 + 1$. For all even numbers in $\mathbb{N}_3$ we apply our concept of the f-structure.

As is easily verified, $f(p,q) = 2p - q + 1$ satisfies the conditions (f1) and (f2) for building a f-structure on $\mathbb{N}_3$. We consider only those f(p,q) which lie in $\mathbb{N}_3$ and we note that by the Bertrand-Chebyshev theorem: $\forall p \in \mathbb{P}_3$, $p > 3$, $\exists\, q \in \mathbb{P}_3$, $q > p$, such that $f(p,q) \in \mathbb{N}_3$.

We now assume that there is an even integer $n > 2$ which is not of the form $n = 2p - q + 1$ with two odd primes p, q. We then consider the multiple nk for any $k \geq 1$ and note that nk belongs to none of the triples ((2p – q + 1)k, pk, qk). This causes a contradiction to (F) and proves the corollary.

□

Note. If we interchange the primes p, q and consider $f'(p,q) = 2q - p + 1$, then f ' also satisfies the condition (f1) for building a f-structure. But for a complete covering of $\mathbb{N}_3$ the number 4 is missing, and we can easily verify that there are other even numbers in $\mathbb{N}_3$ which cannot be represented by f '(p,q).

Another interesting example is $f(p,q) = 2p - q - 3$ versus $f'(p,q) = 2q - p - 3$. f satisfies all conditions, including the covering, and therefore represents all even numbers, whereas f ' satisfies the covering because of $f'(3,5) = 4$, but it violates numerical ordering and distance-preserving. There are even numbers in $\mathbb{N}_3$ which cannot be represented by f ' (p,q).

5. REMARKS

5.1. Due to the unpredictable way that the primes are distributed, all studies on the representation of natural numbers as the sum of primes are problematic when they use approaches based on the distribution of the primes.

Despite tremendous efforts over the centuries, the best result so far was five summands. I was always convinced that the solution must lie in the constructive characteristics of the prime numbers and not in their distribution.

5.2. The statement in the binary Goldbach conjecture actually is nothing more than the symmetric structure (pk, mk, qk) used in the proof. As we have shown, it is in fact a specific case of a general distribution principle within the natural numbers. Furthermore, we note that the multiplicative property of the prime numbers ensures the complete covering of $\mathbb{N}_3$ through the structure.

In order to discard the usual interpretation of the conjecture that focuses on the sums of primes and thus opposes their multiplicative character, we have tackled the problem differently after shifting to the triple form: Instead of searching for primes which determine the needed arithmetic mean equal to a given n, we have approached the issue from the opposite direction. Based on the multiplicative prime decomposition, we identify nk as the component of a structure, in this case determined by the arithmetic mean.

A key point in the proof is the dual role of the numbers k: As multiplier they generate composite numbers while their own multiples in $\mathbb{N}_3$ are strictly set by the used structure.

5.3. In other subject areas, the effect of the formation of new properties after the transition from single items to a whole system is called *emergence* (‘*The whole is more than the sum of its parts.*’). The structure S_g reveals that such principle lies beneath the Goldbach statement: For a given nk, $n \geq 4$, $k \geq 1$, the existence of two odd primes p, q such that nk is the arithmetic mean of pk and qk becomes visible only when we consider all odd primes and all k simultaneously. There is a remarkable aspect of this *emergence*: The two primes which form the so-called Goldbach partition of a given even number 2n are located before 2n, however, the reason for the existence of that partition also involves the primes beyond 2n.

It can be expected that also other questions in number theory own a solution based on this underlying principle.

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